On Jacquet modules of representations of segment type

Ivan Matić and Marko Tadić

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Abstract

Let G_n denote either the group $Sp(n,F)$ or $SO(2n+1,F)$ over a local non-archimedean field F. We study representations of segment type of group G_n, which play a fundamental role in the constructions of discrete series, and obtain a complete description of the Jacquet modules of these representations.

1 Introduction

Let F be a local non-archimedean field characteristic different than two. The representations of reductive groups over F that we shall consider in this paper will be always smooth admissible. In the paper we shall use the standard notation of the representation theory of general linear groups over F introduced by Bernstein and Zelevinsky (see [17]). Recall that Levi factors of maximal parabolic subgroups of general linear groups are direct products of two smaller groups. This fact enables one to consider representation

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parabolically induced by tensor product $\pi_1 \otimes \pi_2$ of two representations of general linear groups, which is denote by

$$\pi_1 \times \pi_2.$$

The parabolic induction that we consider in this paper will be always from the parabolic subgroups standard with respect to the subgroup of upper triangular matrices (the same case will be for Jacquet modules). The Grothendieck group of the category of finite length representations of $GL(n, F)$ is denoted by R_n. The parabolic induction $\times$ defines in a natural way a structure of commutative graded algebra with unit on

$$R = \oplus_{n \in \mathbb{Z}_{\geq 0}} R_n.$$

The induced map from $R \otimes R$ will be denoted by m. (Sums of semi simplifications of) Jacquet modules with respect to the maximal parabolic subgroups define mapping $m^* : R \to R \otimes R$. This gives R the structure of graded coalgebra. Moreover, R is a Hopf algebra.

Denote

$$\nu : GL(n, F) \to \mathbb{R}^\times, g \mapsto |\det(g)|_F$$

where $|\ |_F$ denotes the normalised absolute value. A segment is a set of the form $\{\rho, \nu\rho, \nu^2\rho, \ldots, \nu^k\rho\}$, where ρ is an irreducible cuspidal representation of a general linear group. We denote this set shortly by $[\rho, \nu^k\rho]$. To such a segment is attached a unique irreducible subrepresentation of $\nu^k\rho \times \ldots \times \rho$, which we denote by

$$\delta([\rho, \nu^k\rho]).$$

These are essential square integrable representations, and one gets all such representations in described way.

A very important (and very simple) formula of Bernstein-Zelevinsky theory is

$$m^*(\delta([\rho, \nu^k\rho])) = \sum_{i=-1}^{k} \delta([\nu^{i+1}\rho, \nu^k\rho]) \otimes \delta([\rho, \nu^i\rho]),$$

which by the transitivity of Jacquet modules, describe all the Jacquet modules of irreducible essentially square integrable representations of general linear groups.

One would also like to have such formula for determining Jacquet modules of the representations of classical groups. It is of particular interest to determine Jacquet modules of classes of representations of classical groups whose role in the admissible dual is as important as is the one of essential square integrable representations in the admissible dual of a general linear group. Such description should have an application in the theory of automorphic forms and in the classification of unitary duals.

An analogous problem has been studied for strongly positive representations by the first named author ([6]), and it is mostly based on the fact that the Jacquet module of strongly positive discrete series has a representation of the same type on its classical-group part.

In the present paper we will be concerned with representations of segment type of symplectic and special odd-orthogonal groups over field F. This prominent class of representations, consisting of certain irreducible subquotients of representations induced from an essential square integrable representation and a supercuspidal one, has been introduced by the second named author in [14]. Such representations also appeared as the basic ingredients in the classifications of discrete series and tempered representations of classical groups (we refer the reader to [9] and [16]). Representations of segment type can be viewed as irreducible subquotients of generalized principle series induced from a supercuspidal representation on the classical-group part. We note that composition series of such representations have been obtained by Muić in [10] (in fact, more general class of generalized principle series, having a strongly positive representation on the classical-group part, has been studied there).

In the case of generic reducibilities, representations of segment type are always tempered or discrete series representations. However, for general reducibilities, the representations of segment type might also be the non-tempered ones. The structural formula, which is a version of the geometrical lemma of Bernstein-Zelevinsky, together with certain properties of the representations of segment type obtained in [14], enable us to use an inductive procedure which results with a complete description of Jacquet modules of GL-type and top Jacquet modules of such representations. This results, enhanced by the transitivity of Jacquet modules and some results regarding Jacquet modules of the representations of general linear groups, allow us to determine Jacquet modules of representations of segment type with respect to all standard maximal parabolic subgroups. Since the representations of segment type can appear in several essentially different composition series, we obtain description of their Jacquet modules using case-by-case consideration. However, we impose a convention regarding irreducible constituents of considered composition series, which enables us to write our results in a uniform way.

We note that similar approach has recently been used by the first named author has to provide a description of those Jacquet modules of certain families of discrete series ([7]) which contain an irreducible essentially square integrable representation on the GL-part.

Let us now describe the contents of the paper in more detail. In the following section we introduce some notation which we be used through the paper, while in the third section we recall important properties of the representations of segment type and introduce certain convention which will keep our results uniform. The next three section our devoted to the determination of Jacquet modules of the representations of segment type, using case-by-case consideration. Also, some results obtained in fourth and fifth section help us shorten proofs in the sixth one. In the last two sections we derive some interesting Jacquet modules of discrete series. Firstly, we provide an alternative way for determination of Jacquet modules of strongly positive discrete series. Secondly, we provide a description of top Jacquet modules of general discrete series.

2 Notation

We will first describe the groups that we consider.

Let $J_n = (\delta_{i,n+1-j})_{1 \leq i,j \leq n}$ denote an $n \times n$ matrix, where $\delta_{i,n+1-j}$ stands for the Kronecker symbol. For a square matrix g, we denote by g^t (resp., g^τ) the transposed matrix of g (resp., the transposed matrix of g with respect to the second diagonal). In what follows, we shall fix one of the series of classical groups

$$Sp(n,F) = \left\{ g \in GL(2n,F) : \begin{pmatrix} 0 & -J_n \\ J_n & 0 \end{pmatrix} g^t \begin{pmatrix} 0 & -J_n \\ J_n & 0 \end{pmatrix} = g^{-1} \right\},$$

$$SO(2n+1,F)=\left\{g\in GL(2n+1,F): g^{\tau}=g^{-1}\right\}$$

and denote by G_n a rank n group belonging to the series which we fixed.

The set of standard parabolic subgroups will be fixed in a usual way, i.e., G_n we fix a minimal F-parabolic subgroup consisting of upper-triangular matrices in G_n. Then the Levi factors of standard parabolic subgroups have the form $M \simeq GL(n_1,F)\times\cdots\times GL(n_k,F)\times G_{n'}$. For representations δ_i of $GL(n_i,F)$, $i=1,2,\ldots,k$, and a representation σ of $G_{n'}$, the normalized parabolically induced representation $\mathrm{Ind}_M^{G_n}(\delta_1\otimes\cdots\otimes\delta_k\otimes\sigma)$ will be denoted by $\delta_1\times\cdots\times\delta_k\rtimes\sigma$.

Let $R(G_n)$ denote a Grothendieck group of the category of finite length representations of G_n and define $R(G)=\oplus_{n\geq 0}R(G_n)$. Similarly as in the general linear group case, sums of semisimplifications of Jacquet modules with respect to the maximal parabolic subgroups define mapping $\mu^*: R(G)\to R\otimes R(G)$.

Through the paper, Jacquet module with respect to the minimal standard parabolic subgroup will be called the minimal standard Jacquet module. For representation $\pi\in R(G_n)$ with partial cuspidal support $\sigma\in R(G_{n'})$, the Jacquet module of π with respect to maximal parabolic subgroup having Levi factor equal to $GL(n-n',F)\times G_{n'}$ will be called the Jacquet module of GL-type. Also, the sum of all irreducible subquotients (counted with multiplicities) of $\mu^*(\pi)$ of the form $\tau\otimes\varphi$ where τ is cuspidal, will be denoted by $s_{top}(\pi)$.

We define $\kappa: R\otimes R\to R\otimes R$ by $\kappa(x\otimes y)=y\otimes x$ and extend contragredient $\tilde{}$ to an automorphism of R in a natural way. Let $M^*: R\to R$ be defined by

$$M^*=(m\otimes id)\circ(\tilde{}\otimes m^*)\circ\kappa\circ m^*.$$

We recall the following formulas which hold for ρ not necessary self-dual:

$$M^*\left(\delta([\nu^a\rho,\nu^b\rho])\right)$$

$$=\sum_{i=a-1}^{b}\sum_{j=i}^{b}\delta([\nu^{-i}\tilde{\rho},\nu^{-a}\tilde{\rho}])\times\delta([\nu^{j+1}\rho,\nu^b\rho])\otimes\delta([\nu^{i+1}\rho,\nu^j\rho]).$$

or

$$M^*\left(\delta([\nu^a\rho,\nu^b\rho])\right) = \sum_{k=0}^{b-a+1}\sum_{i=a-1}^{b-k}\delta([\nu^{-i}\tilde{\rho},\nu^{-a}\tilde{\rho}])\times\delta([\nu^{k+i+1}\rho,\nu^b\rho])\otimes\delta([\nu^{i+1}\rho,\nu^{i+k}\rho]).$$

The following lemma, which has been derived in [12], presents a crucial structural formula for our calculations with Jacquet modules.

Lemma 2.1. *Let ρ be an irreducible cuspidal representation of $GL(m,F)$ and $a,b\in\mathbb{R}$ such that $b-a\in\mathbb{Z}_{\geq 0}$. For $\sigma\in R(G_n)$ we write $\mu^*(\sigma)=\sum_{\tau,\sigma'}\tau\otimes\sigma'$. Then the following equalities hold:*

$$\mu^*(\pi\rtimes\sigma)=M^*(\pi)\rtimes\mu^*(\sigma),$$

$$\begin{aligned}\mu^*(\delta([\nu^a\rho,\nu^b\rho])\rtimes\sigma) &= \sum_{i=a-1}^{b}\sum_{j=i}^{b}\sum_{\tau,\sigma'}\delta([\nu^{-i}\widetilde{\rho},\nu^{-a}\widetilde{\rho}])\times\delta([\nu^{j+1}\rho,\nu^b\rho])\times\tau\\ &\otimes\ \delta([\nu^{i+1}\rho,\nu^j\rho])\rtimes\sigma'.\end{aligned}$$

We omit $\delta([\nu^x\rho,\nu^y\rho])$ if $x>y$.

We briefly recall the subrepresentation version of Langlands classification for general linear groups, which is necessary for determination of Jacquet modules of GL-type.

For every irreducible essentially square-integrable representation δ of $GL(n,F)$, there exists an $e(\delta)\in\mathbb{R}$ such that $\nu^{-e(\delta)}\delta$ is unitarizable. Suppose that $\delta_1,\delta_2,\ldots,\delta_k$ are irreducible, essentially square-integrable representations of $GL(n_1,F),GL(n_2,F),\ldots,GL(n_k,F)$ with $e(\delta_1)\leq e(\delta_2)\leq\ldots\leq e(\delta_k)$. Then the induced representation $\delta_1\times\delta_2\times\cdots\times\delta_k$ has a unique irreducible subrepresentation, which we denote by $L(\delta_1,\delta_2,\ldots,\delta_k)$. This irreducible subrepresentation is called the Langlands subrepresentation, and it appears with the multiplicity one in $\delta_1,\delta_2,\ldots,\delta_k$. Every irreducible representation π of $GL(n,F)$ is isomorphic to some $L(\delta_1,\delta_2,\ldots,\delta_k)$. For given π, the representations $\delta_1,\delta_2,\ldots,\delta_k$ are unique up to a permutation.

Since the class of representations which we will study contains certain discrete series representations, we shortly recall basic ingredients of the classification of discrete series for classical groups due to Mœglin and second named author ([8, 9]).

According to this classification, discrete series are in bijective correspondence with admissible Jordan triples. More precisely, discrete series σ of G_n corresponds to the triple of the form $(Jord, \sigma', \epsilon)$, where σ' is a partial cuspidal support of σ, $Jord$ is a finite set (possibly empty) of pairs (c, ρ) (ρ is an irreducible cuspidal self-dual representation of $GL(n_\rho, F)$, $c > 0$ an integer of appropriate parity), while ϵ is a function defined on a subset of $Jord \cup (Jord \times Jord)$ and attains values 1 and -1.

For an irreducible cuspidal self-dual representation ρ of $GL(n_\rho, F)$ we write $Jord_\rho = \{c : (c, \rho) \in Jord\}$. If $Jord_\rho \neq \emptyset$ and $c \in Jord_\rho$, we put $c_- = \max\{d \in Jord_\rho : d < c\}$, if it exists. Now, by definition $\epsilon((c_-, \rho), (c, \rho)) = 1$ if there is some irreducible representation φ such that σ is a subrepresentation of $\delta([\nu^{-\frac{c_- -1}{2}}\rho, \nu^{\frac{c-1}{2}}\rho]) \rtimes \varphi$.

In mentioned classification, every discrete series of G_n is obtained inductively, starting from an alternated Jordan triple which corresponds to strongly positive representation, i.e., to one whose all exponents in the supports of GL-type Jacquet module are strictly greater than zero. In each step one adds a pair of consecutive elements in Jordan block and the expanded ϵ-function equals one on that pair. For more details on this classification we refer the reader to [15] and [16].

3 Representations of segment type

Definition 3.1. *Let ρ and σ be irreducible cuspidal representations of a general linear group and of a classical group, respectively. Let $a, b \in \mathbb{R}$, $b - a \in \mathbb{Z}_{\geq 0}$, such that*

$$0 \leq a + b$$

and

$$\delta([\nu^a\rho, \nu^b\rho]) \rtimes \sigma$$

reduces. Then any irreducible subquotient of the above representation which has $\delta([\nu^a\rho, \nu^b\rho]) \otimes \sigma$ in its Jacquet module (with respect to a standard parabolic subgroups), will be called a representation of segment type.

We emphasize that representations $\delta([\nu^a\rho,\nu^b\rho])\rtimes\sigma$ and $\delta([\nu^{-b}\rho,\nu^{-a}\rho])\rtimes\sigma$ share the same composition series, but the choice $0\leq a+b$ enables us to obtain, using the known formulas for Jacquet modules of GL-type, that representation of segment type is always a subrepresentation of $\delta([\nu^a\rho,\nu^b\rho])\rtimes\sigma$.

Reducibility of the induced representation $\delta([\nu^a\rho,\nu^b\rho])\rtimes\sigma$ implies that ρ is self-dual. Also, let $\alpha\in\mathbb{R}_{\geq 0}$ such that the induced representation $\nu^\alpha\rho\rtimes\sigma$ reduces. For ρ and σ such α is unique, by the results of Silberger [11]. Further, recent results of Arthur imply $\alpha\in(1/2)\mathbb{Z}$ (for more details we refer the reader to [1]).

Note that if $\delta([\nu^a\rho,\nu^b\rho])\rtimes\sigma$ is irreducible, then its Jacquet module contains $\delta([\nu^a\rho,\nu^b\rho])\otimes\sigma$ and in this case we have a complete description of Jacquet modules with respect to maximal parabolic subgroups of this representation.

Further, it has been proved in [13] $\delta([\nu^a\rho,\nu^b\rho])\rtimes\sigma$ reduces if and only if

$$[\nu^a\rho,\nu^b\rho]\cap\{\nu^{-\alpha}\rho,\nu^\alpha\rho\}\neq\emptyset.$$

Thus, in the sequel we shall assume $a,b\in(1/2)\mathbb{Z}$.

Also, we introduce the notion of *proper* Langlands quotient of the induced representation d:

$$L_{proper}(d)=\begin{cases}L(d), & \text{if the corresponding standard module is reducible;}\\ 0, & \text{if the corresponding standard module is irreducible.}\end{cases}$$

In the sequel, we shall take $\delta([\nu^a\rho,\nu^b\rho]_-;\sigma)=0$ if $\delta([\nu^a\rho,\nu^b\rho])\rtimes\sigma$ is a length two representation and if $-a\neq b$.

Definition 3.2. *In the case of reducibility, we define $\delta([\nu^a\rho,\nu^b\rho]_+;\sigma)$ to be an irreducible subquotient of $\delta([\nu^a\rho,\nu^b\rho])\rtimes\sigma$ which has in its Jacquet module of GL-type at least one irreducible subquotient whose all exponents are non-negative.*

Uniqueness of such irreducible subquotient, in general case, is provided by the following lemma.

Lemma 3.3. *Suppose that $\alpha > 0$. There exists a unique irreducible subquotient of the induced representation $\delta([\nu^a\rho, \nu^b\rho]) \rtimes \sigma$ whose Jacquet module of GL-type contains at least one irreducible subquotient with all non-negative exponents.*

Proof. Claim of the lemma obviously holds if the induced representation $\delta([\nu^a\rho, \nu^b\rho]) \rtimes \sigma$ is irreducible. Thus, we may assume $\{-\alpha, \alpha\} \cap [a, b] \neq \emptyset$. It can be directly seen that

$$s_{GL}(\delta([\nu^a\rho, \nu^b\rho]) \rtimes \sigma) = \sum_{i=a}^{b+1} \delta([\nu^{-i+1}\rho, \nu^{-a}\rho]) \times \delta([\nu^i\rho, \nu^b\rho]) \otimes \sigma.$$

It follows immediately that if $a > 0$ then there is unique irreducible subquotient of $s_{GL}(\delta([\nu^a\rho, \nu^b\rho]) \rtimes \sigma)$ with all non-negative exponents, and we obtain such subquotient for $i = a$. Similarly, if $a < 0$ and $a - \frac{1}{2} \in \mathbb{Z}$, we deduce that the unique irreducible subquotient of $s_{GL}(\delta([\nu^a\rho, \nu^b\rho]) \rtimes \sigma)$ having all exponents non-negative is obtained for $i = \frac{1}{2}$ (note that in this case a representation $\delta([\nu^{\frac{1}{2}}\rho, \nu^{-a}\rho]) \times \delta([\nu^{\frac{1}{2}}\rho, \nu^b\rho])$ is irreducible).

Thus, it remains to prove the lemma for $a < 0$, $a \in \mathbb{Z}$. Obviously, $\alpha \leq b$.

In this case, irreducible subquotient of $s_{GL}(\delta([\nu^a\rho, \nu^b\rho]) \rtimes \sigma)$ having all exponents non-negative are $\delta([\nu\rho, \nu^{-a}\rho]) \times \delta([\rho, \nu^b\rho]) \otimes \sigma$ (which appears with multiplicity two) and $L(\delta([\rho, \nu^{-a}\rho]), \delta([\nu\rho, \nu^b\rho])) \otimes \sigma$ (which appears with multiplicity one).

Several possibilities, depending on a, will be considered separately.

Let us first assume $a \leq -\alpha$. By Theorem 2.1 of [10], $\delta([\nu^a\rho, \nu^b\rho]) \rtimes \sigma$ is a length three representation and we denote by π its discrete series subrepresentation whose corresponding ϵ-function ϵ_π satisfies $\epsilon_\pi((-2a+1, \rho), (2b+1, \rho)) = \epsilon_\pi(((-2a+1)_-, \rho), (2a+1, \rho)) = 1$. If we denote by π' a discrete series subrepresentation of $\delta([\nu^a\rho, \nu^b\rho]) \rtimes \sigma$ different than π, it easily follows that its ϵ-function $\epsilon_{\pi'}$ satisfies $\epsilon_{\pi'}((-2a+1, \rho), (2b+1, \rho)) = 1$ and $\epsilon_\pi(((-2a+1)_-, \rho), (2a+1, \rho)) = -1$. Using Lemma 4.1 of [7] and transitivity of Jacquet modules, we obtain that $\delta([\nu\rho, \nu^{-a}\rho]) \times \delta([\rho, \nu^b\rho]) \otimes \sigma$ is not contained in the Langlands quotient of $\delta([\nu^a\rho, \nu^b\rho]) \rtimes \sigma$. Further, using Proposition 7.2 [16] of and transitivity of Jacquet modules, we obtain that $\delta([\nu\rho, \nu^{-a}\rho]) \times \delta([\rho, \nu^b\rho]) \otimes \sigma$ must be in the Jacquet module of π.

Condition $\epsilon_\pi(((-2a+1)_-, \rho), (2a+1, \rho)) = 1$ implies that there is some irreducible representation φ such that π is a subrepresentation of $\delta([\rho, \nu^{-a}\rho]) \rtimes \varphi$. Using Frobenius reciprocity and formula for μ^*, we deduce that φ is an irreducible subquotient of $\delta([\nu\rho, \nu^b\rho]) \rtimes \sigma$. Theorem 4.1 of [10] implies

$$\delta([\nu\rho, \nu^b\rho]) \rtimes \sigma = L_{proper}(\delta([\nu\rho, \nu^b\rho]) \rtimes \sigma) + L_{proper}(\delta([\nu\rho, \nu^{\alpha-1}\rho]) \rtimes \sigma'),$$

where s' is a strongly positive discrete series subrepresentation of $\delta([\nu^\alpha\rho, \nu^b\rho]) \rtimes \sigma$. Since π is square integrable, $\varphi \neq L_{proper}(\delta([\nu\rho, \nu^b\rho]) \rtimes \sigma)$. Consequently, φ equals $L_{proper}(\delta([\nu\rho, \nu^{\alpha-1}\rho]) \rtimes \sigma')$ and it follows immediately that it is a subrepresentation of $\delta([\nu\rho, \nu^b\rho]) \rtimes \sigma$. Thus, Jacquet module of π contains $\delta([\rho, \nu^{-a}\rho]) \otimes \delta([\nu\rho, \nu^b\rho]) \otimes \sigma$. Since Jacquet module of $\delta([\nu\rho, \nu^{-a}\rho]) \times \delta([\rho, \nu^b\rho])$ does not contain $\delta([\rho, \nu^{-a}\rho]) \otimes \delta([\nu\rho, \nu^b\rho])$, transitivity of Jacquet modules forces $L(\delta([\rho, \nu^{-a}\rho]), \delta([\nu\rho, \nu^b\rho])) \otimes \sigma \leq \mu^*(\pi)$.

Let us now assume $-\alpha + 1 < a$. In this case, by Theorem 4.1 of [10], $\delta([\nu^a\rho, \nu^b\rho]) \rtimes \sigma$ is a length two representation and we have

$$\delta([\nu^a\rho, \nu^b\rho]) \rtimes \sigma = L_{proper}(\delta([\nu^a\rho, \nu^b\rho]) \rtimes \sigma) + L_{proper}(\delta([\nu^a\rho, \nu^{\alpha-1}\rho]) \rtimes \sigma'),$$

for strongly positive discrete series subrepresentation σ' of $\delta([\nu^\alpha\rho, \nu^b\rho]) \rtimes \sigma$. It can be directly verified that Jacquet module of $\delta([\nu^a\rho, \nu^{\alpha-1}\rho]) \rtimes \sigma'$ contains $\delta([\nu\rho, \nu^{-a}\rho]) \times \delta([\rho, \nu^b\rho]) \otimes \sigma$ with multiplicity two and $L(\delta([\rho, \nu^{-a}\rho]), \delta([\nu\rho, \nu^b\rho])) \otimes \sigma$ with multiplicity one. Further, applying the same theorem on this induced representation we deduce

$$\delta([\nu^a\rho, \nu^{\alpha-1}\rho]) \rtimes \sigma' = L_{proper}(\delta([\nu^a\rho, \nu^{\alpha-1}\rho]) \rtimes \sigma') + \tau,$$

where τ is a unique common irreducible subquotient of $\delta([\nu^a\rho, \nu^{\alpha-1}\rho]) \rtimes \sigma'$ and $\delta([\nu^a\rho, \nu^{\alpha-2}\rho]) \rtimes \sigma''$, for strongly positive discrete series subrepresentation σ'' of $\nu^{\alpha-1} \times \delta([\nu^\alpha\rho, \nu^b\rho]) \rtimes \sigma$.

Note that Jacquet modules of both $\delta([\nu\rho, \nu^{-a}\rho]) \times \delta([\rho, \nu^b\rho])$ and $L(\delta([\rho, \nu^{-a}\rho]), \delta([\nu\rho, \nu^b\rho]))$ contain irreducible subquotients of the form $\delta([\nu^{\alpha-1}\rho, \nu^b\rho]) \otimes \varphi$.

We will show that Jacquet module of τ does not contain irreducible subquotients of the form $\delta([\nu\rho, \nu^{-a}\rho]) \times \delta([\rho, \nu^b\rho]) \otimes \sigma$ and $L(\delta([\rho, \nu^{-a}\rho]), \delta([\nu\rho, \nu^b\rho])) \otimes \sigma$. Suppose, on the contrary, that some of these representations appears in μ^*. Calculating $\mu^*(\delta([\nu^a\rho, \nu^{\alpha-2}\rho]) \rtimes \sigma'')$ we obtain that $\mu^*(\sigma'')$ contains irreducible subquotient of the form $\delta([\nu^{\alpha-1}\rho, \nu^b\rho]) \otimes \varphi'$ (observe that $a > -\alpha+1$).

This contradicts [6], Theorem 4.6 (or Section 7 of this paper). In consequence, all irreducible subquotient of $s_{GL}(\delta([\nu^a\rho, \nu^b\rho]) \rtimes \sigma)$ having all exponents non-negative appear in Jacquet modules of $L_{proper}(\delta([\nu^a\rho, \nu^{\alpha-1}\rho]) \rtimes \sigma')$.

It remains to consider the case $a = -\alpha + 1$. In this case, again by Theorem 4.1 of [10], we have

$$\delta([\nu^a\rho, \nu^b\rho]) \rtimes \sigma = L_{proper}(\delta([\nu^a\rho, \nu^b\rho]) \rtimes \sigma) + \tau_{temp},$$

where τ_{temp} is a unique common irreducible (tempered) subquotient of induced representations $\delta([\nu^a\rho, \nu^b\rho]) \rtimes \sigma$ and $\delta([\nu^a\rho, \nu^{-a}\rho]) \rtimes \sigma'$, for strongly positive discrete series subrepresentation σ' of $\delta([\nu^\alpha\rho, \nu^b\rho]) \rtimes \sigma$.

Let us first assume $a \neq 0$. Obviously, there is an irreducible representation π (resp., π') such that $\delta([\rho, \nu^{-a}\rho]) \otimes \pi$ (resp., $\delta([\nu\rho, \nu^{-a}\rho]) \otimes \pi'$) is in Jacquet module of τ_{temp}. It is not hard to deduce $\pi \leq \delta([\nu\rho, \nu^b\rho]) \rtimes \sigma$ (resp., $\pi' \leq \delta([\rho, \nu^b\rho]) \rtimes \sigma$). From

$$\delta([\nu\rho, \nu^b\rho]) \rtimes \sigma = L_{proper}(\delta([\nu\rho, \nu^b\rho]) \rtimes \sigma) + L_{proper}(\delta([\nu\rho, \nu^{-a}\rho]) \rtimes \sigma')$$

and

$$\delta([\rho, \nu^b\rho]) \rtimes \sigma = L_{proper}(\delta([\rho, \nu^b\rho]) \rtimes \sigma) + L_{proper}(\delta([\rho, \nu^{-a}\rho]) \rtimes \sigma')$$

for strongly positive discrete series subrepresentation σ' of $\delta([\nu^\alpha\rho, \nu^b\rho]) \rtimes \sigma$, using temperedness of τ_{temp} in the same way as in the previously considered case we deduce $\mu^*(\pi) \geq \delta([\nu\rho, \nu^b\rho]) \otimes \sigma'$ and $\mu^*(\pi') \geq \delta([\rho, \nu^b\rho]) \otimes \sigma$ (with multiplicity two). Transitivity of Jacquet modules implies that all irreducible subquotient of $s_{GL}(\delta([\nu^a\rho, \nu^b\rho]) \rtimes \sigma)$ having all exponents non-negative are contained in Jacquet modules of τ_{temp}.

Now we assume $a = 0$ (i.e., $\alpha = 1$). In this case, since τ_{temp} is a subrepresentation of $\rho \times \delta([\nu\rho, \nu^b\rho]) \rtimes \sigma$, representation $\rho \otimes \delta([\nu^1\rho, \nu^b\rho]) \otimes \sigma$ is in Jacquet module of τ_{temp} and it directly follows $s_{GL}(\tau_{temp}) \geq L(\rho, \delta([\nu\rho, \nu^b\rho])) \otimes \sigma)$. Further, since τ_{temp} is a subrepresentation of $\delta([\rho, \nu^b\rho]) \rtimes \sigma$, using Lemma 4.7 and Corollary 4.9 from [16] we deduce that $\delta([\nu\rho, \nu^b\rho]) \otimes \rho \rtimes \sigma$ is in Jacquet module of τ_{temp}. Since the representation $\rho \rtimes \sigma$ is irreducible, an irreducible subquotient $\delta([\nu\rho, \nu^b\rho]) \otimes \rho \otimes \sigma$ appears with multiplicity two in the Jacquet module of τ_{temp}. Therefore, it easily follows that $\delta([\rho, \nu^b\rho]) \otimes \sigma$ appears with multiplicity two in the Jacquet module of τ_{temp} and Jacquet modules of GL-type of τ_{temp} contain all irreducible subquotient of $s_{GL}(\delta([\rho, \nu^b\rho]) \rtimes \sigma)$ having all exponents non-negative.

This proves the lemma. $\square$

The representations of segment type in the case $\{\nu^{-\alpha}\rho, \nu^{\alpha}\rho\} \subseteq [\nu^a\rho, \nu^b\rho]$, i.e., $[\nu^{-\alpha}\rho, \nu^{\alpha}\rho] \subseteq [\nu^a\rho, \nu^b\rho]$, have been carefully studied in [14]. We take a moment to recall the basic properties of these representations, denoted by $\delta([\nu^a\rho, \nu^b\rho]_{\pm}; \sigma)$ in the case of strictly positive reducibility. They are always non-zero in this case (in other cases that we shall consider below, $\delta([\nu^a\rho, \nu^b\rho]_{\pm}; \sigma)$ will be sometimes 0).

Then in the case of strictly positive reducibility we have

(S1) If $\delta([\nu^a\rho, \nu^b\rho]_{\pm}; \sigma) \neq 0$, then its Jacquet module contains $\delta([\nu^a\rho, \nu^b\rho]) \otimes \sigma$ (by definition).

(S2) $\delta([\nu^a\rho, \nu^b\rho]_{+}; \sigma)$ can be characterized as a subquotient of $\delta([\nu^a\rho, \nu^b\rho]) \rtimes \sigma$ which contains an irreducible subquotient in GL-type Jacquet module whose all exponents in the cuspidal support are non-negative.

(S3) If $\delta([\nu^a\rho, \nu^b\rho]_{-}; \sigma) \neq 0$, then it can be characterized as a subquotient of $\delta([\nu^a\rho, \nu^b\rho]) \rtimes \sigma$ which is a representation of segment type, and which does not contain any irreducible subquotient whose all exponents in the cuspidal support are non-negative in GL-type Jacquet module.

(S4) If $L_{proper}(\delta([\nu^a\rho, \nu^b\rho]); \sigma) \neq 0$, then it does not contain $\delta([\nu^a\rho, \nu^b\rho]) \otimes \sigma$ in its Jacquet module .

(S5) Segment representations are subrepresentations of $\delta([\nu^a\rho, \nu^b\rho]) \rtimes \sigma$.

We shall return to these representations later (i.e., in the case $[-\alpha, \alpha] \subseteq [a, b]$).

To uniformise our notation and results, we introduce the following **Convention**: Suppose $a, b \in (1/2)\mathbb{Z}$, $b - a, a + b \in \mathbb{Z}_{\geq 0}$ such that

$$[\nu^a\rho, \nu^b\rho] \cap \{\nu^{-\alpha}\rho, \nu^\alpha\rho\} = \emptyset.$$

Then $\delta([\nu^a\rho, \nu^b\rho]) \rtimes \sigma$ is irreducible and we define

$$\delta([\nu^a\rho, \nu^b\rho]_+; \sigma) = \begin{cases} 0 & \text{if } [\nu^a\rho, \nu^b\rho] \cap [\nu^{-\alpha}\rho, \nu^\alpha\rho] = \emptyset; \\ \delta([\nu^a\rho, \nu^b\rho]) \rtimes \sigma & \text{if } [\nu^a\rho, \nu^b\rho] \subseteq [\nu^{-\alpha+1}\rho, \nu^{\alpha-1}\rho]. \end{cases}$$

Further, we define

$$L_\alpha(\delta([\nu^a\rho, \nu^b\rho]); \sigma) = \begin{cases} \delta([\nu^a\rho, \nu^b\rho]) \rtimes \sigma & \text{if } [\nu^a\rho, \nu^b\rho] \cap [\nu^{-\alpha}\rho, \nu^\alpha\rho] = \emptyset; \\ 0 & \text{if } [\nu^a\rho, \nu^b\rho] \subseteq [\nu^{-\alpha+1}\rho, \nu^{\alpha-1}\rho]. \end{cases}$$

In the case

$$[\nu^a\rho, \nu^b\rho] \cap \{\nu^{-\alpha}\rho, \nu^\alpha\rho\} \neq \emptyset,$$

we have already defined $\delta([\nu^a\rho, \nu^b\rho]_+; \sigma)$. In this case we simply set

$$L_\alpha(\delta([\nu^a\rho, \nu^b\rho]); \sigma) = L(\delta([\nu^a\rho, \nu^b\rho]); \sigma).$$

Below, positive $\alpha \in (1/2)\mathbb{Z}$ is always the reducibility exponent for ρ and σ.

We now go to study of Jacquet modules of the representations of segment type, with respect to the maximal parabolic subgroups. Several possible cases will be studied separately. We emphasize that the convention introduced earlier enables us to state results obtained in different cases in a uniform way.

4 Representations of segment type corresponding to segments not containing $[\nu^{-\alpha}\rho, \nu^{\alpha}\rho]$

In this section we start our determination of Jacquet modules for representations of segment type. First we study such representations attached to segments which contain exactly one reducibility point.

In what follows, we shall denote segments by

$$[\nu^{-c}\rho, \nu^{d}\rho].$$

Clearly, c (and d) must satisfy $c - \alpha \in \mathbb{Z}$. Further, we assume

$$|c| \leq d.$$

In this section we consider the case when

$$[\nu^{-\alpha}\rho, \nu^{\alpha}\rho] \not\subseteq [\nu^{-c}\rho, \nu^{d}\rho].$$

Theorem 4.1. *Let $c, d \in (1/2)\mathbb{Z}$ such that $d+c, d-c \in \mathbb{Z}_{\geq 0}$, $d - \alpha \in \mathbb{Z}$ and*

$$-\alpha < -c \leq \alpha \leq d.$$

Then

1. *$\delta([\nu^{-c}\rho, \nu^{d}\rho]) \rtimes \sigma$ is a representation of length two, and the composition series consists of $\delta([\nu^{-c}\rho, \nu^{d}\rho]_+; \sigma)$ and $L(\delta([\nu^{-c}\rho, \nu^{d}\rho]); \sigma)$.*
2. *For $-c = \alpha$, $\delta([\nu^{-c}\rho, \nu^{d}\rho]_+; \sigma)$ is square integrable. For $-c = -\alpha + 1$, the representation is tempered, but not square integrable. For $-\alpha + 1 < c$, we have*
$$\delta([\nu^{-c}\rho, \nu^{d}\rho]_+; \sigma) = L(\delta([\nu^{-c}\rho, \nu^{\alpha-1}\rho]); \delta([\nu^{\alpha}\rho, \nu^{d}\rho]_+; \sigma)).$$
3. *If $c < d$, then*
$$s_{top}(\delta([\nu^{-c}\rho, \nu^{d}\rho]_+; \sigma)) = \nu^{d}\rho \otimes \delta([\nu^{-c}\rho, \nu^{d-1}\rho]_+; \sigma) + \nu^{c}\rho \otimes \delta([\nu^{-c+1}\rho, \nu^{d}\rho]_+; \sigma).$$
For $c = d$ $(= \alpha)$, we have $s_{top}(\delta(\nu^{\alpha}\rho_+; \sigma)) = \nu^{\alpha}\rho \otimes \sigma$.

4. *If $c < d$, then*

$$\begin{aligned} s_{top}(L(\delta([\nu^{-c}\rho, \nu^{d}\rho]); \sigma)) &= \nu^{d}\rho \otimes L_{\alpha}(\delta([\nu^{-c}\rho, \nu^{d-1}\rho]); \sigma) + \\ &\quad \nu^{c}\rho \otimes L_{\alpha}(\delta([\nu^{-c+1}\rho, \nu^{d}\rho]); \sigma). \end{aligned}$$

For $c = d$ $(= \alpha)$, we have $s_{top}(L(\nu^{\alpha}\rho; \sigma)) = \nu^{-\alpha}\rho \otimes \sigma$.

5. $s_{GL}(\delta([\nu^{-c}\rho, \nu^{d}\rho]_{+}; \sigma)) = \sum_{i=-c}^{\alpha} \delta([\nu^{-i+1}\rho, \nu^{c}\rho]) \times \delta([\nu^{i}\rho, \nu^{d}\rho]) \otimes \sigma.$

6. *For $c < d$ holds*

$$\begin{aligned} s_{GL}(L(\delta([\nu^{-c}\rho, \nu^{d}\rho]); \sigma)) &= \sum_{i=\alpha+1}^{d+1} \delta([\nu^{-i+1}\rho, \nu^{c}\rho]) \times \delta([\nu^{i}\rho, \nu^{d}\rho]) \otimes \sigma \\ &= \sum_{i=\alpha+1}^{d+1} L(\delta([\nu^{-i+1}\rho, \nu^{c}\rho]), \delta([\nu^{i}\rho, \nu^{d}\rho])) \otimes \sigma. \end{aligned}$$

Proof. For $c = d$ $(= \alpha)$, the theorem holds (this is a very well and very simple known fact). Therefore, In the proof we shall consider only the case

$$c < d.$$

Recall

$$\begin{aligned} & s_{GL}(\delta([\nu^{-c}\rho, \nu^{d}\rho]) \rtimes \sigma) \\ &= \sum_{i=-c}^{d+1} \delta([\nu^{-i+1}\rho, \nu^{c}\rho]) \times \delta([\nu^{i}\rho, \nu^{d}\rho]) \otimes \sigma, \end{aligned} \tag{4.1}$$

Observe that for any $\alpha + 1 \leq j \leq d + 1$ we have

$$\begin{aligned} L(\delta([\nu^{-c}\rho, \nu^{d}\rho]); \sigma) &\hookrightarrow \delta([\nu^{-d}\rho, \nu^{c}\rho]) \rtimes \sigma \\ &\hookrightarrow \delta([\nu^{-j+1}\rho, \nu^{c}\rho]) \times \delta([\nu^{-d}\rho, \nu^{-j}\rho]) \rtimes \sigma \\ &\cong \delta([\nu^{-j+1}\rho, \nu^{c}\rho]) \times \delta([\nu^{j}\rho, \nu^{d}\rho]) \rtimes \sigma, \end{aligned}$$

since the representation $\delta([\nu^{j}\rho, \nu^{d}\rho]) \rtimes \sigma$ is irreducible ([13]).

Now in $s_{GL}(L(\delta([\nu^{-\alpha+1}\rho, \nu^{d}\rho]); \sigma))$ we must have representations with the cuspidal supports which follow from all embeddings of the above type. Such representations are among subquotients of (4.1) and considered cuspidal supports show up precisely in the following part of (4.1):

$$\sum_{i=\alpha+1}^{d+1} \delta([\nu^{-i+1}\rho, \nu^c\rho]) \times \delta([\nu^i\rho, \nu^d\rho]) \otimes \sigma.$$

Observe that all the representations in the above sum are irreducible. This implies

$$s_{GL}(L(\delta([\nu^{-\alpha+1}\rho, \nu^d\rho]); \sigma)) \geq \sum_{i=\alpha+1}^{d+1} \delta([\nu^{-i+1}\rho, \nu^c\rho]) \times \delta([\nu^i\rho, \nu^d\rho]) \otimes \sigma.$$

The formula for μ^* gives in this case

$$s_{top}(\delta([\nu^{-c}\rho, \nu^d\rho]) \rtimes \sigma)) = \nu^d\rho \otimes \delta([\nu^{-c}\rho, \nu^{d-1}\rho]) \rtimes \sigma + \nu^c\rho \otimes \delta([\nu^{-c+1}\rho, \nu^d\rho]) \rtimes \sigma.$$

At this point, we shall show that (2) follows from other statements. The square integrability in the case $-c = \alpha$ and temperedness in the case $-c = -\alpha + 1$, claimed there, follow directly from (5). In the remaining cases it can be seen directly from Theorem 4.1 of [10] that we have the Langlands parameter given in (2) (note that the representation $\delta([\nu^\alpha\rho, \nu^d\rho]; \sigma)$ is strongly positive).

Now we go to the proof of the theorem.

Note that in the case $-c = \alpha$ (1), (2), (3) and (5) follow directly from [5] and [6]. One gets (4) and (6) using additionally two above formulas (for s_{top} and s_{GL} of $\delta([\nu^{-c}\rho, \nu^d\rho]) \rtimes \sigma$). This completes the proof for this case. Therefore, in what follows, we shall always assume

$$-\alpha + 1 \leq -c \leq \alpha - 1.$$

Now we shall prove the theorem for $d = \alpha$. We shall prove it by induction on $-c$. For $-c = \alpha$ we have observed that the theorem holds. Now we shall fix $-c$ as above and assume that the theorem holds for $-c + 1$ (and $d = \alpha$; clearly, $c < \alpha$). Now the inductive assumption implies

$$s_{top}(\delta([\nu^{-c}\rho, \nu^\alpha\rho]) \rtimes \sigma)) = \nu^\alpha\rho \otimes \delta([\nu^{-c}\rho, \nu^{\alpha-1}\rho]_+; \sigma)$$

$$+\nu^c\rho \otimes \delta([\nu^{-c+1}\rho, \nu^\alpha\rho]_+; \sigma) + \nu^c\rho \otimes L_\alpha(\delta([\nu^{-c+1}\rho, \nu^\alpha\rho]); \sigma).$$

Obviously, the first summand must be in the Jacquet module of $\delta([\nu^{-c+1}\rho, \nu^\alpha\rho]_+; \sigma)$, while the last summand must be in the Jacquet module of the Langlands quotient.

Observe that if $c \geq 0$, the second summand must be in the Jacquet module of $\delta([\nu^{-c}\rho, \nu^\alpha\rho]_+; \sigma)$. We prove that this also holds in the case $c < 0$. Obviously, the term $\delta([\nu^{-c}\rho, \nu^\alpha\rho]) \otimes \sigma$ in (4.1) is the only term with such cuspidal support. Thus

$$\delta([\nu^{-c}\rho, \nu^\alpha\rho]_+; \sigma) \hookrightarrow \delta([\nu^c\rho, \nu^\alpha\rho]) \rtimes \sigma \hookrightarrow \delta([\nu^{-c+1}\rho, \nu^\alpha\rho]) \times \nu^{-c} \rtimes \sigma$$

$$\cong \delta([\nu^{-c+1}\rho, \nu^\alpha\rho]) \times \nu^c\rho \rtimes \sigma \cong \nu^c\rho \times \delta([\nu^{-c+1}\rho, \nu^\alpha\rho]) \rtimes \sigma.$$

Using Frobenius reciprocity and transitivity of Jacquet modules we deduce that the second summand in $s_{top}(\delta([\nu^{-c}\rho, \nu^\alpha\rho]) \rtimes \sigma)$ is in the Jacquet module of $\delta([\nu^{-c}\rho, \nu^\alpha\rho]_+; \sigma)$.

This analysis of $s_{top}(\delta([\nu^{-c}\rho, \nu^\alpha\rho]) \rtimes \sigma)$ implies length two of $\delta([\nu^{-c}\rho, \nu^\alpha\rho]) \rtimes \sigma$, and the rest of (1). Also, it implies (3) and (4). Observe that we have just proven

$$s_{top}(L_\alpha(\delta([\nu^{-c}\rho, \nu^\alpha\rho]); \sigma)) = \nu^c\rho \otimes L_\alpha(\delta([\nu^{-c+1}\rho, \nu^\alpha\rho]); \sigma),$$

while the inductive assumption implies

$$s_{GL}(L_\alpha(\delta([\nu^{-c+1}\rho, \nu^\alpha\rho]); \sigma)) = \delta([\nu^{-\alpha}\rho, \nu^{c-1}\rho]) \otimes \sigma.$$

This implies that the minimal Jacquet module of $L_\alpha(\delta([\nu^{-c+1}\rho, \nu^\alpha\rho]); \sigma)$ is $\nu^c\rho \otimes \nu^{c-1}\rho \otimes \ldots \otimes \nu^{-\alpha}\rho \otimes \sigma$. This yields (6), which (6) directly implies (5) (using (1) and (4.1)). This completes the case of $d = \alpha$.

It remains to consider the case

$$-\alpha < -c < \alpha < d.$$

We shall prove the theorem by induction on d. Observe that the theorem holds for $d = \alpha$. We shall fix $d > \alpha$ and suppose that the theorem holds for $d - 1$. We shall show that theorem holds for d by induction on $-c$. Recall that the theorem holds for $-c = -\alpha$. We shall fix $-c > -\alpha$, and suppose that the theorem holds for $-c + 1$.

We consider two cases. First we discuss the case

$$0 < -c.$$

Observe that in the sum

$$s_{GL}(\delta([\nu^{-c}\rho, \nu^{d}\rho]) \rtimes \sigma) = \sum_{i=-c}^{d+1} \delta([\nu^{-i+1}\rho, \nu^{c}\rho]) \times \delta([\nu^{i}\rho, \nu^{d}\rho]) \otimes \sigma$$

all the summands are irreducible since we have $c < i$. Also, all the summands have different cuspidal supports.

In this case $\delta([\nu^{-c}\rho, \nu^{d}\rho]_{+}, \sigma)$ is the subquotient of $\delta([\nu^{-c}\rho, \nu^{d}\rho]) \rtimes \sigma$ which has $\delta([\nu^{-c}\rho, \nu^{d}\rho]) \otimes \sigma$ in its Jacquet module. Observe that the last representation has multiplicity one in the full Jacquet module, and it is direct summand in the Jacquet module (other terms have different cuspidal support). This implies

$$\delta([\nu^{-c}\rho, \nu^{d}\rho]_{+}, \sigma) \hookrightarrow \nu^{d}\rho \times \cdots \times \nu^{-c+1}\rho \times \nu^{-c}\rho \rtimes \sigma$$
$$\cong \nu^{d}\rho \times \cdots \times \nu^{-c+1}\rho \times \nu^{c}\rho \rtimes \sigma \cong \ldots \cong \nu^{c}\rho \times \nu^{d}\rho \times \cdots \times \nu^{-c+1}\rho \rtimes \sigma$$
$$\cong \ldots \cong \nu^{c}\rho \times \nu^{c-1}\rho \times \cdots \times \nu^{\alpha+1}\rho \times \nu^{d}\rho \times \cdots \times \nu^{\alpha}\rho \rtimes \sigma.$$

Now above observations regarding Jacquet modules and cuspidal supports imply that $\delta([\nu^{-c}\rho, \nu^{d}\rho]_{+}, \sigma)$ has in its Jacquet module of GL-type representations with GL-cuspidal support the same as

$$\delta([\nu^{-i+1}\rho, \nu^{c}\rho]) \times \delta([\nu^{i}\rho, \nu^{d}\rho]), \ -c \leq i \leq \alpha.$$

This implies

$$\sum_{i=-c}^{\alpha} \delta([\nu^{-i+1}\rho, \nu^{c}\rho]) \times \delta([\nu^{i}\rho, \nu^{d}\rho]) \leq s_{GL}(\delta([\nu^{-c}\rho, \nu^{d}\rho]_{+}, \sigma)).$$

We have already seen

$$\sum_{i=\alpha+1}^{d+1} \delta([\nu^{i}\rho, \nu^{d}\rho]) \times \delta([\nu^{-i+1}\rho, \nu^{c}\rho])) \otimes \sigma \leq s_{GL}(L(\delta([\nu^{-c}\rho, \nu^{d}\rho]); \sigma)). \quad (4.2)$$

Since the sum of the left hand sides is the whole Jacquet module of GL-type of $\delta([\nu^{-c}\rho, \nu^d\rho]) \rtimes \sigma$, we conclude that $\delta([\nu^{-c}\rho, \nu^d\rho]) \rtimes \sigma$ is a representation of length two and in the above two inequalities we actually have equalities.

In this way we have proved (1), (5) and (6).

Note that the inductive assumptions imply

$$s_{top}(\delta([\nu^{-c}\rho, \nu^d\rho])\rtimes\sigma)) = \nu^d\rho\otimes\delta([\nu^{-c}\rho, \nu^{d-1}\rho]_+;\sigma)+\nu^d\rho\otimes L_\alpha(\delta([\nu^{-c}\rho, \nu^{d-1}\rho]);\sigma)$$

$$+\nu^c\rho \otimes \delta([\nu^{-c+1}\rho, \nu^d\rho]_+;\sigma) + \nu^c\rho \otimes L_\alpha(\delta([\nu^{-c+1}\rho, \nu^d\rho]);\sigma).$$

Obviously, the first summand belongs to $\delta([\nu^{-c}\rho, \nu^d\rho]_+;\sigma)$, while the last one belongs to the Langlands quotient (use an argument of the type that we have already used earlier). From (5) follows that $\delta([\nu^{-c}\rho, \nu^d\rho]_+;\sigma)$ must have the third summand in its Jacquet module (we need to have in Jacquet module something of the form $\nu^c\rho \otimes \ldots \otimes \sigma$). Similarly, (6) implies that the second term must be in the Jacquet module of the Langlands quotient. This proves (3) and (4).

It remains to prove the theorem in the case of

$$-\alpha < -c \leq 0 < \alpha < d.$$

From inductive assumption we know

$$s_{top}(\delta([\nu^{-c}\rho, \nu^d\rho])\rtimes\sigma)) = \nu^d\rho\otimes\delta([\nu^{-c}\rho, \nu^{d-1}\rho]_+;\sigma)+\nu^d\rho\otimes L_\alpha(\delta([\nu^{-c}\rho, \nu^{d-1}\rho]);\sigma)$$

$$+\nu^c\rho \otimes \delta([\nu^{-c+1}\rho, \nu^d\rho]_+;\sigma) + \nu^c\rho \otimes L_\alpha(\delta([\nu^{-c+1}\rho, \nu^d\rho]);\sigma).$$

Observe that all four summands above are non-zero. Looking at the most positive terms, we see that the first and the third term must come from $\delta([\nu^{-c}\rho, \nu^d\rho]_+;\sigma)$, while the last comes one from the Langlands quotient. It remains to determine where belongs the second summand.

Recall that the Langlands quotient embeds into

$$\begin{aligned} L(\delta([\nu^c\rho, \nu^{-d}\rho]);\sigma) &\hookrightarrow \nu^c\rho \times \cdots \times \nu^{-d}\rho \rtimes \sigma \cong \nu^c\rho \times \cdots \times \nu^d \rtimes \sigma \\ &\cong \nu^d\rho \times \nu^c\rho \times \cdots \times \nu^{-d+1}\rho \rtimes \sigma, \end{aligned}$$

since $\alpha < d$, $-\alpha + 1 \leq -c$ and $\alpha < d$. This tells that the second summand is in the Jacquet module of the Langlands quotient.

In this way we get the proof of (1), (3) and (4).

Observe that (5) holds if and only if (6) holds (because of (4.1) and length two of $\delta([\nu^c\rho, \nu^d\rho]) \rtimes \sigma$). Further, by the above estimate (4.2), we also know that the left-hand side of (6) contains the right-hand one.

Let ρ be a representation of $GL(p, F)$. We show that an equality holds on the level of semi simplifications of Jacquet modules $r_{(p,(d+c)p)}$ (from Bernstein-Zelevinsky GL-classification) when applied to the left and the right hand sides of (6).

The result for the left hand side follows from the inductive assumption (4). Using this assumption, we deduce that the corresponding Jacquet module of the left hand side is

$$\sum_{i=\alpha+1}^{d} \nu^d\rho \otimes \delta([\nu^{-i+1}\rho, \nu^c\rho]) \times \delta([\nu^i\rho, \nu^{d-1}\rho]) \otimes \sigma$$

$$+ \sum_{i=\alpha+1}^{d+1} \nu^c\rho \otimes \delta([\nu^{-i+1}\rho, \nu^{c-1}\rho]) \times \delta([\nu^i\rho, \nu^d\rho]) \otimes \sigma.$$

Applying the Jacquet module $r_{(p,(d+c)p)}$ to the right hand side of (6) we obtain

$$\left(\sum_{i=\alpha+1}^{d+1} r_{(p,(d+c)p)}(\delta([\nu^{-i+1}\rho, \nu^c\rho]) \times \delta([\nu^i\rho, \nu^d\rho])) \right) \otimes \sigma$$

$$= \left(\sum_{i=\alpha+1}^{d+1} \nu^c\rho \otimes \delta([\nu^{-i+1}\rho, \nu^{c-1}\rho]) \times \delta([\nu^i\rho, \nu^d\rho]) \right) \otimes \sigma$$

$$+ \left(\sum_{i=\alpha+1}^{d} \nu^d\rho \otimes \delta([\nu^{-i+1}\rho, \nu^c\rho]) \times \delta([\nu^i\rho, \nu^{d-1}\rho]) \right) \otimes \sigma.$$

Observe that we have proved the equality. Therefore, (6) (and (5)) holds.

Observe that in this case $\delta([\nu^{-\alpha+1}\rho, \nu^d\rho]) \otimes \sigma$ has multiplicity two in the Jacquet module of (5).

The proof is now complete. $\square$

In the case when a discrete series subquotient appears we have the following:

Remark 4.2. *If $-\alpha < -c \leq \alpha \leq d$ holds and $\delta([\nu^{-c}\rho, \nu^{d}\rho]_{+}; \sigma)$ is a square integrable, i.e., $-c = \alpha$, we note that $\delta([\nu^{-c}\rho, \nu^{d}\rho]_{+}; \sigma)$ is strongly positive. Further, $Jord(\delta([\nu^{-c}\rho, \nu^{d}\rho]_{+}; \sigma)) = Jord(\sigma) \setminus \{(2\alpha - 1, \rho)\} \cup \{(2d+1, \rho)\}$.*

In the sequel, we shall symmetrize notation, i.e define $\delta([\nu^{-d}\rho, \nu^{c}\rho]_{+}; \sigma)$ to be $\delta([\nu^{-c}\rho, \nu^{d}\rho]_{+}; \sigma)$, $L(\delta([\nu^{-d}\rho, \nu^{c}\rho]); \sigma)$ to be $L(\delta([\nu^{-c}\rho, \nu^{d}\rho]); \sigma)$ etc.

Assume $-\alpha < -c \leq \alpha \leq d$. Consider

$$\mu^{*}\left(\delta([\nu^{-c}\rho, \nu^{d}\rho]) \rtimes \sigma\right)$$

$$= \sum_{i=-c-1}^{d} \sum_{j=i}^{d} \delta([\nu^{-i}\rho, \nu^{c}\rho]) \times \delta([\nu^{j+1}\rho, \nu^{d}\rho]) \otimes \delta([\nu^{i+1}\rho, \nu^{j}\rho]) \rtimes \sigma.$$

$$= \sum_{i=-c-1}^{\alpha-1} \sum_{j=i}^{d} \delta([\nu^{-i}\rho, \nu^{c}\rho]) \times \delta([\nu^{j+1}\rho, \nu^{d}\rho]) \otimes \delta([\nu^{i+1}\rho, \nu^{j}\rho]) \rtimes \sigma.$$

$$+ \sum_{i=\alpha}^{d} \sum_{j=i}^{d} \delta([\nu^{-i}\rho, \nu^{c}\rho]) \times \delta([\nu^{j+1}\rho, \nu^{d}\rho]) \otimes \delta([\nu^{i+1}\rho, \nu^{j}\rho]) \rtimes \sigma.$$

Observe that if on the left hand side of tensor product some exponent less than or equal to $-\alpha$ appears in the cuspidal support, then $[-\alpha, \alpha] \cap [i+1, j] = \emptyset$.

Corollary 4.3. *Let $c, d \in (1/2)\mathbb{Z}$ such that $d + c, d - c \in \mathbb{Z}_{\geq 0}$, $d - \alpha \in \mathbb{Z}$ and $-\alpha < -c \leq \alpha \leq d$. We have*

$$\mu^{*}\left(\delta([\nu^{-c}\rho, \nu^{d}\rho]_{+}; \sigma)\right)$$

$$= \sum_{i=-c-1}^{c} \sum_{j=i+1}^{d} \delta([\nu^{-i}\rho, \nu^{c}\rho]) \times \delta([\nu^{j+1}\rho, \nu^{d}\rho]) \otimes \delta([\nu^{i+1}\rho, \nu^{j}\rho]_{+}; \sigma) +$$

$$\sum_{\substack{-c-1 \leq i \leq c, \\ i+j<-1}} \sum_{i+1 \leq j \leq c,} \delta([\nu^{-i}\rho, \nu^{c}\rho]) \times \delta([\nu^{j+1}\rho, \nu^{d}\rho]) \otimes L_{\alpha}(\delta([\nu^{i+1}\rho, \nu^{j}\rho]); \sigma)$$

$$\sum_{i=-c}^{\alpha} \delta([\nu^{-i+1}\rho, \nu^{c}\rho]) \times \delta([\nu^{i}\rho, \nu^{d}\rho]) \otimes \sigma,$$

$$\mu^*\left(L(\delta([\nu^{-c}\rho,\nu^d\rho]);\sigma)\right)=\mu^*\left(L_\alpha(\delta([\nu^{-c}\rho,\nu^d\rho]);\sigma)\right)=\mu^*\left(L_{proper}(\delta([\nu^{-c}\rho,\nu^d\rho]);\sigma)\right)$$

$$=\sum_{\substack{-c-1\leq i\leq c,\\ 0\leq i+j}}\sum_{i+1\leq j\leq c,} L(\delta([\nu^{-i}\rho,\nu^c\rho]),\delta([\nu^{j+1}\rho,\nu^d\rho]))\otimes L_\alpha(\delta([\nu^{i+1}\rho,\nu^j\rho]);\sigma))+$$

$$\sum_{i=\alpha+1}^{d+1} L(\delta([\nu^{-i+1}\rho,\nu^c\rho]),\delta([\nu^i\rho,\nu^d\rho]))\otimes\sigma.$$

Proof. Note that, by the convention introduced earlier, the second sum appearing in $\mu^*\left(\delta([\nu^{-c}\rho,\nu^d\rho]_+;\sigma)\right)$ is zero, while the first sum in $\mu^*\left(L(\delta([\nu^{-c}\rho,\nu^d\rho]);\sigma)\right)$ equals

$$\sum_{i=-c-1}^{d}\sum_{j=i+1}^{d}\delta([\nu^{-i}\rho,\nu^c\rho])\times\delta([\nu^{j+1}\rho,\nu^d\rho])\otimes L_\alpha(\delta([\nu^{i+1}\rho,\nu^j\rho]);\sigma)).$$

The following identity is direct consequence of the structural formula:

$$\mu^*\left(\delta([\nu^{-c}\rho,\nu^d\rho])\rtimes\sigma\right)$$

$$=\sum_{i=-c-1}^{d}\sum_{j=i}^{d}\delta([\nu^{-i}\rho,\nu^c\rho])\times\delta([\nu^{j+1}\rho,\nu^d\rho])\otimes\left(\delta([\nu^{i+1}\rho,\nu^j\rho]_+;\sigma)+L_\alpha(\delta([\nu^{i+1}\rho,\nu^j\rho]);\sigma)\right).$$

We shall first analyze the case when $\delta([\nu^{i+1}\rho,\nu^j\rho])\rtimes\sigma$ is irreducible. Since $\alpha-1\leq -c\leq i\leq d$, we can have irreducibility only in the following two cases.

The first case is $j<\alpha$. Then $\delta([\nu^{i+1}\rho,\nu^j\rho]_+;\sigma)=\delta([\nu^{i+1}\rho,\nu^j\rho])\rtimes\sigma$ and $L_\alpha(\delta([\nu^{i+1}\rho,\nu^j\rho]);\sigma)=0$ (by our convention). Now it is enough to prove that

$$\delta([\nu^{-i}\rho,\nu^c\rho])\times\delta([\nu^{j+1}\rho,\nu^d\rho])\otimes\delta([\nu^{i+1}\rho,\nu^j\rho])\rtimes\sigma$$

cannot be in the Jacquet module of the Langlands quotient. Suppose that it is. Then the formula for $s_{GL}(\delta([\nu^{i+1}\rho,\nu^j\rho])\rtimes\sigma)$ implies that in the minimal non-zero standard Jacquet module we would have at least one term of the form $\ldots\otimes\nu^\alpha\rho\otimes\sigma$.

On the other side, (6) of the previous theorem tells us such subquotients for the Jacquet module of the Langlands quotient can be only of the form

$$\ldots\otimes\nu^{-i+1}\rho\otimes\sigma\quad\text{where}\quad\alpha+1\leq i\leq d+1,\quad\text{or}\quad\ldots\otimes\nu^i\rho\otimes\sigma,\quad\text{where}\quad\alpha+1\leq i\leq d.$$

Obviously, the term of the form $\ldots \otimes \nu^{\alpha}\rho \otimes \sigma$ cannot be among them, and we get contradiction.

The second case when we have irreducibility of $\delta([\nu^{i+1}\rho, \nu^{j}\rho]) \rtimes \sigma$ is when

$$\alpha < i + 1.$$

Then $\delta([\nu^{i+1}\rho, \nu^{j}\rho]_{+}; \sigma) = 0$ and $L_{\alpha}(\delta([\nu^{i+1}\rho, \nu^{j}\rho]); \sigma) = \delta([\nu^{i+1}\rho, \nu^{j}\rho]) \rtimes \sigma$ (by our convention). Now it is enough to prove that

$$\delta([\nu^{-i}\rho, \nu^{c}\rho]) \times \delta([\nu^{j+1}\rho, \nu^{d}\rho]) \otimes \delta([\nu^{i+1}\rho, \nu^{j}\rho]) \rtimes \sigma$$

cannot be in the Jacquet module of the $\delta([\nu^{-c}\rho, \nu^{d}\rho]_{+}; \sigma)$. Suppose that it is. Then we would have in the minimal non-zero (standard) Jacquet module of $\delta([\nu^{-c}\rho, \nu^{d}\rho]_{+}; \sigma)$ a term of the form $\ldots \otimes \nu^{i+1}\rho \otimes \sigma$, where $\alpha < i + 1$.

From the other side, the formula in (5) of the previous theorem implies that $\delta([\nu^{-c}\rho, \nu^{d}\rho]_{+}; \sigma)$ can have in that Jacquet module only the terms of the form

$$\ldots \otimes \nu^{-i'+1} \otimes \sigma \quad \text{where} \quad -c+1 \leq i' \leq \alpha \quad \text{or} \quad \ldots \otimes \nu^{i'} \otimes \sigma \quad \text{where} \quad -c \leq i' \leq \alpha.$$

Obviously, the above term of the form $\ldots \otimes \nu^{i+1}\rho \otimes \sigma$ with $\alpha < i+1$ can not be among these terms. We again got a contradiction.

It remains to consider the case when $\delta([\nu^{i+1}\rho, \nu^{j}\rho]) \rtimes \sigma$ reduces. In this case we must have $-\alpha + 1 \leq i + 1 \leq \alpha \leq j$. For the proof of the corollary, we need to prove two facts. The first is that

$$\delta([\nu^{-i}\rho, \nu^{c}\rho]) \times \delta([\nu^{j+1}\rho, \nu^{d}\rho]) \otimes L_{\alpha}(\delta([\nu^{i+1}\rho, \nu^{j}\rho]); \sigma)$$

cannot be in the Jacquet module of the $\delta([\nu^{-c}\rho, \nu^{d}\rho]_{+}; \sigma)$. Suppose, on the contrary, that it is in that Jacquet module.

Now (6) of the previous theorem (applied to $L_{\alpha}(\delta([\nu^{i+1}\rho, \nu^{j}\rho]); \sigma)$) tells us that in the minimal non-zero standard Jacquet module of $\delta([\nu^{-c}\rho, \nu^{d}\rho]); \sigma)$ we would have at least one term of the form $\ldots \otimes \nu^{\alpha+1}\rho \otimes \sigma$.

Recall that (5) of the previous theorem tells us that such subquotients for the Jacquet module of $\delta([\nu^{-c}\rho, \nu^{d}\rho]_{+}; \sigma)$ can be only of the form

$$\ldots \otimes \nu^{-i+1}\rho \otimes \sigma \quad \text{where} \quad -c+1 \leq i \leq \alpha, \quad \text{or} \quad \ldots \otimes \nu^{i}\rho \otimes \sigma, \quad \text{where} \quad -c \leq i \leq \alpha.$$

Observe that the above terms cannot contain a term of the form $\ldots \otimes \nu^{\alpha+1}\rho \otimes \sigma$. Therefore, we get a contradiction.

It can be seen in a completely analogous manner that

$$\delta([\nu^{-i}\rho, \nu^{c}\rho]) \times \delta([\nu^{j+1}\rho, \nu^{d}\rho]) \otimes \delta([\nu^{i+1}\rho, \nu^{j}\rho]_{+}; \sigma)$$

cannot be in the Jacquet module of the $L_\alpha(\delta([\nu^{-c}\rho, \nu^{d}\rho]); \sigma)$.

The proof of the corollary is now complete. □

Remark 4.4. *Suppose that $c = d < \alpha$. With the above convention regarding symmetrization we have*

$$\mu^*\left(\delta([\nu^{-c}\rho, \nu^{c}\rho]_{+}; \sigma)\right)$$

$$= \sum_{i=-c-1}^{c} \sum_{j=i}^{c} \delta([\nu^{-i}\rho, \nu^{c}\rho]) \times \delta([\nu^{j+1}\rho, \nu^{c}\rho]) \otimes \delta([\nu^{i+1}\rho, \nu^{j}\rho]_{+}; \sigma).$$

Analogous relation holds if we put on the left hand side L_{proper} (or L_α), and L_α on the right hand side (since all the terms are 0).

5 Representations of segment type corresponding to segments containing $[\nu^{-\alpha}\rho, \nu^{\alpha}\rho]$

We shall first recall of some facts from [14]. More details can be found there.

In this section we suppose

$$-c \leq -\alpha < \alpha \leq d \quad \text{and} \quad c \leq d.$$

First we shall recall the case $c = d$.

We consider the representation

$$\delta([\nu^{-c}\rho, \nu^{c}\rho]) \rtimes \sigma.$$

It is unitarizable, multiplicity one representation, of length at most two, whose each irreducible subquotient is a subrepresentation, and has $\delta([\nu^{-c}\rho, \nu^c\rho]) \otimes \sigma$ in its Jacquet module.

For an irreducible subquotient π of $\delta([\nu^{-c}\rho, \nu^c\rho]) \rtimes \sigma$ we easily see

$$\sum_{i=-c}^{-\alpha} \delta([\nu^{-i+1}\rho, \nu^c\rho]) \times \delta([\nu^i\rho, \nu^c\rho]) \otimes \sigma \leq s_{GL}(\pi).$$

Further, $\delta([\nu^{-c}\rho, \nu^c\rho]) \rtimes \sigma$ and $\delta([\nu^{-c}\rho, \nu^{\alpha-1}\rho]) \rtimes \delta([\nu^\alpha\rho, \nu^c\rho]; \sigma)$ have precisely one irreducible subquotient in common. That subquotient has the most positive part in its Jacquet module and is equal to $\delta([\nu^{-c}\rho, \nu^c\rho]_+; \sigma)$, while the other irreducible subquotient of $\delta([\nu^{-c}\rho, \nu^c\rho]) \rtimes \sigma$ is $\delta([\nu^{-c}\rho, \nu^c\rho]_-; \sigma)$.

Let us now consider the case

$$\alpha \leq c < d.$$

We defined $\delta([\nu^{-c}\rho, \nu^d\rho]_+; \sigma)$ by the most positive term in the Jacquet module. This most positive part is also in the Jacquet module of

$$\delta([\nu^{-c}\rho, \nu^{\alpha-1}\rho]) \times \delta([\nu^\alpha\rho, \nu^d\rho]) \rtimes \sigma.$$

and it follows that the multiplicity of $\delta([\nu^{-c}\rho, \nu^d\rho]) \otimes \sigma$ is at most one in the Jacquet module of $\delta([\nu^{-c}\rho, \nu^c\rho]_+; \sigma)$. We denote the other irreducible subquotient (which is also a subrepresentation) of $\delta([\nu^{-c}\rho, \nu^d\rho]) \rtimes \sigma$ which has in its Jacquet module $\delta([\nu^{-c}\rho, \nu^d\rho]) \otimes \sigma$ by $\delta([\nu^{-c}\rho, \nu^d\rho]_-; \sigma)$.

Since $\delta([\nu^{-d}\rho, \nu^c\rho]) \otimes \sigma$ has the multiplicity one in the Jacquet module of $L(\delta([\nu^{-c}\rho, \nu^d\rho]); \sigma)$, this part of Jacquet module characterizes $L(\delta([\nu^{-c}\rho, \nu^d\rho]); \sigma)$ as an irreducible subquotient of $\delta([\nu^{-c}\rho, \nu^d\rho]) \rtimes \sigma$ for which holds

$$L(\delta([\nu^{-c}\rho, \nu^d\rho]); \sigma) \neq \delta([\nu^{-c}\rho, \nu^d\rho]_\pm; \sigma).$$

Therefore, we have identified three different irreducible subquotients of the induced representation $\delta([\nu^{-c}\rho, \nu^d\rho]) \rtimes \sigma$ (all of multiplicity one).

The following technical lemma will be used several times afterwards in the paper:

Lemma 5.1. *Suppose that $\alpha \geq 0$ is such that $\nu^\alpha\rho \rtimes \sigma$ reduces and that for $c < d$ we have $s_{GL}(L(\delta([\nu^{-c}\rho, \nu^d\rho]); \sigma)) = \sum_{i=\alpha+1}^{d+1} L(\delta([\nu^{-i+1}\rho, \nu^c\rho]), \delta([\nu^i\rho, \nu^d\rho])) \otimes \sigma$. Let $c' \leq d'$ satisfy*

$$-c \leq -c' \leq -\alpha \leq \alpha \leq d' \leq d.$$

Then the Jacquet module of $L(\delta([\nu^{-c}\rho, \nu^d\rho]); \sigma))$ does not contain a representation of the form

$$\pi' \otimes \delta([\nu^{-c'}\rho, \nu^{d'}\rho]) \otimes \sigma$$

for any irreducible representation π' of a general linear group.

Proof. Suppose that the above term is in the Jacquet module of $L(\delta([\nu^{-c}\rho, \nu^{-d}\rho]); \sigma)$. Then $c < d$ (otherwise $L(\delta([\nu^{-c}\rho, \nu^{-d}\rho]); \sigma) = 0$).

Now the assumption on $s_{GL}(L(\delta([\nu^{-c}\rho, \nu^d\rho]); \sigma))$ implies that the representation $\pi' \otimes \delta([\nu^{-c'}\rho, \nu^{d'}\rho]) \otimes \sigma$ appears in the Jacquet module of

$$L(\delta([\nu^{-i'+1}\rho, \nu^c\rho]), \delta([\nu^{i'}\rho, \nu^d\rho])) \otimes \sigma, \quad \text{for some} \quad \alpha + 1 \leq i' \leq d + 1.$$

Using the formula for m^* we see that, to be able to get $\pi' \otimes \delta([\nu^{-c'}\rho, \nu^{d'}\rho]) \otimes \sigma$ in the Jacquet module of the above representation, we must have $-c' = i'$ or $-c' = -i' + 1$. Obviously, the first relation cannot hold (since we have different signs). Therefore, $i' = c' + 1$ and we shall now consider $m^*(L(\delta([\nu^{-c'}\rho, \nu^c\rho]), \delta([\nu^{c'+1}\rho, \nu^d\rho])))$. Let us see how we can get a term of the form $\pi' \otimes \delta([\nu^{-c'}\rho, \nu^{d'}\rho])$ in the above expression.

It can be directly seen that $L(\delta([\nu^{-c'}\rho, \nu^c\rho]), \delta([\nu^{c'+1}\rho, \nu^d\rho]))$ is not fully induces, which implies

$$m^*(L(\delta([\nu^{-c'}\rho, \nu^c\rho]), \delta([\nu^{c'+1}\rho, \nu^d\rho]))) =$$

$$m^*(\delta([\nu^{-c'}\rho, \nu^c\rho])) \times m^*(\delta([\nu^{c'+1}\rho, \nu^d\rho]))) - m^*(\delta([\nu^{-c'}\rho, \nu^d\rho])) \times m^*(\delta([\nu^{c'+1}\rho, \nu^c\rho])). \tag{5.3}$$

The first summand gives terms of the form

$$\big(\delta([\nu^{k+1}\rho, \nu^c\rho]) \otimes \delta([\nu^{-c'}\rho, \nu^k\rho])\big) \times \big(\delta([\nu^{l+1}\rho, \nu^d\rho]) \otimes \delta([\nu^{c'+1}\rho, \nu^l\rho])\big),$$

where $-c' - 1 \leq k \leq c$ and $c' \leq l \leq d$.

Let us first consider the case $d' \leq c$. In this case, there are two possibilities to get $\pi' \otimes \delta([\nu^{-c'}\rho, \nu^{d'}\rho]) \otimes \sigma$ for a subquotient.

One option is $k = d'$ and $l = c'$, which gives

$$\big(\delta([\nu^{d'+1}\rho, \nu^{c}\rho]) \otimes \delta([\nu^{-c'}\rho, \nu^{d'}\rho])\big) \times \big(\delta([\nu^{c'+1}\rho, \nu^{d}\rho]) \otimes 1\big)$$

$$= \delta([\nu^{d'+1}\rho, \nu^{c}\rho]) \times \delta([\nu^{c'+1}\rho, \nu^{d}\rho]) \otimes \delta([\nu^{-c'}\rho, \nu^{d'}\rho]).$$

The remaining option is $k = c'$ and $l = d'$, which gives the term

$$\big(\delta([\nu^{c'+1}\rho, \nu^{c}\rho]) \otimes \delta([\nu^{-c'}\rho, \nu^{c'}\rho])\big) \times \big(\delta([\nu^{d'+1}\rho, \nu^{d}\rho]) \otimes \delta([\nu^{c'+1}\rho, \nu^{d'}\rho])\big)$$

$$= \delta([\nu^{c'+1}\rho, \nu^{c}\rho]) \times \delta([\nu^{d'+1}\rho, \nu^{d}\rho]) \otimes \delta([\nu^{-c'}\rho, \nu^{c'}\rho]) \times \delta([\nu^{c'+1}\rho, \nu^{d'}\rho]).$$

Observe that in the second summand of (5.3) we have the following terms

$$\big(\delta([\nu^{k'+1}\rho, \nu^{d}\rho]) \otimes \delta([\nu^{-c'}\rho, \nu^{k'}\rho])\big) \times \big(\delta([\nu^{l'+1}\rho, \nu^{c}\rho]) \otimes \delta([\nu^{c'+1}\rho, \nu^{l'}\rho])\big),$$

where $-c' - 1 \leq k' \leq d$ and $c' \leq l' \leq c$. We consider the following two terms:

for $k' = d'$ and $l' = c'$ we get

$$\big(\delta([\nu^{d'+1}\rho, \nu^{d}\rho]) \otimes \delta([\nu^{-c'}\rho, \nu^{d'}\rho])\big) \times \big(\delta([\nu^{c'+1}\rho, \nu^{c}\rho]) \otimes 1\big)$$

$$= \delta([\nu^{d'+1}\rho, \nu^{d}\rho]) \times \delta([\nu^{c'+1}\rho, \nu^{c}\rho]) \otimes \delta([\nu^{-c'}\rho, \nu^{d'}\rho]),$$

while for $k' = c'$ and $l' = d'$ we get

$$\big(\delta([\nu^{c'+1}\rho, \nu^{d}\rho]) \otimes \delta([\nu^{-c'}\rho, \nu^{c'}\rho])\big) \times \big(\delta([\nu^{d'+1}\rho, \nu^{c}\rho]) \otimes \delta([\nu^{c'+1}\rho, \nu^{d'}\rho])\big)$$

$$= \delta([\nu^{c'+1}\rho, \nu^{d}\rho]) \times \delta([\nu^{d'+1}\rho, \nu^{c}\rho]) \otimes \delta([\nu^{-c'}\rho, \nu^{c'}\rho]) \times \delta([\nu^{c'+1}\rho, \nu^{d'}\rho]).$$

Let us compute the difference

$$\delta([\nu^{d'+1}\rho, \nu^{c}\rho]) \times \delta([\nu^{c'+1}\rho, \nu^{d}\rho]) \otimes \delta([\nu^{-c'}\rho, \nu^{d'}\rho])$$

$$+\delta([\nu^{c'+1}\rho, \nu^{c}\rho]) \times \delta([\nu^{d'+1}\rho, \nu^{d}\rho]) \otimes \delta([\nu^{-c'}\rho, \nu^{c'}\rho]) \times \delta([\nu^{c'+1}\rho, \nu^{d'}\rho])$$

$$-\delta([\nu^{d'+1}\rho, \nu^{d}\rho]) \times \delta([\nu^{c'+1}\rho, \nu^{c}\rho]) \otimes \delta([\nu^{-c'}\rho, \nu^{d'}\rho])$$

$$-\delta([\nu^{c'+1}\rho, \nu^{d}\rho]) \times \delta([\nu^{d'+1}\rho, \nu^{c}\rho]) \otimes \delta([\nu^{-c'}\rho, \nu^{c'}\rho]) \times \delta([\nu^{c'+1}\rho, \nu^{d'}\rho]).$$

There are two subcases to consider. The first is

$$c' < d'.$$

It follows directly that $[\nu^{c'+1}\rho, \nu^c\rho]$ and $[\nu^{d'+1}\rho, \nu^d\rho]$ are linked. Subtracting the first and the third term, and the second and the fourth term, we get that the above difference equals

$$-L(\delta([\nu^{d'+1}\rho, \nu^d\rho]), \delta([\nu^{c'+1}\rho, \nu^c\rho])) \otimes \delta([\nu^{-c'}\rho, \nu^{d'}\rho])$$

$$+L(\delta([\nu^{c'+1}\rho, \nu^c\rho]), \delta([\nu^{d'+1}\rho, \nu^d\rho])) \otimes \delta([\nu^{-c'}\rho, \nu^{d'}\rho])$$

$$+L(\delta([\nu^{c'+1}\rho, \nu^c\rho]), \delta([\nu^{d'+1}\rho, \nu^d\rho])) \otimes L(\delta([\nu^{-c'}\rho, \nu^{c'}\rho]), \delta([\nu^{c'+1}\rho, \nu^{d'}\rho]))$$

$$= L(\delta([\nu^{c'+1}\rho, \nu^c\rho]), \delta([\nu^{d'+1}\rho, \nu^d\rho])) \otimes L(\delta([\nu^{-c'}\rho, \nu^{c'}\rho]), \delta([\nu^{c'+1}\rho, \nu^{d'}\rho])).$$

This proves that the terms of the form $\ldots \otimes \delta([\nu^{-c'}\rho, \nu^{d'}\rho]) \otimes \sigma$ can not be in the Jacquet module of the Langlands quotient in this subcase.

The remaining subcase is $c' = d'$. Then the difference is 0. So again we can not get a term of the form $\ldots \otimes \delta([\nu^{-c'}\rho, \nu^{d'}\rho]) \otimes \sigma$ for a subquotient.

It remains to consider the case $c < d'$.

We recall that the first summand will give terms of the form

$$\big(\delta([\nu^{k+1}\rho, \nu^c\rho]) \otimes \delta([\nu^{-c'}\rho, \nu^k\rho])\big) \times \big(\delta([\nu^{l+1}\rho, \nu^d\rho]) \otimes \delta([\nu^{c'+1}\rho, \nu^l\rho])\big),$$

where $-c' - 1 \leq k \leq c$ and $c' \leq l \leq d$.

In this case, we have only one option to get a term of the type that we are looking for. This option is $k = c'$ and $l = d'$, which gives the term

$$\big(\delta([\nu^{c'+1}\rho, \nu^c\rho]) \otimes \delta([\nu^{-c'}\rho, \nu^{c'}\rho])\big) \times \big(\delta([\nu^{d'+1}\rho, \nu^d\rho]) \otimes \delta([\nu^{c'+1}\rho, \nu^{d'}\rho])\big)$$

$$= \delta([\nu^{c'+1}\rho, \nu^c\rho]) \times \delta([\nu^{d'+1}\rho, \nu^d\rho]) \otimes \delta([\nu^{-c'}\rho, \nu^{c'}\rho]) \times \delta([\nu^{c'+1}\rho, \nu^{d'}\rho]).$$

On the other hand, in the second summand we have terms

$$\big(\delta([\nu^{k'+1}\rho, \nu^d\rho]) \otimes \delta([\nu^{-c'}\rho, \nu^{k'}\rho])\big) \times \big(\delta([\nu^{l'+1}\rho, \nu^c\rho]) \otimes \delta([\nu^{c'+1}\rho, \nu^{l'}\rho])\big),$$

where $-c'-1 \leq k' \leq d$ and $c' \leq l' \leq c$. Since $c < d'$, the only option to get $\pi' \otimes \delta([\nu^{-c'}\rho, \nu^{d'}\rho]) \otimes \sigma$ for a subquotient is if we take $k' = d'$ and $l' = c'$, which gives

$$\big(\delta([\nu^{d'+1}\rho, \nu^{d}\rho]) \otimes \delta([\nu^{-c'}\rho, \nu^{d'}\rho])\big) \times \big(\delta([\nu^{c'+1}\rho, \nu^{c}\rho]) \otimes 1\big)$$
$$= \delta([\nu^{d'+1}\rho, \nu^{d}\rho]) \times \delta([\nu^{c'+1}\rho, \nu^{c}\rho]) \otimes \delta([\nu^{-c'}\rho, \nu^{d'}\rho]).$$

The difference

$$\delta([\nu^{c'+1}\rho, \nu^{c}\rho]) \times \delta([\nu^{d'+1}\rho, \nu^{d}\rho]) \otimes \delta([\nu^{-c'}\rho, \nu^{c'}\rho]) \times \delta([\nu^{c'+1}\rho, \nu^{d'}\rho])$$
$$-\delta([\nu^{d'+1}\rho, \nu^{d}\rho]) \times \delta([\nu^{c'+1}\rho, \nu^{c}\rho]) \otimes \delta([\nu^{-c'}\rho, \nu^{d'}\rho]),$$

equals

$$\delta([\nu^{c'+1}\rho, \nu^{c}\rho]) \times \delta([\nu^{d'+1}\rho, \nu^{d}\rho]) \otimes L(\delta([\nu^{-c'}\rho, \nu^{c'}\rho]), \delta([\nu^{c'+1}\rho, \nu^{d'}\rho]))$$

if $c' < d'$, and 0 otherwise.

This proves that the (non-zero) terms of the form $\cdots \otimes \delta([\nu^{-c'}\rho, \nu^{d'}\rho] \otimes \sigma$ can not be in the Jacquet module of the Langlands quotient and completes the proof of the lemma. □

The following relations have been obtained in [14]:

$$\sum_{i=-c}^{-\alpha} \delta([\nu^{i}\rho, \nu^{d}\rho]) \times \delta([\nu^{-i+1}\rho, \nu^{c}\rho]) \otimes \sigma \leq s_{GL}(\delta([\nu^{-\alpha-n}\rho, \nu^{\alpha+m}\rho]_{+}, \sigma))$$
$$\leq \sum_{i=-c}^{\alpha} \delta([\nu^{i}\rho, \nu^{d}\rho]) \times \delta([\nu^{-i+1}\rho, \nu^{c}\rho]) \otimes \sigma$$

and

$$s_{GL}(\delta([\nu^{-c}\rho, \nu^{d}\rho]_{-}, \sigma)) = \sum_{i=-c}^{-\alpha} \delta([\nu^{i}\rho, \nu^{d}\rho]) \times \delta([\nu^{-i+1}\rho, \nu^{c}\rho]) \otimes \sigma.$$

The following theorem will give us further details about Jacquet modules of irreducible subquotients of $\delta([\nu^{-c}\rho, \nu^{d}\rho]) \rtimes \sigma$.

To keep the notation of our results uniform, we continue with our convention regarding meaning of $\delta([\nu^{-c}\rho, \nu^{d}\rho]_{+}; \sigma)$ and $L_{\alpha}(\delta([\nu^{-c}\rho, \nu^{d}\rho]); \sigma)$ when $\delta([\nu^{-c}\rho, \nu^{d}\rho]) \rtimes \sigma$ is irreducible.

Theorem 5.2. *Let $c, d \in (1/2)\mathbb{Z}$ such that $d+c, d-c \in \mathbb{Z}_{\geq 0}$, $d-\alpha \in \mathbb{Z}$ and*

$$-c \leq -\alpha < \alpha \leq d.$$

Then

1. *If $c < d$, then $\delta([\nu^{-c}\rho, \nu^{d}\rho]) \rtimes \sigma$ is a representation of length three, and the composition series consists of $\delta([\nu^{-c}\rho, \nu^{d}\rho]_{\pm}; \sigma)$ and $L(\delta([\nu^{-c}\rho, \nu^{d}\rho]); \sigma)$.*

   For $c = d$, $\delta([\nu^{-c}\rho, \nu^{c}\rho]) \rtimes \sigma$ is a representation of length two, and the composition series consists of $\delta([\nu^{-c}\rho, \nu^{c}\rho]_{\pm}; \sigma)$.

2. *For $c < d$, $\delta([\nu^{-c}\rho, \nu^{d}\rho]_{\pm}; \sigma)$ are square integrable. For $c = d$, the representations are tempered, but not square integrable.*

3. $s_{GL}(\delta([\nu^{-c}\rho, \nu^{d}\rho]_{\pm}; \sigma)) = \sum_{i=-c}^{\pm\alpha} \delta([\nu^{-i+1}\rho, \nu^{c}\rho]) \times \delta([\nu^{i}\rho, \nu^{d}\rho]) \otimes \sigma.$

4. *For $c < d$ holds*

   $s_{GL}(L(\delta([\nu^{-c}\rho, \nu^{d}\rho]); \sigma)) = \sum_{i=\alpha+1}^{d+1} L(\delta([\nu^{-i+1}\rho, \nu^{c}\rho]), \delta([\nu^{i}\rho, \nu^{d}\rho])) \otimes \sigma.$

5.

$$\begin{aligned} s_{top}(\delta([\nu^{-c}\rho, \nu^{c}\rho]_{+}; \sigma)) &= 2\nu^{c}\rho \otimes \delta([\nu^{-c+1}\rho, \nu^{c}\rho]_{+}; \sigma) + \\ &\quad \nu^{c}\rho \otimes L_{\alpha}(\delta([\nu^{-c+1}\rho, \nu^{c}\rho]); \sigma), \end{aligned}$$

$$\begin{aligned} s_{top}(\delta([\nu^{-c}\rho, \nu^{c}\rho]_{-}; \sigma)) &= 2\nu^{c}\rho \otimes \delta([\nu^{-c+1}\rho, \nu^{c}\rho]_{-}; \sigma) + \\ &\quad \nu^{c}\rho \otimes L_{\alpha}(\delta([\nu^{-c+1}\rho, \nu^{c}\rho]); \sigma). \end{aligned}$$

6. *For $c < d$ holds*

$$\begin{aligned} s_{top}(\delta([\nu^{-c}\rho, \nu^{d}\rho]_{\pm}; \sigma)) &= \nu^{d}\rho \otimes \delta([\nu^{-c}\rho, \nu^{d-1}\rho]_{\pm}; \sigma) + \\ &\quad \nu^{c}\rho \otimes \delta([\nu^{-c+1}\rho, \nu^{d}\rho]_{\pm}; \sigma). \end{aligned}$$

7. *For $c < d$ holds*

$$\begin{aligned} s_{top}(L(\delta([\nu^{-c}\rho, \nu^{d}\rho]); \sigma)) &= \nu^{d}\rho \otimes L_{\alpha}(\delta([\nu^{-c}\rho, \nu^{d-1}\rho]); \sigma) + \\ &\quad \nu^{c}\rho \otimes L_{\alpha}(\delta([\nu^{-c+1}\rho, \nu^{d}\rho]); \sigma). \end{aligned}$$

Proof. We emphasize that it can be easily seen, using formula for μ^* and well-known composition series of the induced representations of general linear groups, that the sum of two sums on the right hand side of (3) and of the right hand side of (4) equals $s_{GL}(\delta([\nu^{-c}\rho, \nu^d\rho]) \rtimes \sigma)$.

Observe that we know the statements in (2) regarding square-integrability and temperedness of involved representations follows from [14]. From there we also know length two and the statement (3) in the tempered case ([14]). Further, we know that claims (3) and (6) hold for $\delta([\nu^{-c}\rho, \nu^d\rho]_-; \sigma)$. Also, the length three claim in (1) now follows from [10].

We shall prove the rest of the theorem by induction with respect to c and d.

The formula for μ^* implies

$$s_{top}(\delta([\nu^{-c}\rho, \nu^d\rho]) \rtimes \sigma) = \nu^d\rho \otimes \delta([\nu^{-c}\rho, \nu^{d-1}\rho]) \rtimes \sigma + \nu^c\rho \otimes \delta([\nu^{-c+1}\rho, \nu^d\rho]) \rtimes \sigma.$$

First we shall prove the theorem in the case $c = \alpha$. The proof goes by induction over d.

Consider first the case $d = \alpha$. We know that then $\delta([\nu^{-\alpha}\rho, \nu^\alpha\rho]) \rtimes \sigma$ is of length two. The above formula and the previous theorem, together with the symmetrization of notation, imply that

$$s_{top}(\delta([\nu^{-\alpha}\rho, \nu^\alpha\rho]) \rtimes \sigma) = 2\nu^\alpha\rho \otimes \delta([\nu^{-\alpha+1}\rho, \nu^\alpha\rho]_+; \sigma) + 2\nu^\alpha\rho \otimes L(\delta([\nu^{-\alpha+1}\rho, \nu^\alpha\rho]); \sigma).$$

is the decomposition of $s_{top}(\delta([\nu^{-\alpha}\rho, \nu^\alpha\rho]) \rtimes \sigma)$ into the irreducible representations.

From the definition of $\delta([\nu^{-\alpha}\rho, \nu^\alpha\rho]_+; \sigma)$ we know that $2\nu^\alpha\rho \otimes \delta([\nu^{-\alpha+1}\rho, \nu^\alpha\rho]_+; \sigma)$ must belong to its Jacquet module. From the other side, it follows from (3), which we know to hold in this case, that the minimal non-zero Jacquet module of $\delta([\nu^{-\alpha}\rho, \nu^\alpha\rho]_-; \sigma)$ is irreducible. Now from these two facts follows

$$s_{top}(\delta([\nu^{-\alpha}\rho, \nu^\alpha\rho]_+; \sigma)) = 2\nu^\alpha\rho \otimes \delta([\nu^{-\alpha+1}\rho, \nu^\alpha\rho]_+; \sigma) + \nu^\alpha\rho \otimes L(\delta([\nu^{-\alpha+1}\rho, \nu^\alpha\rho]); \sigma),$$

$$s_{top}(\delta([\nu^{-\alpha}\rho, \nu^\alpha\rho]_-; \sigma)) = \nu^\alpha\rho \otimes L(\delta([\nu^{-\alpha+1}\rho, \nu^\alpha\rho]); \sigma).$$

Since $\delta([\nu^{-\alpha+1}\rho, \nu^\alpha\rho]_-; \sigma) = 0)$, this is exactly (5) in this case and implies that the theorem holds in this case (note that (6) and (7) do not apply here).

Fix $d > \alpha$ and assume that the theorem holds for $d-1$. Then the inductive assumption implies the following decomposition into irreducible representations

$$s_{top}(\delta([\nu^{-\alpha}\rho, \nu^d\rho]) \rtimes \sigma) = \nu^d\rho \otimes \delta([\nu^{-\alpha}\rho, \nu^{d-1}\rho]_+; \sigma) + \nu^\alpha\rho \otimes \delta([\nu^{-\alpha+1}\rho, \nu^d\rho]_+; \sigma)$$

$$+\nu^d\rho \otimes \delta([\nu^{-\alpha}\rho, \nu^{d-1}\rho]_-; \sigma)$$

$$+\nu^d\rho \otimes L(\delta([\nu^{-\alpha}\rho, \nu^{d-1}\rho]); \sigma) + \nu^\alpha\rho \otimes L(\delta([\nu^{-\alpha+1}\rho, \nu^d\rho]); \sigma)$$

(note that in this case $\nu^\alpha\rho \otimes \delta([\nu^{-\alpha+1}\rho, \nu^d\rho]_-; \sigma) = 0$ by our notation=.

The first two terms obviously belong to the Jacquet module of $\delta([\nu^{-\alpha}\rho, \nu^d\rho]_+; \sigma)$. Since $\delta([\nu^{-\alpha}\rho, \nu^d\rho]_-; \sigma)$ has in its minimal non-zero Jacquet module the term $\nu^d\rho \otimes \nu^{d-1}\rho \otimes \ldots \otimes \nu^{-\alpha}\rho \otimes \sigma$, which cannot come from the last two terms, the third term must come from $\delta([\nu^{-\alpha}\rho, \nu^d\rho]_-; \sigma)$.

The properties of the Langlands classification imply that we must have in the Jacquet module of $L(\delta([\nu^{-\alpha}\rho, \nu^d\rho]); \sigma)$ a term of the form $\nu^\alpha\rho\otimes\ldots$ Therefore, the last term must be in this Jacquet module. If $d = \alpha + 1$, the fourth term is zero. If not, then the Langlands quotient embeds into

$$\delta([\nu^{-d}\rho, \nu^\alpha\rho]) \rtimes \sigma \hookrightarrow \nu^{-\alpha}\rho \times \cdots \times \nu^{-d}\rho \rtimes \sigma \cong \nu^{-\alpha}\rho \times \cdots \times \nu^d\rho \rtimes \sigma$$

$$\cong \nu^d\rho \times \nu^{-\alpha}\rho \times \cdots \times \nu^{-d+1}\rho \rtimes \sigma.$$

In consequence, we must have a representation of the form $\nu^d\rho \otimes \ldots$ in the Jacquet module of the Langlands quotient. Since $\alpha + 1 \neq d$, we see that the fourth term must be in the Jacquet module of the Langlands quotient. This proves the formulas for the top Jacquet modules claimed in the theorem in this case.

We have already seen that the right hand side of (3) is an upper bound for $s_{GL}(\delta([\nu^{-\alpha}\rho, \nu^d\rho]_+; \sigma))$. Applying Jacquet modules $r_{(p,(d+c)p)}$ on this upper bound and on (6) for $\delta([\nu^{-\alpha}\rho, \nu^d\rho]_+; \sigma)$ (which we have just proved), in the same way as in the proof of Theorem 4.1, we get an equality which proves (3). But this also implies (4), because we have

$$s_{GL}(\delta([\nu^{-c}\rho, \nu^d\rho]) \rtimes \sigma)$$

$$= \sum_{i=-c}^{d+1} \delta([\nu^{-i+1}\rho, \nu^c\rho]) \times \delta([\nu^i\rho, \nu^d\rho]) \otimes \sigma,$$

since $\delta([\nu^{-c}\rho, \nu^d\rho]) \rtimes \sigma$ is a representation of length three, and we know Jacquet modules of the remaining two irreducible subquotients.

This completes the proof for $c = \alpha$.

Now we shall fix

$$c > \alpha,$$

and assume that the theorem holds for $c - 1$. Again, we proceed inductively, similarly as in the case $c = \alpha$.

Let us start with the case $d = c$. We know that then $\delta([\nu^{-c}\rho, \nu^c\rho]) \rtimes \sigma$ is of length two. The above formula and symmetrization of notation imply that

$$s_{top}(\delta([\nu^{-c}\rho, \nu^c\rho]) \rtimes \sigma) = 2\nu^c\rho \otimes \delta([\nu^{-c+1}\rho, \nu^c\rho]_+; \sigma)$$

$$+2\nu^c\rho \otimes \delta([\nu^{-c+1}\rho, \nu^c\rho]_-; \sigma) + 2\nu^c\rho \otimes L(\delta([\nu^{-c+1}\rho, \nu^c\rho]); \sigma).$$

is the decomposition of $s_{top}(\delta([\nu^{-c}\rho, \nu^c\rho]) \rtimes \sigma)$ into the irreducible representations.

Definition of representation $\delta([\nu^{-c}\rho, \nu^c\rho]_+; \sigma)$ directly yields that $2\nu^\alpha\rho \otimes \delta([\nu^{-\alpha+1}\rho, \nu^\alpha\rho]_+; \sigma)$ belongs to its Jacquet module. Further, since $\delta([\nu^{-c}\rho, \nu^c\rho]) \otimes \sigma$ is in the Jacquet module of both $\delta([\nu^{-c}\rho, \nu^c\rho]_+; \sigma)$ and $\delta([\nu^{-c}\rho, \nu^c\rho]_-; \sigma)$, using transitivity of Jacquet modules and the fact that $\nu^{-c}\rho$ does not appear in the cuspidal support of $s_{GL}(\delta([\nu^{-c+1}\rho, \nu^c\rho]_+; \sigma))$, we obtain that $\nu^\alpha\rho \otimes L(\delta([\nu^{-c+1}\rho, \nu^c\rho]); \sigma)$ must be in each of the Jacquet modules.

From the formula for s_{GL} of $\delta([\nu^{-c}\rho, \nu^c\rho]_\pm; \sigma)$ it follows directly that the multiplicity of the representation $\nu^c\rho \otimes \nu^c\rho \otimes \nu^{c-1}\rho \otimes \ldots \otimes \nu^{-c+1}\rho \otimes \sigma$ is the same for both representations (since $c > \alpha$). Observe that $\nu^c\rho \otimes \nu^c\rho \otimes \nu^{c-1}\rho \otimes \ldots \otimes \nu^{-c+1}\rho \otimes \sigma$ has positive multiplicity in $\nu^c\rho \otimes \delta([\nu^{-c+1}\rho, \nu^c\rho]_-; \sigma)$ and multiplicity zero in $\nu^c\rho \otimes L(\delta([\nu^{-c+1}\rho, \nu^c\rho]); \sigma)$. It follows immediately that $2\nu^c\rho \otimes \delta([\nu^{-c+1}\rho, \nu^c\rho]_-; \sigma)$ appears in the Jacquet module of $\delta([\nu^{-c}\rho, \nu^c\rho]_-; \sigma)$. This completes the proof of the theorem in this case.

Let us now fix $d > c$ and assume that the theorem holds for $d - 1$ and c, and also for $c - 1$ and all $d \geq c - 1$. The inductive assumption gives the following decomposition into irreducible representations

$$s_{top}(\delta([\nu^{-c}\rho,\nu^{d}\rho])\rtimes\sigma)=\nu^{d}\rho\otimes\delta([\nu^{-c}\rho,\nu^{d-1}\rho]_{+};\sigma)+\nu^{c}\rho\otimes\delta([\nu^{-c+1}\rho,\nu^{d}\rho]_{+};\sigma)$$
$$+\nu^{d}\rho\otimes\delta([\nu^{-c}\rho,\nu^{d-1}\rho]_{-};\sigma)+\nu^{c}\rho\otimes\delta([\nu^{-c+1}\rho,\nu^{d}\rho]_{-};\sigma)$$
$$+\nu^{d}\rho\otimes L(\delta([\nu^{-c}\rho,\nu^{d-1}\rho]);\sigma)+\nu^{c}\rho\otimes L(\delta([\nu^{-c+1}\rho,\nu^{d}\rho]);\sigma).$$

In the same way as in the case $c=\alpha$ we deduce that the first two terms belong to the Jacquet module of $\delta([\nu^{-c}\rho,\nu^{d}\rho]_{+};\sigma)$, while $s_{top}(\delta([\nu^{-\alpha}\rho,\nu^{d}\rho]_{-};\sigma))$ contains the third one. Also, since $\nu^{-d}\rho$ appears only in the last term, that term belongs to Jacquet module of the Langlands quotient.

If $d=c+1$, then the fifth representation is zero. If not, i.e., if $d>c+1$, in the same way as before we see that the fifth representation is also in the Jacquet module of the Langlands quotient.

Observe that the GL-type of Jacquet module of $\delta([\nu^{-c}\rho,\nu^{d}\rho]_{-};\sigma)$ contains $\delta(\nu^{-c+1}\rho,\nu^{d}\rho])\times\nu^{c}\rho\otimes\sigma$. This implies that we must have in the Jacquet module of this representations subquotients of the form $\nu^{c}\rho\otimes\dots$. Therefore, the fourth representation is in the Jacquet module of $\delta([\nu^{-\alpha}\rho,\nu^{d}\rho]_{-};\sigma)$. This proves the formulas for the top Jacquet modules claimed in the theorem.

In completely analogous manner as in the case $c=\alpha$ we prove (3) for representation $\delta([\nu^{-\alpha}\rho,\nu^{d}\rho]_{+};\sigma)$, by showing equality on the level of Jacquet modules $r_{(p,(d+c)p)}$ (applying this Jacquet modules to the right hand side of (3), which we know is an upper bound for $s_{GL}(\delta([\nu^{-\alpha}\rho,\nu^{d}\rho]_{+};\sigma))$, and on (6)). This also implies (4) and completes the proof. □

We take a moment to provide an interpretation of the results obtained in the previous theorem in terms of admissible triples.

Remark 5.3. *If $-c\leq-\alpha<\alpha\leq d$ and $\delta([\nu^{-c}\rho,\nu^{d}\rho]_{\pm};\sigma)$ are square integrable (i.e., $c<d$) then Jord$(\delta([\nu^{-c}\rho,\nu^{d}\rho]_{\pm};\sigma))=$Jord$(\sigma)\cup\{(2c+1,\rho),(2d+1,\rho)\}$. Further, if we denote the ϵ-function corresponding to $\delta([\nu^{-c}\rho,\nu^{d}\rho]_{+};\sigma)$ (resp., $\delta([\nu^{-c}\rho,\nu^{d}\rho]_{-};\sigma)$) by ϵ_{+} (resp., ϵ_{-}), then we obviously have $d_{-}=c$ and $\epsilon_{\pm}((c,\rho),(d,\rho))=1$. Further, by (3) of the previous theorem we have $\epsilon_{+}((c_{-},\rho),(c,\rho))=1$ and $\epsilon_{-}((c_{-},\rho),(c,\rho))=-1$.*

In the following corollary we determine all Jacquet modules for the representations of segment type in this case.

Corollary 5.4. *Let $c, d \in (1/2)\mathbb{Z}$ such that $d+c, d-c \in \mathbb{Z}_{\geq 0}$, $d-\alpha \in \mathbb{Z}$ and $-c \leq -\alpha < \alpha \leq d$. Then*

1.

$$\mu^* \big(\delta([\nu^{-c}\rho, \nu^d\rho]_{\pm}; \sigma)\big)$$

$$= \sum_{i=-c-1}^{d} \sum_{j=i+1}^{d} \delta([\nu^{-i}\rho, \nu^c\rho]) \times \delta([\nu^{j+1}\rho, \nu^d\rho]) \otimes \delta([\nu^{i+1}\rho, \nu^j\rho]_{\pm}; \sigma)$$

$$+ \sum_{\substack{-c-1\leq i\leq c,\\ i+j<-1}} \sum_{i+1\leq j\leq c,} \delta([\nu^{-i}\rho, \nu^c\rho]) \times \delta([\nu^{j+1}\rho, \nu^d\rho]) \otimes L_\alpha(\delta([\nu^{i+1}\rho, \nu^j\rho]); \sigma)$$

$$+ \sum_{i=-c}^{\pm\alpha} \delta([\nu^{-i+1}\rho, \nu^c\rho]) \times \delta([\nu^i\rho, \nu^d\rho]) \otimes \sigma.$$

2. *For $c < d$ we have*

$$\mu^* \big(L(\delta([\nu^{-c}\rho, \nu^d\rho]); \sigma)\big)$$

$$= \sum_{\substack{-c-1\leq i\leq d,\\ 0\leq i+j}} \sum_{i+1\leq j\leq d,} L(\delta([\nu^{-i}\rho, \nu^c\rho]), \delta([\nu^{j+1}\rho, \nu^d\rho])) \otimes L_\alpha(\delta([\nu^{i+1}\rho, \nu^j\rho]); \sigma))$$

$$+ \sum_{i=\alpha+1}^{d+1} L(\delta([\nu^{-i+1}\rho, \nu^c\rho]), \delta([\nu^i\rho, \nu^d\rho])) \otimes \sigma.$$

Proof. We have

$$\mu^* \big(\delta([\nu^{-c}\rho, \nu^d\rho]) \rtimes \sigma\big)$$

$$= \sum_{i=-c-1}^{d} \sum_{j=i}^{d} \delta([\nu^{-i}\rho, \nu^c\rho]) \times \delta([\nu^{j+1}\rho, \nu^d\rho])$$

$$\otimes \big(\delta([\nu^{i+1}\rho, \nu^j\rho]_+; \sigma) + \delta([\nu^{i+1}\rho, \nu^j\rho]_-; \sigma) + L_\alpha(\delta([\nu^{i+1}\rho, \nu^j\rho]); \sigma)\big).$$

In the previous theorem we have determined the Jacquet modules of GL-type which coincide with the approveriate terms in the above formulas. Therefore, it remains to consider the case $i < j$.

We shall first analyze the case of the summands $\ldots \otimes \delta([\nu^{i+1}\rho, \nu^j\rho]_+; \sigma)$ when this representation is non-zero (i.e., when $[i+1, j] \cap [-\alpha, \alpha] \neq \emptyset$). Then Theorem 4.1 and Theorem 5.2 imply that the minimal non-zero standard Jacquet module of this representation contains at least one term of the form $\ldots \otimes \nu^\alpha\rho \otimes \sigma$.

By transitivity of Jacquet modules, the representation of above type must also be contained in the Jacquet module of the representation whose Jacquet module contains $\ldots \otimes \delta([\nu^{i+1}\rho, \nu^j\rho]_+; \sigma)$.

Using Theorems 4.1 and 5.2, we see that minimal non-zero standard Jacquet modules of $\delta([\nu^{-c}\rho, \nu^d\rho]_-; \sigma)$ and of $L(\delta([\nu^{-c}\rho, \nu^{-d}\rho]); \sigma)$ do not contain the representation of the form $\ldots \otimes \nu^\alpha\rho \otimes \sigma$.

Consequently, all the terms of the form $\ldots \otimes \delta([\nu^{i+1}\rho, \nu^j\rho]_+; \sigma)$ are in the Jacquet module of $\delta([\nu^{-c}\rho, \nu^{-d}\rho]_+; \sigma)$.

Let us now analyze the case of the summands

$$\delta([\nu^{-i}\rho, \nu^c\rho]) \times \delta([\nu^{j+1}\rho, \nu^d\rho]) \otimes \delta([\nu^{i+1}\rho, \nu^j\rho]_-; \sigma)$$

when this representation is non-zero, i.e., when $[-\alpha, \alpha] \subseteq [i+1, j]$.

Then $i \leq -\alpha - 1$ and $\alpha \leq j$. Now Lemma 5.1 implies that there are no irreducible subquotients of this term in the Jacquet module of the Langlands quotient.

Suppose that some subquotient of the form $\pi' \otimes \delta([\nu^{i+1}\rho, \nu^j\rho]_-; \sigma)$ appears in the Jacquet module of $\delta([\nu^{-c}\rho, \nu^d\rho]_+; \sigma)$.

Two possibilities will be studied separately.

Consider first the case $|i+1| \leq j$, i.e., $-i-1 \leq j$.

Let φ denote any irreducible subquotient of the minimal non-zero Jacquet module of $\pi' \otimes \delta([\nu^{i+1}\rho, \nu^j\rho]) \otimes \sigma$. Then φ must be in Jacquet module of $\delta([\nu^{-c}\rho, \nu^d\rho]_+; \sigma)$.

By the Theorem 5.2, $s_{GL}(\delta([\nu^{-c}\rho,\nu^d\rho]_-;\sigma)) \le s_{GL}(\delta([\nu^{-c}\rho,\nu^d\rho]_+;\sigma)$, and the difference is $\sum_{i=-\alpha+1}^{\alpha}\delta([\nu^{-i+1}\rho,\nu^c\rho])\times\delta([\nu^i\rho,\nu^d\rho])\otimes\sigma$. Passing to the minimal non-zero Jacquet module, this will give terms of the form

$$\ldots\otimes\nu^{-\alpha+1}\rho\otimes\sigma,\ldots\otimes\nu^{-\alpha+2}\rho\otimes\sigma,\ldots,\ldots\otimes\nu^{\alpha}\rho\otimes\sigma$$

and of the form

$$\ldots\otimes\nu^{\alpha}\rho\otimes\sigma,\ldots\otimes\nu^{\alpha-1}\rho\otimes\sigma,\ldots,\ldots\otimes\nu^{-\alpha+1}\rho\otimes\sigma.$$

Since $i+1\le-\alpha$, we see that φ can not be in the Jacquet module of the difference. Therefore, φ is also contained in the Jacquet module of $\delta([\nu^{-c}\rho,\nu^d\rho]_-;\sigma)$.

In the same way we see that the multiplicity of φ in the Jacquet modules of both $\delta([\nu^{-c}\rho,\nu^d\rho]_\pm;\sigma)$ is the same (and strictly positive).

Now we shall study the multiplicity of φ in the Jacquet modules of $\delta([\nu^{-c}\rho,\nu^{-d}\rho]_\pm;\sigma)$ using the transitivity of Jacquet modules, through the parabolic subgroup corresponding to $\pi'\otimes\delta([\nu^{i+1}\rho,\nu^j\rho])\otimes\sigma$. Obviously, we only need to study

$$\delta([\nu^{-i}\rho,\nu^c\rho])\times\delta([\nu^{j+1}\rho,\nu^d\rho])\otimes\delta([\nu^{i+1}\rho,\nu^j\rho]_+;\sigma)$$

$$+\delta([\nu^{-i}\rho,\nu^c\rho])\times\delta([\nu^{j+1}\rho,\nu^d\rho])\otimes\delta([\nu^{i+1}\rho,\nu^j\rho]_-;\sigma)$$

for $c<j$.

Applying Theorem 5.2 to $\delta([\nu^{i+1}\rho,\nu^j\rho]_\pm;\sigma)$, we get that the multiplicity of φ in the first summand is greater than or equal to the multiplicity in the second summand.

We have already proved that the first term belongs entirely to the Jacquet module of $\delta([\nu^{-c}\rho,\nu^{-d}\rho]_+;\sigma)$. Now our assumption that the subquotient $\pi'\otimes\delta([\nu^{i+1}\rho,\nu^j\rho]_-;\sigma)$ of the second sum, in which φ has positive multiplicity, belongs to the Jacquet module of $\delta([\nu^{-c}\rho,\nu^d\rho]_+;\sigma)$ implies that the multiplicity of φ in $\delta([\nu^{-c}\rho,\nu^d\rho]_+;\sigma)$ is strictly greater then the multiplicity of φ in the Jacquet module of $\delta([\nu^{-c}\rho,\nu^d\rho]_-;\sigma)$. This contradicts the fact that multiplicities are the same.

The case $j\le c$ can be handled in the same way, but more easily.

This completes the proof that all the terms of the form $\pi' \otimes \delta([\nu^{i+1}\rho, \nu^j\rho]_-; \sigma)$ are in the Jacquet of $\delta([\nu^{-c}\rho, \nu^{-d}\rho]_\pm; \sigma)$ if $|i+1| \leq j$.

The case $j < |i+1|$ can be handled in completely analogous way, but working with the segment $[\nu^{-j}\rho, \nu^{-i-1}\rho]$ instead of $[\nu^{i+1}\rho, \nu^j\rho]$.

It remains to determine where do the non-zero representations of the form $\pi' \otimes L(\delta([\nu^{i+1}\rho, \nu^j\rho]); \sigma)$ belong. Clearly, we can assume $[i+1, j] \not\subseteq [-\alpha + 1, \alpha - 1]$.

First observe that if $c + 1 \leq j$, then the right hand side contains a representation of the form $\nu^{-j}\rho$ in the cuspidal support. This is not in the discrete series by previous theorem and in this case the above term belongs to the Jacquet module of $L(\delta([\nu^{-c}\rho, \nu^d\rho]); \sigma)$.

It remains to consider the case $j \leq c$.

Let us first assume $i + j < -1$.

Since $i+1 \leq j$, we obtain $|j| \leq -i-1$. Now the condition $L(\delta([\nu^{i+1}\rho, \nu^j\rho]); \sigma) \neq 0$ gives $\alpha \leq -i-1$, i.e. $i+1 \leq -\alpha$.

By Theorem 5.2, $\delta(([\nu^{i+1}\rho, \nu^d\rho]) \times \delta(([\nu^{-i}\rho, \nu^c\rho])) \otimes \sigma$ appears in the Jacquet module of $\delta([\nu^{-c}\rho, \nu^d\rho]_\pm; \sigma)$.

Looking at $m^*(\delta(([\nu^{i+1}\rho, \nu^{-i}\rho]) \times \delta(([\nu^{-i}\rho, \nu^c\rho])) \otimes \sigma$ we deduce that

$$\delta([\nu^{j+1}\rho, \nu^d\rho]) \times \delta([\nu^{-i}\rho, \nu^c\rho]) \otimes L(\delta([\nu^{-j}\rho, \nu^{-i-1}\rho]); \sigma)$$

$$= \delta([\nu^{j+1}\rho, \nu^d\rho]) \times \delta([\nu^{-i}\rho, \nu^c\rho]) \otimes L(\delta([\nu^{i+1}\rho, \nu^j\rho]); \sigma)$$

appears in Jacquet modules of both $\delta([\nu^{-c}\rho, \nu^d\rho]_\pm; \sigma)$.

Now we assume $-1 < i + j$.

We immediately get $|i+1| \leq j$, while the non-triviality of $L(\delta([\nu^{i+1}\rho, \nu^j\rho]); \sigma)$ implies $\alpha \leq j$. By Theorem 5.2, Jacquet module of the Langlands quotient $L(\delta([\nu^{-c}\rho, \nu^d\rho]); \sigma)$ contains $L(\delta([\nu^{-j}\rho, \nu^c\rho]), \delta([\nu^{j+1}\rho, \nu^d\rho])) \otimes \sigma$. It is not hard to see that

$$L(\delta([\nu^{-i}\rho, \nu^c\rho]), \delta([\nu^{j+1}\rho, \nu^d\rho])) \otimes L(\delta([\nu^{i+1}\rho, \nu^j\rho]); \sigma).$$

is contained in the Jacquet module of $L(\delta([\nu^{-c}\rho, \nu^d\rho]); \sigma)$.

Representation of the form $\ldots \otimes L(\delta([\nu^{i+1}\rho, \nu^j\rho]); \sigma)$ can be obtained either from indexes i and j, or from indexes $-j+1$ and $-i+1$. In one case on the left hand side is an irreducible representation, while in the other case is a representation of length two and for all three irreducible pieces we have determined in which Jacquet modules they are contained.

This completes the proof. □

6 Reducibility at zero

The purpose of this section is to provide a complete and uniform treatment of the Jacquet modules of the representations of segment type in the case when the point of rank-one reducibility α equals zero. In this case the induced representation $\rho \rtimes \sigma$ reduces and we have the following decomposition:

$$\rho \rtimes \sigma = \tau_1 \oplus \tau_{-1}$$

into irreducible non-equivalent representations.

We shall fix non-negative integers c and d satisfying $c \leq d$. Consider the representation

$$\begin{aligned} & (\nu\rho \times \nu^2\rho \times \ldots \times \nu^d\rho) \times (\nu\rho \times \nu^2\rho \times \ldots \times \nu^c\rho) \times \rho \rtimes \sigma \\ \cong \quad & \oplus_{i=-1}^{1} (\nu\rho \times \nu^2\rho \times \ldots \times \nu^d\rho) \times (\nu\rho \times \nu^2\rho \times \ldots \times \nu^c\rho) \rtimes \tau_i. \end{aligned}$$

Remark 6.1. *A direct consequence of the formula for μ^* is that any irreducible sub quotient of $(\nu\rho \times \nu^2\rho \times \ldots \times \nu^d\rho) \times (\nu\rho \times \nu^2\rho \times \ldots \times \nu^c\rho) \rtimes \tau_i$ cannot have in its Jacquet module a term of the form $\cdots \otimes \tau_{-i}$, for $i \in \{1, -1\}$.*

The multiplicity of $\delta([\nu\rho, \nu^d\rho]) \times \delta([\nu\rho, \nu^c\rho]) \otimes \tau_i$ in the GL-type Jacquet module of the above full induced representation is one (the same holds for $\delta([\nu^{-c}\rho, \nu^d\rho]) \rtimes \sigma)$. We denote by $\delta([\nu^{-c}\rho, \nu^d\rho]_{\tau_i}; s)$ the unique irreducible subquotient which have this representation in its Jacquet module (it is also a subquotient of $\delta([\nu^{-c}\rho,\nu^d\rho])\rtimes\sigma)$. Such subquotient contains $\delta([\nu^{-c}\rho,\nu^d\rho])\otimes\sigma$ in its Jacquet module.

We continue with conventions that we have introduced for positive reducibility. Therefore, $L_0(\delta([\nu^a\rho,\nu^b\rho]);\sigma)$ denotes the usual Langlands quotient if $a\neq -b$, and $L_0(\delta([\nu^a\rho,\nu^b\rho]);\sigma)=0$ if $a=-b$. Further, $L_0(\emptyset;\sigma)=0$, $\delta(\emptyset;\sigma)=\sigma$. If $0\notin[a,b]$, then we take $\delta([\nu^a\rho,\nu^b\rho]_{\tau_i};\sigma)=0$. We continue also with symmetrization of the notation.

Theorem 6.2. *Let $c,d\in\mathbb{Z}$ such that $c\le d$ and $-c\le 0\le d$. Then*

1. *If $c<d$, then $\delta([\nu^{-c}\rho,\nu^d\rho])\rtimes\sigma$ is a representation of length 3, and the composition series consists of $\delta([\nu^{-c}\rho,\nu^d\rho]_{\tau_k};\sigma)$, $k=-1,1$, and $L(\delta([\nu^{-c}\rho,\nu^d\rho]);\sigma)$.*

   For $c=d$, $\delta([\nu^{-c}\rho,\nu^c\rho])\rtimes\sigma$ is a representation of length 2, and the composition series consists of $\delta([\nu^{-c}\rho,\nu^c\rho]_{\tau_k};\sigma)$, $k=-1,1$.

2. *For $c<d$, $\delta([\nu^{-c}\rho,\nu^d\rho]_{\tau_k};\sigma)$ are square integrable. For $c=d$, the representations are tempered, but not square integrable.*

3. $s_{GL}(\delta([\nu^{-c}\rho,\nu^d\rho]_{\tau_k};\sigma))=\sum_{i=-c}^{0}\delta([\nu^{-i+1}\rho,\nu^c\rho])\times\delta([\nu^i\rho,\nu^d\rho])\otimes\sigma.$

4. *For $c<d$ holds*
   $s_{GL}(L(\delta([\nu^{-c}\rho,\nu^d\rho]);\sigma))=\sum_{i=1}^{d+1}L(\delta([\nu^{-i+1}\rho,\nu^c\rho]),\delta([\nu^i\rho,\nu^d\rho]))\otimes\sigma.$

5. *For $c>0$ holds*
   $$\begin{aligned} s_{top}(\delta([\nu^{-c}\rho,\nu^c\rho]_{\tau_k};\sigma)) &= 2\nu^c\rho\otimes\delta([\nu^{-c+1}\rho,\nu^c\rho]_{\tau_i};\sigma)+\\ &\nu^c\rho\otimes L_0(\delta([\nu^{-c+1}\rho,\nu^c\rho]);\sigma).\end{aligned}$$

   Further $s_{top}(\delta([\rho,\rho]_{\tau_k};\sigma))=\rho\otimes\sigma$.

6. *For $c<d$ holds*
   $$\begin{aligned} s_{top}(\delta([\nu^{-c}\rho,\nu^d\rho]_{\tau_k};\sigma)) &= \nu^d\rho\otimes\delta([\nu^{-c}\rho,\nu^{d-1}\rho]_{\tau_k};\sigma)+\\ &\nu^c\rho\otimes\delta([\nu^{-c+1}\rho,\nu^d\rho]_{\tau_k};\sigma).\end{aligned}$$

7. *For $c<d$ holds*
   $$\begin{aligned} s_{top}(L(\delta([\nu^{-c}\rho,\nu^d\rho]);\sigma)) &= \nu^d\rho\otimes L_0(\delta([\nu^{-c}\rho,\nu^{d-1}\rho]);\sigma)+\\ &\nu^c\rho\otimes L_0(\delta([\nu^{-c+1}\rho,\nu^d\rho]);\sigma).\end{aligned}$$

Proof. Regarding (1), in [14] is proved length two in the tempered case. Also, in the same paper have been proved (2) and (3). On the other hand, the length three claim in (1) follows from [10].

Observe that it can be proved in the same way as in the proof of previous theorem that the sum of three sums on the right hand side of (3) and (4) equals $s_{GL}(\delta([\nu^{-c}\rho, \nu^d\rho]) \rtimes \sigma)$. Thus, the claim (4) follows. It remains to prove claims (5), (6) and (7).

Note that $s_{top}(\delta([\rho,\rho]_{\tau_i};\sigma)) = \rho \otimes \sigma$. In the rest of the proof is enough to consider the case $0 < c + d$.

The formula for μ^* now gives

$$s_{top}(\delta([\nu^{-c}\rho, \nu^d\rho]) \rtimes \sigma) = \nu^d\rho \otimes \delta([\nu^{-c}\rho, \nu^{d-1}\rho]) \rtimes \sigma + \nu^c\rho \otimes \delta([\nu^{-c+1}\rho, \nu^d\rho]) \rtimes \sigma.$$

First we shall prove the theorem in the case $c = 0$. The proof will go by induction over d.

Consider first the case $d = 1$. The above formula and the previous theorem, together with symmetrization of the notation, imply

$$s_{top}(\delta([\rho, \nu\rho]) \rtimes \sigma) = \nu\rho \otimes \tau_1 + \nu\rho \otimes \tau_{-1} + \rho \otimes L_0(\nu\rho; \sigma).$$

Therefore, the theorem holds in this situation ($\nu\rho \otimes \tau_i$ is obviously in the Jacquet module of $\delta([\rho, \nu\rho]_{\tau_i}; \sigma)$).

Fix $d > 1$ and assume that the theorem holds for $d - 1$. Then the inductive assumption implies the following decomposition into irreducible representations

$$s_{top}(\delta([\rho, \nu^d\rho]) \rtimes \sigma) = \nu^d\rho \otimes \delta([\rho, \nu^{d-1}\rho]) \rtimes \sigma + \rho \otimes \delta([\nu\rho, \nu^d\rho]) \rtimes \sigma$$

$$= \nu^d\rho \otimes \delta([\rho, \nu^{d-1}\rho]_{\tau_1}; \sigma) + \nu^d\rho \otimes \delta([\rho, \nu^{d-1}\rho]_{\tau_{-1}}; \sigma)$$

$$+\nu^d\rho \otimes L_0(\delta([\rho, \nu^{d-1}\rho]); \sigma) + \rho \otimes L_0(\delta([\nu\rho, \nu^d\rho]); \sigma).$$

The first two terms obviously belong to the Jacquet modules of $\delta([\nu^{-\alpha}\rho, \nu^d\rho]_{\tau_i}; \sigma)$, $i = 1, -1$, since neither $\nu^{-(d-1)}\rho$ nor $\nu^{-d}\rho$ show up on the cuspidal support of the discrete series (and $\nu^d\rho \otimes \delta([\rho, \nu^{d-1}\rho]_{\tau_i}; \sigma)$ is obviously in the Jacquet module of $\delta([\rho, \nu^d\rho]_{\tau_i}; \sigma))$. Further, considering $\nu^{-d}\rho$, we get that the last summand is in the Jacquet module of the Langlands quotient. Observe that the Langlands quotient embeds into $\delta([\nu^{-d}\rho, \rho]) \rtimes \sigma \hookrightarrow \nu^d\rho \times \delta([\nu^{-d+1}\rho, \rho]) \rtimes \sigma$ and this implies that the third summand is in the Jacquet module of the Langlands quotient. This proves formulas (5), (6) and (7).

Now we shall fix $c > \alpha$, and assume that the theorem holds for $c - 1$. We shall proceed with induction, similarly as in the case $c = 0$.

We start with the case $d = c$. We know that then $\delta([\nu^{-c}\rho, \nu^c\rho]) \rtimes \sigma$ is of length two. The above formula and symmetrization of notation yield

$$s_{top}(\delta([\nu^{-c}\rho, \nu^c\rho]) \rtimes \sigma) = 2\nu^c\rho \otimes \delta([\nu^{-c+1}\rho, \nu^c\rho]_{\tau_1}; \sigma)$$

$$+2\nu^c\rho \otimes \delta([\nu^{-c+1}\rho, \nu^c\rho]_{\tau_{-1}}; \sigma) + 2\nu^c\rho \otimes L(\delta([\nu^{-c+1}\rho, \nu^c\rho]); \sigma).$$

Directly follows that $2\nu^c\rho \otimes \delta([\nu^{-c+1}\rho, \nu^c\rho]_{\tau_i}; \sigma)$ must belong to the Jacquet module of $\delta([\nu^{-c}\rho, \nu^c\rho]_{\tau_i}; \sigma)$ (recall that we know length two of the induced representation). Further, the fact that $\delta([\nu^{-c}\rho, \nu^c\rho]) \otimes \sigma$ is in the Jacquet module of both $\delta([\nu^{-c}\rho, \nu^c\rho]_{\tau_i}; \sigma)$ and transitivity of Jacquet modules imply that $\nu^c\rho \otimes L(\delta([\nu^{-c+1}\rho, \nu^c\rho]); \sigma)$ must be in each of the Jacquet modules.

This completes the proof of the theorem in this case.

Fix $d > c$ and assume that the theorem holds for $d - 1$ (for that c), and for $c - 1$ (for all $d \leq c - 1$). The inductive assumption implies the following decomposition into irreducible representations

$$s_{top}(\delta([\nu^{-c}\rho, \nu^d\rho]) \rtimes \sigma) = \nu^d\rho \otimes \delta([\nu^{-c}\rho, \nu^{d-1}\rho]) \rtimes \sigma + \nu^c\rho \otimes \delta([\nu^{-c+1}\rho, \nu^d\rho]) \rtimes \sigma$$

$$= \nu^d\rho \otimes \delta([\nu^{-c}\rho, \nu^{d-1}\rho]_{\tau_1}; \sigma) + \nu^c\rho \otimes \delta([\nu^{-c+1}\rho, \nu^d\rho]_{\tau_1}; \sigma)$$
$$+\nu^d\rho \otimes \delta([\nu^{-c}\rho, \nu^{d-1}\rho]_{\tau_{-1}}; \sigma) + \nu^c\rho \otimes \delta([\nu^{-c+1}\rho, \nu^d\rho]_{\tau_{-1}}; \sigma)$$
$$+\nu^d\rho \otimes L(\delta([\nu^{-c}\rho, \nu^{d-1}\rho]); \sigma) + \nu^c\rho \otimes L(\delta([\nu^{-c+1}\rho, \nu^d\rho]); \sigma),$$

and the rest of the proof follows in the same way as in the proof of Theorem 5.2. □

Again, we have an interpretation in terms of admissible triples:

Remark 6.3. *Suppose that we have $c \neq d$ and $-c \leq 0 \leq d$. Then $Jord(\delta([\nu^{-c}\rho, \nu^{d}\rho]_{\tau_k}; \sigma)) = Jord(\sigma) \cup \{(2c+1, \rho), (2d+1, \rho)\}$. Further, if we denote by ϵ_k the ϵ-function corresponding to $\delta([\nu^{-c}\rho, \nu^{d}\rho]_{\tau_k}; \sigma)$ we have $\epsilon_k((2c+1, \rho), (2d+1, \rho)) = 1$ and $\epsilon_k((2d+1, \rho)) = k$.*

Using previous theorem and Lemma 5.1, which also holds when reducibility point equals zero, we obtain a complete description of Jacquet modules in this case. Proof of the following corollary can be obtained in the same manner as the proof of Corollary 5.4, details being left to the reader.

Corollary 6.4. *Let $c, d \in \mathbb{Z}$ such that $c \leq d$ and $-c \leq 0 \leq d$. Then*

1.
$$\mu^* \left(\delta([\nu^{-c}\rho, \nu^{d}\rho]_{\tau_i}; \sigma)\right)$$
$$= \sum_{-c-1 \leq i \leq c,} \sum_{i+1 \leq j \leq d?} \delta([\nu^{-i}\rho, \nu^{c}\rho]) \times \delta([\nu^{j+1}\rho, \nu^{d}\rho]) \otimes \delta([\nu^{i+1}\rho, \nu^{j}\rho]_{\tau_i}; \sigma)$$
$$+ \sum_{\substack{-c-1 \leq i \leq c, \\ i+j<-1}} \sum_{i+1 \leq j \leq c,} \delta([\nu^{-i}\rho, \nu^{c}\rho]) \times \delta([\nu^{j+1}\rho, \nu^{d}\rho]) \otimes L_0(\delta([\nu^{i+1}\rho, \nu^{j}\rho]); \sigma)$$
$$+ \sum_{i=-c}^{0} \delta([\nu^{-i+1}\rho, \nu^{c}\rho]) \times \delta([\nu^{i}\rho, \nu^{d}\rho]) \otimes \sigma.$$

2. *For $c < d$ we have*
$$\mu^* \left(L(\delta([\nu^{-c}\rho, \nu^{d}\rho]); \sigma)\right) = \mu^* \left(L_0(\delta([\nu^{-c}\rho, \nu^{d}\rho]); \sigma)\right) = \mu^* \left(L_{proper}(\delta([\nu^{-c}\rho, \nu^{d}\rho]); \sigma)\right)$$
$$= \sum_{\substack{-c-1 \leq i \leq d, \\ 0 \leq i+j}} \sum_{i+1 \leq j \leq d,} L(\delta([\nu^{-i}\rho, \nu^{c}\rho]), \delta([\nu^{j+1}\rho, \nu^{d}\rho])) \otimes L_0(\delta([\nu^{i+1}\rho, \nu^{j}\rho]); \sigma))$$
$$+ \sum_{i=1}^{d+1} L(\delta([\nu^{-i+1}\rho, \nu^{c}\rho]), \delta([\nu^{i}\rho, \nu^{d}\rho])) \otimes \sigma.$$

7 Jacquet modules of strongly positive representations

In this section we present an alternative way of determining the formula for Jacquet modules of strongly positive representations, which can be viewed as a certain generalization of representations of segment type studied in the fourth section. An analogous formula is obtained in [6].

We fix self-dual cuspidal representation ρ of $GL(n_\rho, F)$ (this defines n_ρ) and cuspidal representation σ of G_{n_σ} (this defines n_σ). We assume that $\nu^\alpha\rho \rtimes \sigma$ reduces for $\alpha > 0$ and put $\epsilon = 1$ if α is integer and $\epsilon = 1/2$ otherwise . Fix

$$n_\epsilon < n_{\epsilon+1} < \cdots < n_\alpha$$

such that $\epsilon - 1 \leq n_\epsilon$ and $n_i - \alpha$ is an integer for $i = \epsilon, \epsilon + 1, \ldots, \alpha$. Observe that then also $i - 1 \leq n_i$ for all indexes.

It has been proved in [5] that the induced representation

$$\delta([\nu^\epsilon\rho, \nu^{n_\epsilon}\rho]) \times \cdots \times \delta([\nu^\alpha\rho, \nu^{n_\alpha}\rho]) \rtimes \sigma$$

has a unique irreducible subrepresentation, which we will denote by $DS_{\rho;\sigma}(n_\alpha, \ldots, n_\epsilon)$. This representation is strongly positive. Further, it has been proved in [8, 9] and separately in [5] that every strongly positive discrete series which contains only twists of the representation ρ in its cuspidal support is isomorphic to some $DS_{\rho;\sigma}(n_\alpha, \ldots, n_\epsilon)$.

We will denote the unique irreducible subrepresentation of

$$\delta([\nu^\epsilon\rho, \nu^{n_\epsilon}\rho]) \times \cdots \times \delta([\nu^\alpha\rho, \nu^{n_\alpha}\rho])$$

by $Lad_\rho(n_\alpha, \ldots, n_\epsilon)$. This is the ladder representation $L(\delta([\nu^\epsilon\rho, \nu^{n_\epsilon}\rho]), \ldots, \delta([\nu^\alpha\rho, \nu^{n_\alpha}\rho]))$, as introduced in [4].

Uniqueness of irreducible subrepresentation of $\delta([\nu^\epsilon\rho, \nu^{n_\epsilon}\rho]) \times \cdots \times \delta([\nu^\alpha\rho, \nu^{n_\alpha}\rho]) \rtimes \sigma$ implies

$$DS_{\rho;\sigma}(n_\alpha, \ldots, n_\epsilon) \hookrightarrow Lad_\rho(n_k, \ldots, n_\epsilon) \rtimes \sigma.$$

This implies

$$s_{GL}(DS_{\rho;\sigma}(n_\alpha, \ldots, n_\epsilon)) \leq s_{GL}(Lad_\rho(n_\alpha, \ldots, n_\epsilon) \rtimes \sigma).$$

Calculating the right-hand side of the previous inequality, we obtain that the only term which has all positive exponents in cuspidal support is $Lad_\rho(n_\alpha, \dots, n_e) \otimes \sigma$. Therefore, $s_{GL}(DS_{\rho;\sigma}(n_\alpha, \dots, n_\epsilon)) \leq Lad_\rho(n_\alpha, \dots, n_\epsilon) \otimes \sigma$, which implies

$$s_{GL}(DS_{\rho;\sigma}(n_\alpha, \dots, n_\epsilon)) = Lad_\rho(n_\alpha, \dots, n_\epsilon) \otimes \sigma.$$

Using Lemma 3.5 of [6] we see that this Jacquet module characterizes the strongly positive representation.

Using the formula for Jacquet modules of ladder representations from [3], we deduce

$$(m^* \otimes id)(s_{GL}(DS_{\rho;\sigma}(n_\alpha, \dots, n_\epsilon))) = m^*(Lad_\rho(n_\alpha, \dots, n_\epsilon)) \otimes \sigma =$$
$$\sum_{\substack{c_\epsilon < \dots < c_\alpha, \\ i-1 \leq c_i \leq n_i}} L(\delta([\nu^{c_\epsilon+1}\rho, \nu^{n_\epsilon}\rho]), \dots, \delta([\nu^{c_\alpha+1}\rho, \nu^{n_\alpha}\rho])) \otimes Lad_\rho(c_\alpha, \dots, c_\epsilon) \otimes \sigma,$$

which directly gives, using the above characterization of strongly positive representations by GL-type Jacquet modules

$$\mu^*(DS_{\rho;\sigma}(n_\alpha, \dots, n_\epsilon)) =$$
$$\sum_{\substack{c_\epsilon < \dots < c_\alpha, \\ i-1 \leq c_i \leq n_i}} L(\delta([\nu^{c_\epsilon+1}\rho, \nu^{n_\epsilon}\rho]), \dots, \delta([\nu^{c_\alpha+1}\rho, \nu^{n_\alpha}\rho])) \otimes DS_{\rho;\sigma}(c_\alpha, \dots, c_\epsilon).$$

Now we shall try to give Jordan blocks interpretation of the above formula.

Fix ρ and σ as above, and a sequence of integers

$$k_{\lceil\alpha\rceil} > \dots > k_1 \geq 0.$$

These integers are taken to be odd if α is integral. Otherwise, we take them to be even.

Denote $Jord_{(\rho;k_\alpha,\dots,k_1)} = \{(\rho, k_{\lceil\alpha\rceil}), \dots, (\rho, k_1)\}$, where we drop $(\rho, 0)$ is it shows up on the right hand side. From

As we have seen before, the induced representation

$$\delta([\nu^{\lceil\alpha\rceil-\alpha+1}\rho, \nu^{(k_1-1)/2}\rho]) \times \dots \times \delta([\nu^{\lceil\alpha\rceil}\rho, \nu^{(k_{\lceil\alpha\rceil}-1)/2}\rho]) \rtimes \sigma$$

contains a unique irreducible subrepresentation, which will be denoted by $\lambda_{\{(\rho,k_{\lceil\alpha\rceil}),\dots,(\rho,k_1)\},\epsilon_+,\sigma}$. It is a discrete series reresentation and, by [Moe-exhaus], it is attached to an admissible triple. But for such admissible triple we have an alternated partially defined function and there is at most one such function, which is already determined by Jordan blocks and partial cuspidal support σ. Now the above formula in this notation becomes

$$\mu^*(\lambda_{\{(\rho,k_{\lceil\alpha\rceil}),\dots,(\rho,k_1)\},\epsilon_+,\sigma}) = \sum_{\substack{l_1<\dots<l_{\lceil\alpha\rceil} \\ 2(\lceil\alpha\rceil-\alpha+i)-1\le l_i\le k_i+1}}$$

$$L(\delta([\nu^{(l_1+1)/2}\rho,\nu^{(k_1-1)/2}\rho]),\dots,\delta([\nu^{(l_{\lceil\alpha\rceil}+1)/2}\rho,\nu^{(k_{\lceil\alpha\rceil}-1)/2}\rho]))\otimes\lambda_{\{(\rho,l_{\lceil\alpha\rceil}),\dots,(\rho,l_1)\},\epsilon_+,\sigma},$$

where $k_i - l_i$ is an integer for all i.

8 Top Jacquet modules

This section is devoted to determination of top Jacquet modules of general discrete series of classical groups.

Let use denote a discrete series representation by π, corresponding to an admissible triple $(Jord(\pi), \epsilon_\pi, \pi_{cusp})$.

The facts that we collect in the following lemma are well known (see [8] and [9]).

Lemma 8.1. *Suppose that $\tau\otimes\varphi$ is an irreducible representation contained in $\tau\otimes\varphi \le s_{top}(\pi)$ ans that ρ is an irreducible self-dual representation of a general linear group. Let $\tau = \nu^{e(\tau)}\tau_u$, with τ_u unitarizable. Then*

1. *τ_u is self-dual, $e(\tau)\in(1/2)\mathbb{Z}$ and $e(\tau)>0$.*

2. *$(\tau_u, 2e(\tau)+1)\in Jord(\pi)$.*

3. If $2 \in Jord_\rho(\pi)$ *and* $\epsilon_\pi((\rho, 2)) = -1$*, then*

$$\tau \neq \nu^{1/2}\rho$$

for any $\tau \otimes \varphi$ *as above.*

4. If $a, a-2 \in Jord_\rho(\pi)$ *and* $\epsilon_\pi((\rho, a-2), (\rho, a)) = -1$*, then*

$$\tau \neq \nu^{(a-1)/2}\rho$$

for any $\tau \otimes \varphi$ *as above.* □

It follows from the previous lemma that if some irreducible representation is in $s_{top}(\pi)$, then it must be of the form $\nu^{(a-1)/2}\rho \otimes \varphi$, where the following holds:

- $a \geq 2$.
- If $a = 2$, then $\epsilon_\pi((\rho, 2)) = 1$.
- If $a - 2 \in Jord_\rho(\pi)$, then $\epsilon_\pi((\rho, a-2), (\rho, a)) = 1$.

First we shall consider the situation that

$$2 \in Jord_\rho(\pi) \ \text{ and } \ \epsilon_\pi((\rho, 2)) = 1.$$

In this case $\pi^{(\rho,2\downarrow\emptyset)}$ or $\pi^{(\rho,2\downarrow 0)}$ will denote the irreducible square integrable representation determined by an admissible triple

$$(Jord_\rho(\pi)\backslash\{(\rho, 2)\}, \epsilon'_\pi, \pi_{cusp}),$$

where ϵ'_π denotes the partially defined function which one gets by restriction of ϵ_π to $Jord_\rho(\pi)\backslash\{(\rho, 2)\}$.

Lemma 8.2. *Let* ρ *be an irreducible self-dual representation of a general linear group. Suppose that* $2 \in Jord_\rho(\pi)$*,* $\epsilon_\pi((\rho, 2)) = 1$ *and* $\nu^{1/2}\rho \otimes \varphi \leq s_{top}(\pi)$ *for some irreducible* φ*. Then* $\varphi \cong \pi^{\delta(\rho,2\downarrow 0)}$ *and the multiplicity of* $\nu^{1/2}\rho \otimes \varphi$ *in* $s_{top}(\pi)$ *is one.*

Proof. Since $\epsilon_\pi((\rho,2))=1$, we have

$$\pi \hookrightarrow \nu^{1/2}\rho \rtimes \pi^{(\rho,2\downarrow 0)}$$

(see Lemma 9.1 of [16], or [2]). Thus

$$\nu^{1/2}\rho\otimes\varphi \leq \mu^*(\nu^{1/2}\rho \rtimes \pi^{(\rho,2\downarrow 0)}),$$

which implies directly (by the formula for μ^*)

$$\nu^{1/2}\rho\otimes\varphi \leq (\nu^{1/2}\rho\otimes 1)\rtimes \mu^*(\pi^{(\rho,2\downarrow 0)}) + (1\otimes \nu^{1/2}\rho)\rtimes \mu^*(\pi^{(\rho,2\downarrow 0)}).$$

We have two possibilities. First is $\nu^{1/2}\rho\otimes\varphi \leq (\nu^{1/2}\rho\otimes 1)\rtimes \mu^*(\pi^{(\rho,2\downarrow 0)})$, which implies $\varphi\cong\pi^{(\rho,2\downarrow 0)}$. The second is $\nu^{1/2}\rho\otimes\varphi \leq (1\otimes\nu^{1/2}\rho)\rtimes \mu^*(\pi^{(\rho,2\downarrow 0)})$, which directly implies that $(\rho,2)$ is in the Jordan block of $\pi^{(\rho,2\downarrow 0)}$, a contradiction.

In consequence, $\varphi\cong\pi^{(\rho,2\downarrow 0)}$.

Further, assumption $2\nu^{1/2}\rho\otimes\pi^{(\rho,2\downarrow 0)} \leq s_{top}(\pi)$ would yield $2\nu^{1/2}\rho\otimes\pi^{(\rho,2\downarrow 0)} \leq \nu^{1/2}\rho\otimes\pi^{(\rho,2\downarrow 0)}$, which is impossible. The proof is now complete. □

Now we shall consider the situation when

$$a\geq 4,\ a\in Jord_\rho(\pi) \text{ and } a-2\notin Jord_\rho(\pi).$$

Then $\pi^{(\rho,a\downarrow a-2)}$ will denote the irreducible square integrable representation determined by admissible triple given in the following way. The Jordan blocks are obtained by replacing in $Jord_\rho(\pi)$ the representation (ρ,a) by $(\rho,a-2)$ and keeping all other representations unchanged. The new partially defined function is obtained from the old one by replacing everywhere (ρ,a) by $(\rho,a-2)$, while the cuspidal support remains unchanged.

Lemma 8.3. *Let ρ be an irreducible self-dual representation of a general linear group. Suppose that $a\geq 3$, $a\in Jord_\rho(\pi)$ and $a-2\notin Jord_\rho(\pi)$ and*

$$\nu^{(a-1)/2}\rho\otimes\varphi\leq s_{top}(\pi)$$

for some irreducible φ. Then

$$\varphi\cong\pi^{(\rho,a\downarrow a-2)},$$

and the multiplicity of $\nu^{(a-1)/2}\rho\otimes\varphi$ in $s_{top}(\pi)$ is one.

Proof. Lemma 8.1 of [16] gives

$$\pi \hookrightarrow \nu^{(a-1)/2}\rho \rtimes \pi^{(\rho,a\downarrow a-2)}.$$

Now the rest of the proof runs in the same way as in the proof of previous lemma. □

At the end, we shall consider the situation

$$a, a-2 \in Jord_\rho(\pi) \text{ and } \epsilon_\pi((\rho,a)) = \epsilon_\pi((\rho,a-2)).$$

Here we shall need the parameterisation of tempered duals. We choose to work with the one from [2].

Jantzen parameters are very similar to the parameters of the square integrable representations, but here parameters are quadruples, where the additional parameter is the multiplicity function on Jordan blocks. However, we can interpret these parameters as triples, in a way that we interpret Jordan blocks $Jord(\tau)$ as multisets. When we consider the set determined by $Jord(\tau)$ (this is the case when one considers the partially defined function attached to τ in [2]), then it will be determined by $|Jord(\tau)|$.

Let us denote by $\pi_0 = \pi^{(\rho,a,),(\rho,a-2)\mapsto\emptyset}$ an irreducible discrete series determined by admissible triple given in the following way: the Jordan blocks are obtained by removing in $Jord_\rho(\pi)$ the representations (ρ, a) and $(\rho, a-2)$, the new partially defined function is obtained from the old one by restriction and the cuspidal support remains unchanged.

Consider now two inequivalent tempered irreducible subrepresentations of

$$\delta([\nu^{-\frac{a-1}{2}}\rho, \nu^{\frac{a-1}{2}}\rho]) \rtimes \pi_0 = \tau_1 + \tau_{-1}.$$

For precisely one $i_0 \in \{1,-1\}$, we have

$$\pi \hookrightarrow \nu^{(a-1)/2}\rho \rtimes \tau_{i_0}.$$

Now we shall discuss the Jantzen parameters of representations τ_i. The cuspidal supports of both τ_i's are the same as of π_0 (and π). Further, one gets Jordan blocks of both τ_i's by adding two times $(\rho, a-2)$ to $Jord(\pi_0)$. In another words, one gets $Jord(\tau_i)$ from $Jord(\pi)$ by replacing (ρ, a) by $(\rho, a-2)$ (not forgetting that we have now the multiplicity two of $(\rho, a-2)$).

One has two possibilities for the partially defined functions corresponding to representations τ_i. Let us denote the partially defined function on $|Jord(\tau)|$ which one gets from ϵ_π replacing (ρ, a) by $(\rho, a-2)$ everywhere in the definition of ϵ_π by ϵ'. Now for precisely one of the τ_i's as above, the partially defined function of τ_i is equal to ϵ'. We denote τ_i corresponding to this partially defined function by $\pi^{(\rho,a\downarrow a-2)}$.

Now Corollary 3.2.3 of [2] implies

$$\pi \hookrightarrow \nu^{(a-1)/2}\rho \rtimes \pi^{(\rho,a\downarrow a-2)}.$$

Lemma 8.4. *Let ρ be an irreducible self-dual representation of a general linear group. Suppose that $a \geq 3$, $a, a-2 \in Jord_\rho(\pi)$ and $\epsilon_\pi((\rho, a)) = \epsilon_\pi((\rho, a-2))$ and*

$$\nu^{(a-1)/2}\rho \otimes \varphi \leq s_{top}(\pi)$$

for some irreducible φ. Then $\varphi \cong \pi^{(\rho,a\downarrow a-2)}$. The multiplicity of $\nu^{(a-1)/2}\rho \otimes \varphi$ in $s_{top}(\pi)$ is one.

Proof. We know

$$\nu^{(a-1)/2}\rho \otimes \varphi \leq \mu^*(\pi) \leq \mu^*(\nu^{(a-1)/2} \rtimes \pi^{(\rho,a\downarrow a-2)}),$$

which directly implies

$$\nu^{(a-1)/2}\rho\otimes\varphi \leq (\nu^{(a-1)/2}\rho\otimes 1) \rtimes \mu^*(\pi^{(\rho,a\downarrow a-2)}) + (1\otimes\nu^{(a-1)/2}\rho) \rtimes \mu^*(\pi^{(\rho,a\downarrow a-2)}).$$

Again have two possibilities. The first is $\nu^{(a-1)/2}\rho \otimes \varphi \leq (\nu^{(a-1)/2}\rho \otimes 1) \rtimes \mu^*(\pi^{(\rho,a\downarrow a-2)})$, which implies $\varphi \cong \pi^{(\rho,a\downarrow a-2)}$.

The remaining possibility is $\nu^{(a-1)/2}\rho \otimes \varphi \leq (1 \otimes \nu^{(a-1)/2}\rho) \rtimes \mu^*(\pi^{(\rho,a\downarrow a-2)})$. This implies

$$(\nu^{(a-1)/2}\rho \otimes \varphi) \leq (1 \otimes \nu^{(a-1)/2}\rho) \times M^*(\delta([\nu^{-\frac{a-3}{2}}\rho, \nu^{\frac{a-3}{2}}\rho])) \rtimes \mu^*(\pi_0).$$

The formula for $M^*(\delta([\nu^{-\frac{a-3}{2}}\rho, \nu^{\frac{a-3}{2}}\rho]))$ forces

$$\nu^{(a-1)/2}\rho \otimes \varphi' \leq \mu^*(\pi_0)$$

for some φ', which further implies that (ρ, a) is in the Jordan block of π_0, which is not the case. Thus, we got a contradiction.

Consequently, $\varphi \cong \pi^{(\rho,a\downarrow a-2)}$.

The assumption $2\nu^{(a-1)/2}\rho \otimes \varphi \leq s_{top}(\pi)$ would imply that

$$\nu^{(a-1)/2}\rho \otimes \delta([\nu^{-\frac{a-3}{2}}\rho, \nu^{\frac{a-3}{2}}\rho]) \rtimes \pi'$$

is not a multiplicity one representation, which is impossible since $\delta([\nu^{-\frac{a-3}{2}}\rho, \nu^{\frac{a-3}{2}}\rho]) \rtimes \pi'$ is a multiplicity one representation.

This completes the proof. □

From the above four lemmas we obtain the following

Theorem 8.5. *Let π be an irreducible square integrable representation of a classical group. Then*

$$s_{top}(\pi) = \sum \nu^{(a-1)/2}\rho \otimes \pi^{(\rho,a\downarrow a-2)},$$

where the sum runs over all

$$(\rho, a) \in Jord(\pi)$$

which satisfy the following two conditions:

$$a - 2 \in Jord_\rho(\pi) \implies \epsilon_\pi((\rho, a)) = \epsilon_\pi((\rho, a-2));$$

$$a = 2 \implies \epsilon_\pi((\rho, 2)) = 1.$$

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Ivan Matić
Department of Mathematics, University of Osijek
Osijek, Croatia
E-mail: imatic@mathos.hr

Marko Tadić
Department of Mathematics, University of Zagreb
Zagreb, Croatia
E-mail: tadic@math.hr