On some properties of the Lyapunov equation for damped systems

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Abstract
We consider a damped linear vibrational system whose dampers depend linearly on the viscosity parameter $v$. We show that the trace of the corresponding Lyapunov solution can be represented as a rational function of $v$ whose poles are the eigenvalues of a certain skew symmetric matrix. This makes it possible to derive an asymptotic expansion of the solution in the neighborhood of zero (small damping).

1 Introduction
We consider a damped linear vibrational system

\[ M\ddot{x} + C\dot{x} + Kx = 0 \]  \hspace{1cm} (1.1)

where the matrices $M, C, K$ (mass, damping, stiffness) are symmetric, $M, K$ are positive definite and $C$ is positive semidefinite. If internal damping is neglected $C$ has often small rank as it describes a few dampers built in to calm down dangerous vibrations. Often $C$ has the form

\[ C = vC_0 \]

where $v$ is a variable viscosity and $C_0$ describes the geometry of a damper. $C_0$ will have rank 1, 2 or 3 according to whether the damper can exhibit linear, planar or spatial displacements.

1 A part of this work was written while the author was visiting researcher on the Lehrgebiet Mathematische Physik, Fernuniversität, Hagen
An example is the so-called *n-mass oscillator* or *oscillator ladder* (Fig. 1) where

\[
M = \text{diag}(m_1, m_2, \ldots, m_n)
\]

\[
K = \begin{bmatrix}
  k_1 & -k_1 \\
  -k_1 & k_1 + k_2 & -k_2 \\
  & \ddots & \ddots & \ddots \\
  & & -k_{n-2} & k_{n-2} + k_{n-1} & -k_{n-1} \\
  & & & -k_{n-1} & k_{n-1} + k_n
\end{bmatrix},
\]

\[
C = ve_1^T + v(e_3 - e_2)(e_3 - e_2)^T.
\]

(1.2)

![Figure 1: The n-mass oscillator with two dampers](image)

Here \(m_i > 0\) are the masses, \(k_i > 0\) the spring constants or stiffnesses, \(e_i\) is the \(i\)-th canonical basis vector, and \(v\) is the viscosity of the damper applied on the \(i\)-th mass.

After the substitution

\[
y_1 = \Omega \Phi^{-1} x, \quad y_2 = \Phi^{-1} \dot{x},
\]

(1.3)

where

\[
K \Phi = M \Phi \Omega^2, \quad \Phi^T M \Phi = I
\]

\[
\Omega = \text{diag}(\omega_1, \ldots, \omega_n), \quad \omega_1 < \ldots < \omega_n
\]

(1.4)

(1.5)

is the eigenreduction of the symmetric positive definite matrix pair \(K, M\), the system (1.1) goes over into

\[
\dot{y} = Ay, \quad y = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix},
\]

(1.6)

\[
A = \begin{bmatrix} 0 & \Omega \\ -\Omega & -D \end{bmatrix}, \quad D = \Phi^T C \Phi.
\]

(1.7)

Then

\[
E = \frac{1}{2} \int_0^\infty (x^T M \dot{x} + x^T K x) \, dt = \int_0^\infty \|y\|^2 \, dt = \int_0^\infty \|c^A t \, y_0\|^2 \, dt,
\]

(1.8)
where \( y_0 \) is the initial data. Thus,

\[
E \equiv E(y_0) = y_0^T X y_0 ,
\]

where

\[
X = \int_0^\infty e^{A^T t} e^{A t} dt
\]

solves the Lyapunov equation

\[
A^T X + X A = -I .
\]

Our penalty function is obtained by averaging \( E \) over all initial data with the equal energy, that is, we form

\[
\bar{E} = \int_{\|y_0\|=1} y_0^T X y_0 d\sigma
\]

where \( d\sigma \) is a probability measure on the unit sphere in \( \mathbb{R}^{2n} \). For any given measure there is a unique positive semidefinite matrix \( Z \) such that

\[
\bar{E} = Tr(Z X) .
\]

To achieve better stability, we may want to minimize \( \bar{E} \) over the free parameter \( v \). (For more details on this derivation see [6], [5], [3] or [4]). A typical choice of \( Z \) is

\[
Z = Z_s = \begin{bmatrix}
I_s & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & I_s & 0 \\
0 & 0 & 0 & 0
\end{bmatrix},
\]

where \( I_s \) is the identity of size \( s \). This corresponds to covering the external excitations whose frequencies lie below the \( s \)-th frequency \( \omega_s \).

In [6] a simple solution of the problem (1.11) has been presented for \( A \) from (1.7) and \( \text{rank}(C) = 1 \). In particular,

\[
Tr(Z X(v)) = \text{const} + \frac{a}{v^2} + bv, \quad a, b > 0 ,
\]

which made it possible to find the minimum by a simple formula explicitly. The case \( \text{rank}(C) > 1 \) seems to be essentially more difficult to handle.

The main result of this paper is the explicit formula:

\[
X(v) = \Psi_{-1} - \Psi_0 - v \tilde{\Psi}_1 + \sum_{i=1}^s \frac{\lambda_i (\lambda_i \Phi_i - v \Upsilon_i)}{\lambda_i^2 + v^2} .
\]

where \( \Psi_{-1}, \Psi_0, \tilde{\Psi}_1, \Phi_i \) and \( \Upsilon_i \) are \( m \times m \) matrices and \( \lambda_i \) are eigenvalues of the pencil \((\mathcal{A}_0, \mathcal{D})\) where \( \mathcal{A}_0 \) and \( \mathcal{D} \) are matrices which correspond to the linear operators

\[
\begin{align*}
X &\mapsto -\mathcal{A}_0 X + X \mathcal{A}_0 , \\
X &\mapsto \mathcal{D} X + X \mathcal{D} ,
\end{align*}
\]
respectively. Thus, technically, we have turned the viscosity into the spectral parameter.

Further, we have obtained a simple formula for the \( \Psi_{-1} \) in (1.14). This matrix is responsible for the behavior of the solution \( X(v) \) in the neighborhood of zero (small damping):

\[
\Psi_{-1} = \begin{bmatrix} D_{\Delta} & 0 \\ 0 & D_{\Delta} \end{bmatrix},
\]

where

\[
D_{\Delta} = (\text{diag}(D))^{-1}.
\]

We will use the following notation: matrices written in the simple Roman fonts, \( M, D \) or \( K \) for example will have \( n^2 \) entries. Matrices written in the mathematical bold fonts, \( A, B \) will have \( m^2 \) entries, where \( m = 2n \) (that is \( A, B \) are matrices defined on the \( 2n \)-dimensional phase space). Finally, matrices written in the Blackboard bold fonts \( \mathbb{A}, \mathbb{D} \) will have more than \( O(m^2) \) entries.

### 2 The main result

As we have said in the Introduction, our aim is to obtain the solution \( X \) of the Lyapunov equation

\[
A(v)^T X + XA(v) = -I,
\]

where \( Z \) is defined by (1.12) and \( I \) is \( m \times m \) identity matrix.

From (1.7) it follows that \( A(v) \) can be written as

\[
A(v) \equiv A_0 - vD,
\]

where \( A_0 = \begin{bmatrix} \Omega_1 \\ \Omega_2 \\ \vdots \\ \Omega_n \end{bmatrix}, \quad \Omega_i = \begin{bmatrix} 0 & \omega_i \\ -\omega_i & 0 \end{bmatrix} \]

and \( D = D_0 D_0^T \), where

\[
D_0 = \begin{bmatrix}
0 & 0 & \cdots & 0 \\
0 & 0 & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & 0 \\
d_{11} & d_{12} & \cdots & d_{1r} \\
d_{21} & d_{22} & \cdots & d_{2r} \\
\vdots & \vdots & \ddots & \vdots \\
\end{bmatrix},
\]

\( d_{ij} \) are entries of the matrix

\[
D_0 = P \begin{bmatrix} 0 \\ L_C \end{bmatrix}, \quad C_0 = L_C L_C^T,
\]
and \( P \) is the “perfect shuffling” permutation.

Now, we proceed with solving equation (2.1). As it is well known, Lyapunov equation (2.1) is equivalent to ([2, Theorem 12.3.1])

\[
(I \otimes (A_0 - vD)^T + (A_0 - vD)^T \otimes I) \cdot \text{vec}(X) = -\text{vec}(I),
\]

(2.3)

where \( L \otimes T \) denotes the Kronecker product of \( L \) and \( T \), and \( \text{vec}(I) \) is the vector formed by ”stacking” the columns of \( I \) into one long vector.

Further, we will need the following two \( m^2 \times m^2 \) matrices defined by

\[
A_0 = I \otimes A_0^T + A_0^T \otimes I, \quad D = I \otimes D_0D_0^T + D_0D_0^T \otimes I.
\]

(2.4)

It is easy to show that \( D = D_F D_F^T \), where

\[
D_F = \begin{bmatrix} I \otimes D_0 & D_0 \otimes I \end{bmatrix}.
\]

(2.5)

Now, using (2.5) and (2.4) it follows that solution \( \text{vec}(X) \) of equation (2.3) can be written as

\[
\text{vec}(X) = - (A_0 - vD_F D_F^T)^{-1} \text{vec}(I).
\]

(2.6)

Obviously, there exists a unitary matrix \( U \) such that

\[
U^T A_0 U = \begin{bmatrix} 0 \\ \hat{A}_0 \end{bmatrix},
\]

(2.7)

where \( \hat{A}_0 \) is the skew-symmetric matrix corresponds with linear operator defined in (2.4) and \( \hat{A}_0 \) is a non-singular block diagonal matrix defined by

\[
\hat{A}_0 = \text{diag}(\Xi_1, \ldots, \Xi_{m_2}) \quad \text{where} \quad \Xi_i = \begin{bmatrix} 0 & -\mu_i \\ \mu_i & 0 \end{bmatrix},
\]

(2.8)

where \( \pm \mu_i, i = 1, \ldots, m_2 \) are non-zero eigenvalues of matrix \( A_0 \), that is, \( 0 \neq \mu_i = (\pm \omega_i) - (\pm \omega_j) \) for \( i, j = 1, \ldots, n \) (see [2, Corollary 12.2.2]). Note that \( m_2 = (m^2 - m)/2 \). Set

\[
\begin{bmatrix} D_1 \\ D_2 \end{bmatrix} = U^T D_F,
\]

where \( D_F \) is defined in (2.5).

Now,

\[
A_0 - vD_F D_F^T = U \left( \begin{bmatrix} 0 \\ \hat{A}_0 \end{bmatrix} - v \begin{bmatrix} D_1D_1^T & D_1D_2^T \\ D_2D_1^T & D_2D_2^T \end{bmatrix} \right) U^T.
\]

(2.9)

Taking

\[
G = \begin{bmatrix} I \\ -D_2D_1^T (D_1D_1^T)^{-1} I \end{bmatrix}
\]

(2.10)

we obtain

\[
A_0 - vD_F D_F^T = U G^{-1} \begin{bmatrix} -vD_1D_1^T \\ \hat{A}_0 \end{bmatrix} G^{-T} U^T,
\]

(2.11)
where
\[ \hat{A} = \hat{A}_0 - v D_2 (I - D_1^T (D_1 D_1^T)^{-1} D_1) D_2^T. \] (2.12)

Note that we can write
\[ D_2 (I - D_1^T (D_1 D_1^T)^{-1} D_1) D_2^T = FF^T. \]

Further, we have to find the inverse of \( \tilde{A} = \hat{A}_0 - v FF^T \). This is obtained by using the Sherman-Morrison-Woodbury formula (\([1, (2.1.4), pg.51.])\), that is,
\[ (\hat{A}_0 - v FF^T)^{-1} = \hat{A}_0^{-1} + v \hat{A}_0^{-1} F (I - v FF^T \hat{A}_0^{-1} F)^{-1} F^T \hat{A}_0^{-1}. \] (2.13)

The inverse of \( I - v FF^T \hat{A}_0^{-1} F \) remains to be found. Let
\[ \Lambda = \text{diag} \begin{bmatrix} 0 & \lambda_1 \\ -\lambda_1 & 0 \\ \vdots & \vdots \\ 0 & \lambda_s \\ -\lambda_s & 0 \end{bmatrix}, \]
where \( \pm i\lambda_1, \pm i\lambda_2, \ldots, \pm i\lambda_s \) are non-vanishing finite eigenvalues of the problem
\[ (\hat{A}_0 - \lambda D) \text{vec}(Y) = 0. \]

Since \( F^T \hat{A}_0^{-1} F \) is skew-symmetric, then there exists an orthogonal matrix \( U_S \) of order \( 2(r - 1)m \) such that
\[ U_S^T F^T \hat{A}_0^{-1} F U_S = \begin{bmatrix} 0 & 0 \\ 0 & \Gamma \end{bmatrix}, \] (2.14)
where \( \Gamma = \Lambda^{-1}. \)

Using (2.11), (2.13), (2.14) and (2.6) it follows
\[ \text{vec}(X) = \begin{bmatrix} V^{-1} - \hat{A}_0^{-1} F U_S \\ \Gamma U_S^T \text{vec}(I) \end{bmatrix}, \] (2.15)
where
\[ \Delta_1 = \begin{bmatrix} 0 & 0 \\ -\lambda_1 & 0 \\ \vdots & \vdots \\ 0 & -\lambda_s \end{bmatrix}, \]
\[ \Delta_2 = \hat{A}_0^{-1} + v \hat{A}_0^{-1} F U_S \begin{bmatrix} 0 \\ (I - v \Gamma)^{-1} \end{bmatrix} U_S^T F^T \hat{A}_0^{-1}. \]

Since \( \Gamma \) is block diagonal we have
\[ (I - v \Gamma)^{-1} = \text{diag} \left( \frac{1}{\lambda_1^2 + v^2} \begin{bmatrix} \lambda_1^2 & -v \lambda_1 \\ v \lambda_1 & \lambda_1^2 \end{bmatrix}, \ldots, \frac{1}{\lambda_s^2 + v^2} \begin{bmatrix} \lambda_s^2 & -v \lambda_s \\ v \lambda_s & \lambda_s^2 \end{bmatrix} \right). \] (2.17)

Using (2.16) and (2.15) it follows
\[ \text{vec}(X) = \begin{bmatrix} \frac{V^{-1} - \hat{A}_0^{-1} F U_S}{v} - \hat{V} \end{bmatrix} \text{vec}(I) + \sum_{i=1}^s \frac{\lambda_i (\lambda_i \hat{W}_i - v \hat{Z}_i)}{\lambda_i^2 + v^2} \text{vec}(I), \] (2.18)
where matrices $V_{-1}$, $V_0$, $\hat{V}_1$, $W_i$ and $Z_i$ are constructed using (2.10), (2.16) and (2.15).

By ”reshaping” vectors in (2.18) back into $m \times m$ matrices we obtain the solution of equation (2.1)

$$
X = \frac{\Psi_{-1}}{v} - \Psi_0 - v\hat{\Psi}_1 + v \sum_{i=1}^{s} \frac{\lambda_i (\lambda_i \Phi_i - v Y_i)}{\lambda_i^2 + v^2}.
$$

(2.19)

If one is interested in deriving an optimal damping, then according to (1.11) one has to minimize the function

$$
tr(ZX(v)) = vec(Z)^T vec(X) = -vec(Z)^T (A_0 - vD_F W_F^T)^{-1} vec(I).
$$

This gives

$$
Tr(ZX) = \frac{X_{-1}}{v} - X_0 - v\hat{X}_1 + v \sum_{i=1}^{s} \frac{\lambda_i^2 X_i}{\lambda_i^2 + v^2},
$$

(2.20)

where

$$
X_{-1} = vec(Z)^T V_{-1} vec(I),
$$

$$
X_0 = vec(Z)^T V_0 vec(I),
$$

$$
\hat{X}_1 = vec(Z)^T \hat{V}_1 vec(I),
$$

and

$$
X_i = vec(Z)^T W_i vec(I), \quad Y_i = vec(Z)^T Z_i vec(I).
$$

(2.21)

(2.22)

Note that the function $Tr(ZX)$ from (2.20) is a generalization of the function $Tr(X)$ defined in (1.13).

**Remark 2.1** In the case when $Z$ is a diagonal matrix, the function $Tr(ZX)$ from (2.20) has the following simpler form:

$$
Tr(ZX) = \frac{X_{-1}}{v} - v\hat{X}_1 + v \sum_{i=1}^{s} \frac{\lambda_i^2 X_i}{\lambda_i^2 + v^2}.
$$

Finally we derive an explicit formula for the matrix $\Psi_{-1}$. Let $U_0$ be that part of $U$ corresponding to the null-space of $A_0$. From (2.18) it follows that

$$
vec(\Psi_{-1}) = V_{-1} vec(I) = -U_0 (D_1 D_1^T)^{-1} U_0^T vec(I).
$$

(2.23)

It is easily seen that $i$-th column of the matrix $U_0$ can be written as $vec(O_i)$, where $O_i$ is a block-diagonal matrix and

$$
O_{2i-1} = \text{diag}(0, \ldots, 0, O_{ii}, 0, \ldots, 0)
$$

$$
O_{2i} = \text{diag}(0, \ldots, 0, \hat{O}_{ii}, 0, \ldots, 0)
$$
where $O_{ii}$ and $\hat{O}_{ii}$ are "orthogonal" solutions of

$$-\Omega_i O_{ij} + O_{ij} \Omega_j = 0 \quad i, j = 1, \ldots, n,$$

that is

$$O_{ii} = \begin{bmatrix} \frac{1}{\sqrt{2}} & 0 \\ 0 & \frac{1}{\sqrt{2}} \end{bmatrix}, \quad \text{and} \quad \hat{O}_{ii} = \begin{bmatrix} 0 & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & 0 \end{bmatrix}.$$

Let $\langle A, B \rangle = Tr(A^T B)$ be the usual Frobenius scalar product. Then, we can write the $(p, q)$-th element of $U_0^T D_F D_F^T U_0$ as

$$(U_0^T (p, :)) D_F D_F^T (U_0)(q, :) = (D_{pq} O_p + O_p D_{pq}, O_q) \quad p, q = 1, \ldots, m,$$

where $D_{pq} = (D_{pq})$ and $D_{pq} = \begin{bmatrix} 0 & 0 \\ 0 & D_{(p, :) (q, :)} \end{bmatrix}$.

The orthonormality property

$$\langle O_p, O_q \rangle = \delta_{pq},$$

implies

$$(U_0^T D_F D_F^T U_0)^{-1} = \text{diag}(1/(D)_{22}, 1/(D)_{22}, \ldots, 1/(D)_{mm}, 1/(D)_{mm}).$$

Using the fact that $U_0 U_0^T \text{vec}(I) = \text{vec}(I)$, from (2.23) it follows that

$$\Psi_{-1} = \text{diag} \left( \frac{1}{D_{22}}, \frac{1}{D_{22}}, \ldots, \frac{1}{D_{mm}}, \frac{1}{D_{mm}} \right). \quad (2.24)$$

After applying a perfect shuffle permutation we have

$$\Psi_{-1} = \begin{bmatrix} D_{\Delta} & 0 \\ 0 & D_{\Delta} \end{bmatrix}, \quad (2.25)$$

where

$$D_{\Delta} = (\text{diag} \,(D))^{-1}.$$

The explicitness of the obtained formulas is attractive for possible numerical computation. Our first attempts to perform this task did not succeed due to unexpected complexity problems. We will come back to this issue in our future research.

References


