On estimation of uniform distribution if data are measured with additive error

Mirta Benšić, Kristian Sabo
On estimation of uniform distribution if data are measured with additive error

Mirta Benšić, Kristian Sabo
Department of Mathematics, University of Osijek, Gajev trg 6, HR-31 000 Osijek, Croatia

Abstract. The paper considers the problem of estimating the edges of the symmetric uniform distribution on the line when data are measured with normal additive error. The main purpose is to compare the efficiency of the maximum likelihood estimator and the moment method estimator. It is proven that the model is regular and that the maximum likelihood estimator is more efficient than the moment method estimator. There is also given a sufficient condition for existence of both estimators.

Key words. maximum likelihood estimator, method of moments estimator, measurement error, uniform distribution

1 Introduction

The problem of estimating the uniform distribution on the line is naturally formulated in terms of a parameter estimation problem. In this context we can concentrate on an estimation of the unknown parameter $a$ if the uniform distribution is on the line segment $[0, a]$ or if it is on the symmetric line segment $[-a, a]$. In the classical statistical theory the problem of estimating the parameter $a$ has been well studied and now we know several estimators for $a$ and their properties (see eg. [7], [8], [12], [13]). But, if the data are measured with additive error, the statistical model is drastically changed.

The model which is studied here is only the simplest model of the type

$$X = B + E$$

where $B$ is a bounded continuous random variable (or a vector), $E$ is a random variable (or a vector) that describes measurement (or some other) error, and it is of interest to determine a precise border of the density support for $B$. If we spread our consideration to two and three dimensional cases, such type of problem can be recognize in many real situations. A simulated data set from uniform distribution on elliptical domain plus normal errors (two-dimensional with independent margins) is shown in Figure 1 together with the real border of the elliptical domain.

Thus, if it is of interest to recognize the precise border of an object observed...
a) $D(p) = \{(t, f) \in \mathbb{R}^2 : (t - p)^2 + (f - q)^2 \leq r^2\}$  
b) $D(p) = \{(t, f) \in \mathbb{R}^2 : \frac{(t-p)^2}{\alpha^2} + \frac{(f-q)^2}{\beta^2} \leq 1\}$

Fig. 1. Data from two-dimensional uniform distribution on elliptical domain plus two-dimensional normally distributed error (independent from uniform and with i.i.d. margins) and the true border of the domain with a fluorescent microscope ([9]) or with a ground penetrating radar the problem can be set in the way presented with expression (1). The same type of model can be used in the problem of a protein secondary structure assignment ([2], [6]) or in $\alpha$-particle determination on solid state nuclear track detector based on pattern recognition, etc.

In [10] Schneeweiss also mentions the usefulness of one-dimensional models in a number of applications (i.e. not only as an example in the theory of statistics) in particular in the field of image analysis (see e.g. [11]). He discusses properties of maximum likelihood (ML) and moment method (MM) estimators if the bounded part of the model (1) is uniformly distributed on the line segment $[0, a]$ and the additive error is normal with zero mean and known variance $\sigma^2$. It has been noted in this paper that, although the computational procedure of the ML estimator in such a model can be very hard compared with the first order MM estimator (which is simply $\hat{a} = 2\bar{x}$), the ML estimator is more efficient.

For our purpose, we consider the uniform distribution on the symmetric line segment $[-a, a]$ rather than on the line segment $[0, a]$ and we add a normal error with zero mean and known variance. This model is a symmetric one and for it the first order MM estimator does not have sense as the expectation is always zero (does not depend on the parameter). Also, such a model can serve as a base for an estimation in asymmetric models of the same type but with the uniform distribution on the line segment $[a, b]$ if the expectation $(a + b)/2$ is supposed to be known. Indeed, centering data with a known expectation will result with data from a symmetric model which is going to be described in this paper.
The main purpose of the present paper is to compare the efficiency of the ML estimator and the second order MM estimator in the model of the described type, to give sufficient conditions for the existence of the ML estimator and to suggest a numerical procedure for its computation.

In Section 2 we briefly introduce the model and study the properties of the ML and the second order MM estimators. In Section 3 we comment on examples for which the ML estimator does not exist (if \( a > 0 \)) and we give sufficient conditions for the existence. As it happens very often that numerical problems arise while computing the ML estimator in this model, in Section 4 we give some useful suggestions for the numerical computation of the ML estimator.

Although the model we consider is one dimensional, it can be used for two or three dimensional problems if the bounded domain is rectangular and additive error with independent normal margins. Also, it is possible to start solving similar two-dimensional problems even if the domain is not rectangular using slicing and one-dimensional models in combination with curve fitting procedure. The last section contains some suggestions for further research in these directions.

### 2 The model and properties of the ML and MM estimators

Let \( U \) be a uniformly distributed random variable on the interval \([-a, a]\) for some \( a > 0 \) which is to be estimated but its realizations cannot be observed directly. In fact, we observe the variable

\[
X = U + N
\]

where \( N \) is normal with zero mean and variance \( \sigma^2 \), independent of \( U \).

The density function for such type of a random variable has the form

\[
f(x|a, \sigma) = \frac{1}{2a} \left( \Phi \left( \frac{a-x}{\sigma} \right) - \Phi \left( \frac{-a-x}{\sigma} \right) \right),
\]

(3)

where \( \Phi(x) \) is a standard normal distribution function, i.e.

\[
\Phi(x) = \int_{-\infty}^{x} \varphi(t) \, dt, \quad \varphi(t) = \frac{1}{\sqrt{2\pi}} e^{-\frac{t^2}{2}}.
\]

The graph of this density is shown in Figure 1 for values \( a = 2 \) and \( \sigma = 0.02 \).
It can be easily shown that the limiting density for \( \sigma \) tending to zero is uniform on \([-a, a]\) and for \( a \) tending to zero it is normal \( \mathcal{N}(0, \sigma^2) \).

Here we suppose that the parameter \( \sigma \) is known and we treat the problem as a one-parameter estimation problem.

### 2.1 Method of moments estimator

The first order MM estimator cannot be applied here as the expectation of \( X \) is zero. But, as we supposed that \( \sigma^2 \) is known a second order MM estimator is feasible. The model distribution second moment has the form:

\[
\mu_2 = \frac{a^2}{3} + \sigma^2. \tag{4}
\]

Thus, the second order MM estimator can be calculated using (4) and a second sample moment \( \hat{\mu}_2 = \frac{1}{n} \sum_{i=1}^{n} x_i^2 \) as:

\[
\hat{a}_{mm} = \sqrt{3(\hat{\mu}_2 - \sigma^2)}. \tag{5}
\]

The estimator is asymptotically normal i.e. the sequence of the second order MM estimators \( \hat{a}_{mm}(n) \) satisfy:

\[
\sqrt{n}(\hat{a}_{mm}(n) - \tilde{a}) \rightarrow \mathcal{N}(0, V),
\]
\[ V = \frac{\hat{a}^2}{15} + \sigma^2 + \frac{3\sigma^4}{2\hat{a}^2} . \]

(\(\hat{a}\) is the true value of the parameter \(a\).) The asymptotic variance of \(\hat{a}_{mm}\) is then given by

\[ V_{mm} = \frac{1}{n} \left( \frac{\hat{a}^2}{15} + \sigma^2 + \frac{3\sigma^4}{2\hat{a}^2} \right). \]  

(6)

2.2 Maximum likelihood estimator

For ML estimation the likelihood function has to be considered for a random sample \((X_1, X_2, \ldots, X_n)\) from the supposed distribution. Let us denote a realization of the sample by \(x = (x_1, \ldots, x_n)\). Thus, the likelihood function of the random sample model has the form

\[ L(a) := L(x; a) = \frac{1}{(2\alpha)^n} \prod_{i=1}^{n} \int_{-\frac{a-x_i}{\alpha}}^{\frac{a-x_i}{\alpha}} \varphi(x) \, dx \]

\[ = \frac{1}{(2\sigma)^n} \prod_{i=1}^{n} \int_{-1}^{1} \varphi \left( \frac{a x - x_i}{\sigma} \right) \, dx, \]  

and the log-likelihood function for the random sample model as well as its first derivative are as follows:

\[ l(a) := \log L(a) = -n \log(2\alpha) + \sum_{i=1}^{n} \log \left( \Phi \left( \frac{a - x_i}{\sigma} \right) - \Phi \left( \frac{-a - x_i}{\sigma} \right) \right) , \]  

\[ l'(a) = -\frac{n}{a} + \frac{1}{\sigma} \sum_{i=1}^{n} \varphi \left( \frac{a-x_i}{\sigma} \right) + \varphi \left( \frac{-a-x_i}{\sigma} \right). \]

To prove the regularity of this model ([7], Theorem 3.10, pg. 449) let us mention that the parameter space is an open interval; the model density support \(A = \{x : f(x|a) > 0\}\) is the whole \(\mathbb{R}\) and it does not depend on \(a\); the model density has derivatives of all orders regarding \(a\) and the integral \(\int f(x|a) \, dx\) can be thrice differentiated under the integral sign. Also, the second and the third derivatives of the model log-likelihood function have the following forms:

\[ \frac{\partial^2 \log f(x|a)}{\partial a^2} = \frac{1}{a^2} + \frac{1}{\sigma^2} \frac{\varphi' \left( \frac{a-x}{\sigma} \right) - \varphi' \left( \frac{-a-x}{\sigma} \right)}{\Phi \left( \frac{a-x}{\sigma} \right) - \Phi \left( \frac{-a-x}{\sigma} \right)} - \frac{1}{\sigma^2} \left[ \frac{\varphi \left( \frac{a-x}{\sigma} \right) + \varphi \left( \frac{-a-x}{\sigma} \right)}{\Phi \left( \frac{a-x}{\sigma} \right) - \Phi \left( \frac{-a-x}{\sigma} \right)} \right]^2 \]
\[
\frac{\partial^3 \log(f(x|a))}{\partial a^3} = -\frac{2}{a^3} + \frac{1}{\sigma^3} \frac{\varphi''(\frac{a-x}{\sigma}) + \varphi''(\frac{-a-x}{\sigma})}{\Phi(\frac{a-x}{\sigma}) - \Phi(\frac{-a-x}{\sigma})} \\
- \frac{3}{\sigma^3} \frac{\varphi'(\frac{a-x}{\sigma}) - \varphi'(\frac{-a-x}{\sigma})(\varphi(\frac{a-x}{\sigma}) + \varphi(\frac{-a-x}{\sigma}))}{(\Phi(\frac{a-x}{\sigma}) - \Phi(\frac{-a-x}{\sigma}))^2} \\
+ \frac{2}{\sigma^3} \left[ \frac{\varphi(\frac{a-x}{\sigma}) + \varphi(\frac{-a-x}{\sigma})}{\Phi(\frac{a-x}{\sigma}) - \Phi(\frac{-a-x}{\sigma})} \right]^3.
\]

It is clear from the above expressions that, to prove the last demand for regularity, one must investigate the behavior of the quantities

\[
\frac{\varphi(\frac{a-x}{\sigma}) + \varphi(\frac{-a-x}{\sigma})}{\Phi(\frac{a-x}{\sigma}) - \Phi(\frac{-a-x}{\sigma})}, \quad \frac{\varphi'(\frac{a-x}{\sigma}) - \varphi'(\frac{-a-x}{\sigma})}{\Phi(\frac{a-x}{\sigma}) - \Phi(\frac{-a-x}{\sigma})}, \quad \frac{\varphi''(\frac{a-x}{\sigma}) + \varphi''(\frac{-a-x}{\sigma})}{\Phi(\frac{a-x}{\sigma}) - \Phi(\frac{-a-x}{\sigma})}
\]

in plus and minus infinity, as these expressions are defined on the whole \(\mathbb{R}\).

It can be shown (for instance, using the L’Hospital’s rule) that

\[
\frac{\varphi(\frac{a-x}{\sigma}) + \varphi(\frac{-a-x}{\sigma})}{\Phi(\frac{a-x}{\sigma}) - \Phi(\frac{-a-x}{\sigma})} \sim \begin{cases} \frac{x-a}{\sigma}, & x \to +\infty \\ \frac{-x-a}{\sigma}, & x \to -\infty \end{cases}
\]

\[
\frac{\varphi'(\frac{a-x}{\sigma}) - \varphi'(\frac{-a-x}{\sigma})}{\Phi(\frac{a-x}{\sigma}) - \Phi(\frac{-a-x}{\sigma})} \sim \begin{cases} (\frac{x-a}{\sigma})^2, & x \to +\infty \\ -(\frac{x+a}{\sigma})^2, & x \to -\infty \end{cases}
\]

\[
\frac{\varphi''(\frac{a-x}{\sigma}) + \varphi''(\frac{-a-x}{\sigma})}{\Phi(\frac{a-x}{\sigma}) - \Phi(\frac{-a-x}{\sigma})} \sim \begin{cases} \frac{(x-a)^3}{\sigma}, & x \to +\infty \\ -(\frac{x+a}{\sigma})^3, & x \to -\infty \end{cases}
\]

Here we have used the symbol ”\(\sim\)” in the following context: \(f_1(x) \sim f_2(x)\) for \(x \to \infty \Leftrightarrow \lim_{x \to \infty} \frac{f_1(x)}{f_2(x)} = 1\). It is obvious that \(f_1(x) \sim f_2(x)\) for \(x \to \infty\) has the consequence that we can find a constant \(K\) and bound \(|f_1(x)|\) with \(K|f_2(x)|\) for \(|x|\) big enough. As all functions from the right-hand side of the above expressions as well as their powers, are integrable regarding the model measure for all \(a > 0\), if we restrict our attention to \(a \in (\bar{a} - c, \bar{a} + c)\) (\(\bar{a}\) is the true value of the parameter \(a\) and \(\bar{a} > c > 0\)) we can find the uniform (in \(a\)) upper bound function for the second as well as for the third derivative which is integrable regarding the model measure. This will assure regularity conditions.
Thus, any consistent sequence \( \hat{a}_n = \hat{a}_n(X_1, \ldots, X_n) \) of roots of the likelihood equation satisfy

\[
\sqrt{n}(\hat{a}_n - \tilde{a}) \to \mathcal{N}(0, \frac{1}{I(\tilde{a})}).
\]

Here

\[
I(a) = \frac{1}{a^2} + \frac{2}{\sigma^2} \int_0^\infty \left( \frac{\varphi\left(\frac{a-x}{\sigma}\right) + \varphi\left(-\frac{a-x}{\sigma}\right)}{\Phi\left(\frac{a-x}{\sigma}\right) - \Phi\left(-\frac{a-x}{\sigma}\right)} \right) dx
\]

and the asymptotic variance of the ML estimator \( \hat{a}_{ml} \) has the form

\[
V_{ml} = \frac{1}{n} \left[ -\frac{1}{\tilde{a}^2} + \frac{2}{\sigma^2} \int_0^\infty \left( \frac{\varphi\left(\frac{\tilde{a}-x}{\sigma}\right) + \varphi\left(-\frac{\tilde{a}-x}{\sigma}\right)}{\Phi\left(\frac{\tilde{a}-x}{\sigma}\right) - \Phi\left(-\frac{\tilde{a}-x}{\sigma}\right)} \right) dx \right]^{-1}
\]

### 3 Existence theorem

It can be easily seen using the form of the MM estimator (5) that the MM estimator \( \hat{a}_{mm} \) exists if the data set satisfy the following condition:

\[
\sigma < \sqrt{\hat{\mu}_2} = \sqrt{\frac{1}{n} \sum_{i=1}^n x_i^2}.
\]

The answer to the question if the ML estimator \( \hat{a}_{ml} \) also exists in this case is not so obvious. But, as the ML estimator has to be computed using a numerical algorithm it is of great importance to know in advance if the solution exists or not and what is a good choice for an initial approximation. It is known from the theory of ML estimation that the set of data for which the solution exists is not small, i.e. it has a probability tending to one, as \( n \to \infty \), and the likelihood equation has a root that tends to the true value of the parameter in probability. But, practically, this does not exclude the possibility of collecting data for which the ML estimate does not exist. For instance, if we have only two data \( x_1 = 0.5 \) and \( x_2 = 0.3 \), the likelihood functions with \( \sigma_1 = 0.01 \), \( \sigma_2 = 0.06 \) are shown in Figure 2(i) and with \( \sigma_2 = 1 \) in Figure 2(ii). It is obvious from the graphs that the functions in Figure 2(i) have an extreme point at the interval \((0, \infty)\), but the function in Figure 2(ii) does not have it, which can also be easily confirmed using classical techniques of real function analysis. For our model, the following theorem proves that if the MM estimator
exists, the ML estimator exists too.

**Theorem 1** Let \( x_i, i = 1, \ldots, n \) be the data, i.e. real numbers, and \( \sigma > 0 \) such that

\[
\sigma < \sqrt{\frac{1}{n} \sum_{i=1}^{n} x_i^2}. \tag{13}
\]

Then there exists \( a^* \in (0, \infty) \) satisfying

\[
L(a^*) = \sup_{a \in (0, \infty)} L(a),
\]

where \( L : (0, \infty) \to (0, \infty) \) is defined with (7).

**PROOF.** In the proof we follow some ideas from the papers [4], [5].

To prove the assertion let us first note that \( L(a) \) is a bounded function on \((0, \infty)\). Namely, it is obvious that the function is bounded from below, and the fact

\[
L(a) = \frac{1}{(2\sigma)^n} \prod_{i=1}^{n} \int_{-1}^{1} \varphi \left( \frac{a x - x_i}{\sigma} \right) \, dx < \left( \frac{1}{\sqrt{2\pi\sigma}} \right)^n.
\]

assures that the function is bounded from above.
Since $L(a)$ is bounded, there exists $L^* := \sup_{a \in (0, \infty)} L(a)$. Obviously $L^* > 0$. Let $(a_k)$ be a sequence in $(0, \infty)$ such that

$$L^* = \lim_{k \to \infty} L(a_k).$$

We first show that the sequence $(a_k)$ is bounded. We prove this by contradiction. Suppose it is not, i.e. $a_k$ is a subsequence that goes to infinity. Then, taking the limit $k \to \infty$, from

$$L(a_k) < \frac{1}{(2a_k)^n} \prod_{i=1}^{\infty} \int_{-\infty}^{\infty} \varphi(x) \, dx = \left( \frac{1}{2a_k} \right)^n,$$

we obtain $L^* = \lim_{k \to \infty} L(a_k) \leq 0$. This contradicts the fact that $L^* > 0$.

We proved that $(a_k)$ is bounded, so we may assume it is also convergent (otherwise we take a convergent subsequence). Let $a_k \to a^*$. Note that $a^* \geq 0$, because of $a_k > 0$, $k \in \mathbb{N}$.

Now we are going to show that $a^* > 0$. If $a^* = 0$, then

$$L^* = \lim_{k \to \infty} L(a_k) = \lim_{k \to \infty} \frac{1}{(2\sigma)^n} \prod_{i=1}^{1} \int_{-1}^{1} \varphi \left( \frac{a_k x - x_i}{\sigma} \right) \, dx$$

$$= \left( \frac{1}{\sqrt{2\pi}\sigma} \right)^n \exp \left( - \sum_{i=1}^{n} \frac{x_i^2}{2\sigma^2} \right) =: L_1.$$

Let us find a point in $(0, \infty)$, at which the function $L$ attains a value greater than $L_1$, thus showing that $a^* > 0$. For this purpose let us define a continuous function $g : [0, \infty) \to (0, \infty)$ by the formula

$$g(a) := \begin{cases} \frac{1}{(2\sigma)^n} \prod_{i=1}^{1} \int_{-1}^{1} \varphi \left( \frac{ax - x_i}{\sigma} \right) \, dx, & a \neq 0 \\ \left( \frac{1}{\sqrt{2\pi}\sigma} \right)^n \exp \left( - \sum_{i=1}^{n} \frac{x_i^2}{2\sigma^2} \right), & a = 0. \end{cases}$$

Note that $g(a) = L(a)$ for $a > 0$, and $g(0) = L_1$. 

9
The derivative of $g$ equals:

$$g'(a) = -\frac{1}{2\sigma^2}g(a) \sum_{i=1}^{n} \frac{1}{\int_{-1}^{1} \exp \left(-\frac{(a x - x_i)^2}{(2\sigma^2)}\right) \, dx} \int_{-1}^{1} (a x - x_i) x \exp \left(-\frac{(a x - x_i)^2}{(2\sigma^2)}\right) \, dx.$$

By using the L’Hospital’s rule we obtain

$$\lim_{a \to 0^+} \frac{1}{a} g'(a) = \lim_{a \to 0^+} -\frac{1}{2\sigma^2}g(a) \sum_{i=1}^{n} \frac{1}{\int_{-1}^{1} \exp \left(-\frac{(a x - x_i)^2}{(2\sigma^2)}\right) \, dx} \int_{-1}^{1} \frac{(a x - x_i)x}{a} \exp \left(-\frac{(a x - x_i)^2}{(2\sigma^2)}\right) \, dx$$

$$= \frac{1}{3\sigma^{n+4}} \left( \sum_{i=1}^{n} x_i^2 - n\sigma^2 \right) \exp \left(-\frac{\sum_{i=1}^{n} x_i^2}{(2\sigma^2)} \right),$$

wherefrom using inequality (12) we obtain $\lim_{a \to 0^+} \frac{1}{a} g'(a) > 0$. This means that the derivative of the function $g$ is strictly positive whenever $a$ is small enough. Therefore there is a real number $\delta$ such that the function $g$ is strictly increasing on $[0, \delta)$. Hence for every $a \in (0, \delta)$ we have

$$L_1 = g(0) < g(a) = L(a).$$

Thus, we proved that $a^* > 0$. By the continuity of the function $L$ we have

$$\sup_{a \in (0, \infty)} L(a) = \lim_{k \to \infty} L(a_k) = L(a^*).$$

This completes the proof of the theorem. \( \square \)

It is important to notice that this sufficient condition is natural and that it is not very strong. Indeed, the statistic

$$S^2 = \frac{1}{n} \sum_{i=1}^{n} X_i^2$$

is an unbiased and consistent estimator for the second moment of the underlying distribution. As the expectation is supposed to be zero, it is also the
estimator for the variance. But the model variance has the form:

$$Var(X) = \frac{a^2}{3} + \sigma^2$$

and it is greater than $\sigma^2$. So, if we collect data from our model distribution, they will meet our condition with the probability tending to one as we increase the dimension of the sample. In fact, given $a$ and $\sigma$ the probability that the second sample moment is less than $\sigma^2$ in a sample of size $n$ can be obtained from the non-central $\chi^2$ distribution. Indeed,

$$E\left\{I\left\{\sum_{i=1}^{n} X_i^2 \geq n\sigma^2\right\} | U_1 = u_1, \ldots, U_n = u_n \right\} = P\{\chi^2(n, \sum_{i=1}^{n} u_i^2 / \sigma^2) \geq n\},$$

and

$$P\left\{\sum_{i=1}^{n} X_i^2 \geq n\sigma^2\right\} = E_{(u_1, \ldots, u_n)}[E\{I\left\{\sum_{i=1}^{n} X_i^2 \geq n\sigma^2\right\} | U_1, \ldots, U_n \}].$$

4 Numerical experiments

To illistrate the behaviour of the asymptotic variance of the ML estimator and the asymptotic variance of the MM estimator we show the graphs of $nV_{mm}(\sigma)$ (dashed line) and $nV_{ml}(\sigma)$ (solid line) in Figure 4 for $a = 1$.

![Graph](image)

Fig. 4. $nV_{mm}(\sigma)$ and $nV_{ml}(\sigma)$ for $a = 1$

For calculation of the integral in (11) we used the Matlab subroutine quad. The function quad implements a low order method using an adaptive recursive Simpson’s rule (see [3]).
Also, it can be useful for application, to give some guidelines for numerical computation of the ML estimator in the described model, i.e. the maximum point of the function $L(a)$ defined in (7) which is the same as the maximum point of its logarithm (8).

Namely, as it was mentioned in [10] for a similar model, that the Newton algorithm for finding zeros of the score function (which was suggested in the same paper) could sometimes fail. We found out that in our model this algorithm fails very often. The reason lies in the fact that the roots of the first and the second derivatives of the function $l$ could be close to each other, as it can be seen in Figure 3.

![Graph of $l'$ and $l''$ for $x_1 = -0.335676$, $x_2 = 1.07986$ and $\sigma = 0.2$.](image)

Fig. 5. Graphs of $l'$ and $l''$ for $x_1 = -0.335676$, $x_2 = 1.07986$ and $\sigma = 0.2$.

There are a lot of numerical examples with data simulated from the described model where this happens. A solution for avoiding these problems can be to modify the Newton algorithm (for instance to include a check that the negative log likelihood decreases at each iteration) or to use some other method for one dimensional maximization. As it was confirmed in our simulations, Brent’s method, which is based on the golden section search and parabolic interpolation (see [1]), gives excellent results. For starting value $a_0$ we can choose the ML estimator of the parameter $a$ from the model without additive error ($\sigma = 0$), i.e.

$$a_0 := \max\{x_i : i = 1, \ldots, n\}$$

or the MM estimator which can be calculated easily using (5).

![Graph of $l$ for $x_1 = -0.335676$, $x_2 = 1.07986$ and starting value $(a_0, l(a_0))$, where $a_0 = x_2$.](image)

Fig. 6. Graph of $l$ for $x_1 = -0.335676$, $x_2 = 1.07986$ and starting value $(a_0, l(a_0))$, where $a_0 = x_2$. 
Through simulations and numerical computations we used options of the Matlab package. For the simulations of the function $\Phi$ we used Matlab Statistics Toolbox function \texttt{cdf} and for maximization Matlab function \texttt{fminbnd} which is based on Brent’s method.

It can be interesting to mention that, using these procedures, we successfully calculated the estimator in every simulation which fulfills the sufficient condition (12).

In a simulation study the value of the parameter $a$ was kept fixed at $a = 1$, and $\sigma$ varied in the set $\{0.01, 0.05, 0.1, 0.5\}$. A relatively small ($n = 50$), medium ($n = 100$), relatively large ($n = 500$) and large ($n = 1000$) sample size were chosen. The number of replications was $N = 1000$.

The results of the Monte Carlo study are shown in the following four tables ($V_{mn}$ and $V_{ml}$ are the estimators' variance calculated with equations (6) and (11) respectively):

### Table 1. Results of 1000 experiments and $\sigma = 0.01$.

<table>
<thead>
<tr>
<th>$n$</th>
<th>$V_{mn}$</th>
<th>$V_{ml}$</th>
<th>$\hat{a}_{mn}$</th>
<th>$\hat{a}_{ml}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>50</td>
<td>0.0013353</td>
<td>0.0000400</td>
<td>0.9987</td>
<td>0.9881</td>
</tr>
<tr>
<td>100</td>
<td>0.0006677</td>
<td>0.0000200</td>
<td>0.9977</td>
<td>0.9958</td>
</tr>
<tr>
<td>500</td>
<td>0.0001335</td>
<td>0.0000040</td>
<td>1.0004</td>
<td>0.9993</td>
</tr>
<tr>
<td>1000</td>
<td>0.0000668</td>
<td>0.0000020</td>
<td>1.0001</td>
<td>0.9998</td>
</tr>
</tbody>
</table>

### Table 2. Results of 1000 experiments and $\sigma = 0.05$.

<table>
<thead>
<tr>
<th>$n$</th>
<th>$V_{mn}$</th>
<th>$V_{ml}$</th>
<th>$\hat{a}_{mn}$</th>
<th>$\hat{a}_{ml}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>50</td>
<td>0.0013835</td>
<td>0.0001920</td>
<td>0.9997</td>
<td>0.9940</td>
</tr>
<tr>
<td>100</td>
<td>0.0006918</td>
<td>0.0000960</td>
<td>0.9979</td>
<td>0.9970</td>
</tr>
<tr>
<td>500</td>
<td>0.0001384</td>
<td>0.0000192</td>
<td>0.9992</td>
<td>0.9991</td>
</tr>
<tr>
<td>1000</td>
<td>0.0000692</td>
<td>0.0000096</td>
<td>1.0001</td>
<td>0.9997</td>
</tr>
</tbody>
</table>

### Table 3. Results of 1000 experiments and $\sigma = 0.1$.
Table 4. Results of 1000 experiments and $\sigma = 0.5$.

<table>
<thead>
<tr>
<th>$n$</th>
<th>$V_{mm}$</th>
<th>$V_{ml}$</th>
<th>$\tilde{a}_{mm}$</th>
<th>$\tilde{a}_{ml}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>50</td>
<td>0.0015363</td>
<td>0.0003880</td>
<td>0.9951</td>
<td>0.9913</td>
</tr>
<tr>
<td>100</td>
<td>0.0007682</td>
<td>0.0001940</td>
<td>0.9980</td>
<td>0.9967</td>
</tr>
<tr>
<td>500</td>
<td>0.0001536</td>
<td>0.0000388</td>
<td>1.0007</td>
<td>0.9990</td>
</tr>
<tr>
<td>1000</td>
<td>0.0000768</td>
<td>0.0000194</td>
<td>1.0002</td>
<td>0.9996</td>
</tr>
</tbody>
</table>

Although, the variance of ML estimator is less than the variance of MM estimator, it can be seen from the tables that there are some values for $n$ and $\sigma$ for which $V_{mm}$ and $V_{ml}$ of the same order of magnitude (e.g. $\sigma = 0.5$). This fact is also confirmed in variability of ML and MM estimators in Monte Carlo study as it can be seen in Figure 4.

Fig. 7. Box plot of estimated values $\tilde{a}_{mm}$ and $\tilde{a}_{ml}$ for $n = 500$ and $\sigma = 0.01$

5 Conclusion

Once we have data for which we know that they should be concentrated in the bounded region but they are measured with an additive error, the estimation of this region’s edges becomes a problem. Here we consider the simplest
symmetric one dimensional case, i.e. data which are collected from the independent identically distributed random variable which is a sum of the uniform on \([-a, a]\) and normal with zero mean. The reason why we choose the symmetric uniform distribution is the fact that it can serve as a base for estimation if the considered uniform is on any \([\alpha, \beta]\) and the expectation is supposed to be known. The estimation problem has been stated in a parametric form with the unknown parameter \(a\). We have shown model’s regularity and gave the sufficient condition for the existence of the MM and ML estimators for the unknown parameter \(a\).

To consider such a problem in the context of applications it is of primary interest to generalize it to two and three dimensions. It is obvious that the one dimensional model can be directly used in two or three-dimensional problems if the bounded domain of interest is rectangular. Also, if two or three-dimensional domain can be expressed in a parametric way (for instance, circular or elliptical) it is possible to start analyzing a new model in the parametric form. But, if the domain does not have a prescribed shape in the parametric form then we can start estimating procedure, for instance, using a one-dimensional model in the following way:

1. Slice the data set into the strips;
2. Apply the one-dimensional model to the strips and pick up the estimates for points at the boundary;
3. Apply one of the known curve-fitting (or surface-fitting) procedures to estimate the border.

It is confirmed in our simulations that such an algorithm can give good results in combination with the least squares curve-fitting procedure, especially if it is obvious from the data how to choose slicing direction, but this approach has to be studied in details in further research.
References


