Estimation of the optimal damping for mechanical vibrating systems

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ESTIMATION OF THE OPTIMAL DAMPING FOR MECHANICAL VIBRATING SYSTEMS

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ABSTRACT

This paper is concerned with the efficient algorithm for dampers’ and viscosity optimization in mechanical systems. Our algorithm optimize simultaneously the dampers’ positions and their viscosities. For the criterion for optimization we use minimization of the average total energy of the system which can be done by the minimization of the trace of the solution of the corresponding Lyapunov equation. Efficiency of the algorithm is obtained by new heuristics for finding the optimal dampers’ positions and for the approximation of the trace of the solution of the Lyapunov equation.

Keywords: Linear vibrating system, Lyapunov equation, damped-vibration, multi-variable optimization of dampers’ viscosities, passive damping

1 INTRODUCTION

We consider damped linear vibrational system described by the differential equation

$$M \ddot{x} + D \dot{x} + K x = 0,$$

$$x(0) = x_0, \quad \dot{x}(0) = \dot{x}_0,$$

where $M, D, K$ (mass, damping, stiffness matrix, respectively) are real, symmetric matrices of order $n$ with $M, K$ positive definite and $D = C_u + C$, where $C_u$ is a positive definite and presents the internal damping which is usually taken as 2–10 % of the critical damping (see pp. 26, 260 (Paz 1991)), and $C$ is positive semidefinite.

Very important question arises in considerations of such systems: for given mass and stiffness determine the available dampers’ positions and viscosities as to insure an optimal evanescence.

This problem has been considered by many authors, for example in (Milman and Chu 1994) the question of placement of damping elements was investigated, while in (Kasturi and Dupont 1998) the problem of a periodic optimal control, which maximize the energy dissipation, was considered.

Recently, the optimization problem which optimize only viscosities was considered in the following papers: (Cox et al. 2004), (Truhar and Veselić 2004), (Truhar 2004), (Truhar and Veselić 2007), (Truhar and Veselić) and (Veselić et al. 2001).
We are going to use optimization criterion which was considered in (Veselić 1990), (Veselić et al. 2001), (Truhar 2004), (Cox et al. 2004), (Truhar and Veselić), which is given by requirement of the minimization of the total energy of the system, that is

\[ \int_{0}^{\infty} E(t) \, dt \rightarrow \min. \]  

(1)

Since, criterion (1) depends on the initial condition, the simplest way to correct this is to take mean value over all initial data with the unit energy. As it was described in (Truhar 2004), (Truhar and Veselić 2004), (Veselić et al. 2001) and (Cox et al. 2004), the minimization of the average total energy is equivalent to minimization of the trace of the solution of corresponding Lyapunov equation.

Since an efficient general algorithm for the optimization of damping does not exist, that is, available algorithms optimize only viscosities of dampers, not their positions, we propose a simple and efficient approach. With this approach one can find optimal positions and corresponding dampers’ viscosities efficiently with satisfactory accuracy.

For estimation of the optimal viscosity for given dampers’ positions we propose a new algorithm which is based on the simple “reduction” (truncation) of the corresponding Lyapunov equation, which usually produces acceleration of more than 100 times. For the optimization of dampers’ positions we propose a new heuristic.

Both algorithms are based on certain heuristic and unfortunately we do not have their accuracy bounds, but, as it will be illustrated in the last section in examples with the Lyapunov equation of modest dimensions (\( n \leq 200 \)), they perform very good. Thus we assume that the obtained results will be even better for bigger dimensions.

Currently, two types of algorithms are in use for the estimation of the optimal viscosity (for a given dampers’ positions). The first type are the Newton-type algorithms for higher-dimensional (constrained or unconstrained) problems which use some Lyapunov solvers, and the second type are the algorithms which explicitly calculate the trace of the solution of the corresponding Lyapunov equation.

Algorithms of the second type were presented in (Veselić 1990), (Truhar 2004) or (Truhar and Veselić) and they consider the case with one or more dampers with the same viscosity.

On the other hand, in (Brabender 1998) the Newton-type algorithm for the case with \( r \geq 1 \) different dampers was proposed. As it was shown in paper (Truhar and Veselić), the algorithm proposed in (Brabender 1998) can produce a poor result due to the problems with determining the starting point.

Our method for viscosity optimization uses the minimizer of a constrained nonlinear multivariable function \texttt{fmincon}, which is incorporated in Matlab, while for the solving of the reduced Lyapunov equation we used function \texttt{lrllyapchol}, which is a block variant of the Hammarling’s method proposed in (Kressner 2006). This method was chosen because it can use the small rank structure of the right-hand side of the Lyapunov equation.

The paper is organized as follows: Section 2 contains problem definition, after which we presented the procedure for the approximation of the solution of the Lyapunov equation. Section 4 includes heuristic algorithm for determining the optimal damper’s position, while the last section describes numerical example. We will use the following notation: matrices written in
the simple Roman fonts, \( M, D \) or \( K \) for example, will have \( \mathcal{O}(n^2) \) entries. Matrices written in the mathematical bold fonts, \( \mathbf{A}, \mathbf{B} \), will have \( \mathcal{O}(m^2) \) entries, where \( m = 2n \). The symbol \( \| \cdot \| \) stands for the standard 2-norm, while \( \| \cdot \|_F \) denotes Frobenius norm and \( \sigma(A) \) denotes the spectrum of the matrix \( A \).

## 2 PROBLEM DEFINITION

Let

\[
M \ddot{x} + D \dot{x} + K x = 0
\]  

(2)

be the differential equation describing a damped linear vibrational system, where \( M, D, K \) are mass, damping and stiffness matrix, respectively. An example is the so-called \( n \)-mass oscillator or oscillator ladder (Figure 1) where

\[
M = \text{diag}(m_1, m_2, \ldots, m_n),
\]

\[
K = \begin{bmatrix}
    k_0 + k_1 & -k_1 & & \\
    -k_1 & k_1 + k_2 & -k_2 & \\
    & \ddots & \ddots & \ddots \\
    & & -k_{n-2} & k_{n-2} + k_{n-1} & -k_{n-1} \\
    & & & -k_{n-1} & k_{n-1} + k_n
\end{bmatrix},
\]

\[
D \equiv C_u + C = C_u + v_1 e_1 e_1^T + v_3 e_3 e_3^T.
\]

Here \( m_i > 0 \) are the masses, \( k_i > 0 \) the spring constants or stiffnesses and \( e_i \) is the \( i \)-th canonical basis vector, and \( v_i \) is the viscosity of the \( i \)-th damper applied on the \( i \)-th mass. Note that the rank of the matrix \( C \) is two. In this paper we study the system with \( r \) dampers where we assume that \( r \ll n \) (usually \( r = 2, 3, 4 \)).

To (2) corresponds the eigenvalue problem

\[
(\lambda^2 M + \lambda D + K) x = 0.
\]  

(3)

Obviously all eigenvalues of (3) lie in the left complex plane.

Using the eigenvalue decomposition

\[
\Phi^T K \Phi = \Omega^2, \quad \Phi^T M \Phi = I,
\]

where

\[
\Omega = \text{diag}(\omega_1, \ldots, \omega_n), \quad \omega_1 < \ldots < \omega_n,
\]  

(4)

Figure 1: The \( n \)-mass oscillator with two dampers
and setting
\[ y_1 = \Omega \Phi^{-1} x, \quad y_2 = \Phi^{-1} \dot{x}, \]
equation (2) can be written as
\[ \dot{y} = Ay, \]
\[ y = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}, \quad A = \begin{bmatrix} 0 & \Omega \\ -\Omega & -\Phi^T D \Phi \end{bmatrix}, \]
(we are now in the \(2n\)-dimensional phase space), with the solution
\[ y = e^{At} y_0, \quad \text{where } y_0 \text{ is the initial data}. \tag{5} \]

It can be shown that the criterion of the minimization of the average total energy over the set of the initial conditions is equivalent to
\[ Tr(ZX) = \min, \]
where \(X\) is solution of the following Lyapunov equation
\[ A^T X + X A = -I. \]
and \(Z\) is the symmetric positive semidefinite matrix which may be normalized to have a unit trace. If we take for the measure for example the measure generated by Lebesgue measure on \(\mathbb{R}^{2n}\), we obtain \(Z = \frac{1}{2n} I\).

Further it is easy to show that
\[ Tr(ZX) = Tr(Y), \]
where \(Y\) is a solution of the so-called ”dual Lyapunov equation”
\[ AY + YA^T = -Z. \tag{6} \]

The structure of the matrix \(Z\) was studied in detail in (Nakić 2002) and some of the results are presented in (Truhar 2004).

Through this paper we will assume that the matrix \(Z\) has the following form
\[ Z = \begin{bmatrix} 0_{t_1} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & I_s & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0_{t_2} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0_{t_1} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & I_s & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0_{t_2} \end{bmatrix}, \tag{7} \]
where \(I_s\) is the \(s\)-dimensional identity matrix, and \(O_{t_i}\) is the \(t_i\)-dimensional \((i = 1, 2)\) zero matrix, where \(t_1\) and \(s\) are defined such that the eigenfrequencies from (4) smaller than \(\omega_{t_1}\) and greater than \(\omega_{t_1+s}\) are not dangerous (observe that \(t_2 = n - t_1 - s\)).

Note that the solution of Lyapunov equation (6) is a function of several variables, dampers positions and corresponding viscosities. Thus, the simultaneous optimization of dampers’ positions and viscosity can be computationally very demanding. Furthermore we will propose a new approach for dampers optimization.

In the next section we will present a new algorithm which approximate the solution (as well as its trace) of the corresponding Lyapunov equation.
3 APPROXIMATION OF THE SOLUTION OF THE LYAPUNOV EQUATION

As we mentioned in the previous section, we consider the following Lyapunov equation

\[ AX + XA^T = -Z. \]  

(8)

After the perfect shuffle permutation, \( A \) has the following form:

\[ A \equiv A_0 - D_0 V D_0^T, \]  

(9)

where

\[ A_0 = \Omega_1 \oplus \ldots \oplus \Omega_n, \quad \Omega_i = \begin{bmatrix} 0 & \omega_i \\ -\omega_i & 0 \end{bmatrix}, \quad V = \text{diag}(v_{i_1}, \ldots, v_{i_r}), \]

and \( D_0 \in \mathbb{R}^{m \times r} \), with \( \text{rank}(D_0) = r \ll m \), \( \text{rank}(Z) = 2s \ll m \) and \( v_{i_k} \in \mathbb{R}_+^* \), for \( k = 1, \ldots, r \). For more details about the structure of the equation (8) see (Truhar 2004), (Truhar and Veselić 2004), (Veselić et al. 2001), (Cox et al. 2004).

During the process of optimization (regardless which optimization routine is used) one has to calculate the trace of Lyapunov equation (8) for chosen viscosities \( v_{i_1}, \ldots, v_{i_r} \).

For simpler following of the main ideas, without losing the generality, we will assume that the matrix \( Z \) from (8) has the following form:

\[ Z = \begin{bmatrix} I_{2s} & 0 \\ 0 & 0 \end{bmatrix}, \]

where \( I_{2s} \) is identity matrix of dimension \( 2s \). This corresponds to the calming down of the first \( s \) eigenfrequencies \( \omega_1, \ldots, \omega_s \).

Let

\[
\begin{bmatrix}
A_{11} & A_{12} \\
A_{21} & A_{22}
\end{bmatrix}
\begin{bmatrix}
X_{11} & X_{12} \\
X_{T21} & X_{T22}
\end{bmatrix}
+ \begin{bmatrix}
X_{11} & X_{12} \\
X_{T12} & X_{T22}
\end{bmatrix}
\begin{bmatrix}
A_{11}^T & A_{21}^T \\
A_{12}^T & A_{22}^T
\end{bmatrix}
= -\begin{bmatrix}
Z_{11} & 0 \\
0 & 0
\end{bmatrix},
\]

be the block version of Lyapunov equation (8), where \( A_{11}, X_{11} \) and \( Z_{11} \) are \( l \times l \) matrices \( (l > 2s) \), and all other block matrices have corresponding dimensions. We assume that \( \sigma(A_{11}) \cap \sigma(A_{22}) = \emptyset \). Note that

\[ Z_{11} = \begin{bmatrix} I_{2s} & 0 \\ 0 & 0_{l-2s} \end{bmatrix}. \]  

(10)

The new approach is based on the following simple idea: the solution of the Lyapunov equation (8) will be approximated with the solution

\[ X_0 = \begin{bmatrix} \tilde{X}_{11} & 0 \\ 0 & 0 \end{bmatrix}, \]  

(11)

where \( \tilde{X}_{11} \) is \( l \times l \) solution of the following Lyapunov equation

\[ A_{11} \tilde{X}_{11} + \tilde{X}_{11} A_{11}^T = -Z_{11}, \]  

(12)

and \( Z_{11} \) is given by (10).
Note that the above means that we have the following approximation for the trace:

\[ Tr(X) \approx Tr(\tilde{X}_{11}) . \]  \hfill (13)

It is important to note that \( X_0 \) from (11) is the exact solution of the following Lyapunov equation:

\[ A_0 \tilde{X}_0 + \tilde{X}A_0^T = -Z , \]  \hfill (14)

where

\[ A_0 = \begin{bmatrix} A_{11} & 0 \\ 0 & A_{22} \end{bmatrix} \]  \hfill (15)

That is, if \( V = 0 \) (i.e. \( v_i = 0 \) for \( i = 1, \ldots, r \)), then \( X_0 \) is the exact solution of Lyapunov equation (8).

This further means that Lyapunov equation (8) can be written as perturbed equation

\[ (A_0 + \Delta A)(X_0 + \Delta X) + (X_0 + \Delta X)(A_0 + \Delta A)^T = -Z , \]  \hfill (16)

where

\[ X_0 = \begin{bmatrix} \tilde{X}_{11} & 0 \\ 0 & 0 \end{bmatrix}, \quad \Delta X = \begin{bmatrix} (\Delta X)_{11} & (\Delta X)_{12} \\ (\Delta X)_{12}^T & (\Delta X)_{22} \end{bmatrix} , \]

\[ A_0 = \begin{bmatrix} A_{11} & 0 \\ 0 & A_{22} \end{bmatrix}, \quad \Delta A = \begin{bmatrix} 0 & A_{12} \\ A_{21} & 0 \end{bmatrix} . \]

The question arises: \textit{What can we say about the accuracy of our approximation?}

As we have already mentioned in the Introduction, the answer is \textit{not too much}!

The simplest way to check the quality of our approximation is to calculate the relative error for the trace approximation:

\[ re_X = \frac{\left| Tr(X) - Tr(X_{11}) \right|}{Tr(X)} \]  \hfill (17)

If error (17) is small enough we can assume that the approximation for trace (13) can be useful. In various different applications it has been noted that the solution of the Lyapunov equation have the small ”numerical rank”, or equivalently, the eigenvalues are strongly decaying towards zero. There are some results which consider this phenomenon under certain structural assumptions on matrix \( A \) for example see (Antoulas, Sorensen, and Zhou 2002), (Penzl 2000). Thus, for such a system it can be assumed that the error (17) will be small enough.

If one is interested in approximation of the solution of Lyapunov equation (8), with the right-hand side defined as in (7) then we propose algorithm similar to one presented in (Truhar and Veselić 2007) for deriving a suboptimal set of ADI-parameters.

Thus, let \( Z \) be obtained from (7) by the perfect shuffle permutation with rank \( 2s \) and \( s \ll n \). The following procedure has to be applied for the construction of matrices \( A_{11}, \tilde{X}_{11} \) and \( Z_{11} \).  

### Procedure for the approximate Lyapunov equation
(1) Find the indices of 1’s on the right hand side (i.e. find positions of ones in the matrix \( Z \)).

(2) Find the corresponding submatrix of \( A \) using these indices.

(3) Take a “little bit bigger block” (which depends on a particular problem) which includes the submatrix chosen in the above step. This block will be \( A_{11} \).

(4) Corresponding block in the matrix \( Z \) will be \( Z_{11} \).

The method for forming the matrix \( A_{11} \) can be seen from (Figure 2).

After the above procedure is finished, we can simply apply the algorithm described in the first part of this section, that is:

let \( l > 2s \) be the chosen dimension of the matrix \( A_{11} \). We are going to approximate the solution of Lyapunov equation (8) with solution

\[
X_0 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & \tilde{X}_{11} & 0 \\ 0 & 0 & 0 \end{bmatrix},
\]

which gives a trace approximation:

\[
Tr(X) \approx Tr(\tilde{X}_{11}),
\]

where

\[
A_{11}\tilde{X}_{11} + \tilde{X}_{11}A_{11}^T = -Z_{11}.
\]

Recall, in the first step of our optimization we have to find optimal viscosities \( v_1, \ldots, v_r \) defined in (9), for a given dampers’ positions, such that corresponding trace of the solution of the Lyapunov equation reaches its minimal value. Thus, the optimization will be carried out in the
suitable chosen region, that is we will assume that \( v_i \in [0, \xi] \), where choice of the upper bound \( \xi > 0 \) depends on a particular problem.

As one can see from (14)–(15) the quality of the obtained approximation of the solution of Lyapunov equation (11) depends on magnitudes of viscosities \( v_1, \ldots, v_r \). Thus, the quality of this approximation can be checked using error (17) for a couple of different sets \( v_1, \ldots, v_r \). For example if \( r = 2 \), (which is the number of dampers in our examples in the last section) and if we know the upper bound \( \xi \) for viscosity of each damper, that is, if \( 0 < v_i < \xi \) holds, it is enough to check the error (17) for the following viscosities:

\[
\{v_1^*, v_2^*\} \in \left\{ \{0.1, 0.1\}, \{0.1, \frac{\xi}{2}\}, \{\frac{\xi}{2}, 0.1\}, \{\frac{\xi}{2}, \frac{\xi}{2}\} \right\}.
\]

For some applications one can assume feasible quantities for optimal viscosities. In such cases if for any of the chosen pairs \( \{v_1^*, v_2^*\} \) we obtain the approximation which is not good enough, (is not feasible) then we have to use the bigger block in the step (3) of our procedure (to increase dimension \( l \)).

4 THE OPTIMIZATION OF DAMPERS’ POSITIONS

As we mentioned in Introduction, the problem of dampers’ optimization is still an open problem. The existing algorithms optimize the dampers’ viscosities for the given positions. The reason for this lies in the fact that the calculation of the optimal position is usually computationally very demanding.

Thus, in this section we present a new heuristic for the calculating of the optimal dampers’ positions.

Although we consider the mechanical system with \( r \) different dampers, described in Section 2, for the sake of simplicity we will assume that \( r = 2 \). (The system with 2 dampers will be considered in the next section too).

Thus, we have to find the best position \((i_0, j_0)\) for two dampers where, \( i_0 \) and \( j_0 \) are defined with the following loop:

for \( i = 1 : n - 1 \)
for \( j = i + 1 : n \)
\[
[i_0, j_0] = [i, j];
\]
end
end

\[\text{(18)}\]

The best position means that we have to find two positions \( i_0 \) and \( j_0 \) such that the corresponding optimal traces of the solution of Lyapunov equation (8) achieve for this position their minimal value. That is, for each position we have to calculate the optimal trace (viscosity optimization) of the solution of the Lyapunov equation, and the best position \( i_0 \) and \( j_0 \) will be the one with the minimal value of the optimal trace. This means that in the case when \( r = 2 \) we will have \( n \cdot (n - 1)/2 \) optimizations.

For one optimization one has to solve around 50 corresponding Lyapunov equations, thus if we use the approximation of the solution of the Lyapunov equation proposed in the previous
section, the whole optimization will take \(50\mathcal{O}(n^2)\mathcal{O}(l^3)\) operation. This can be still very demanding, especially if \(n\) is big.

Thus, for the reduction of the number of viscosity optimizations we propose the following heuristic procedure:

First we divide the set of indices \(\{1, 2, \ldots, n\}\) into \(n_0\) “appropriate” subsets. The number of subsets \(n_0\) depends on the dimension of the system \(n\). For example if \(n = 100\), one can set \(n_0 = 10\), which gives the following subsets \(\{1, 2, \ldots, 10\}, \{11, 12, \ldots, 20\}, \ldots, \{91, 92, \ldots, 100\}\). The same configuration will be considered in the next section. Let \(k_0\) be the number of elements in each subset, that is \(k_0 = n/n_0\).

Instead of considering all positions, as in (18), we will take into consideration the following two sets of positions:

\[
\begin{align*}
(i) & \quad \text{for } i = 1 : k_0 : n - 1 \\
& \quad \text{for } j = i + 1 : k_0 : n \\
& \quad [i_0, j_0] = [i, j]; \\
& \quad \text{end}
\end{align*}
\]

\[
\begin{align*}
(ii) & \quad \text{for } i = 1 : k_0 : n - 1 \\
& \quad \text{for } j = i + 1 : k_0 : n \\
& \quad [i_0, j_0] = [i + k_0/2, j]; \\
& \quad \text{end}
\end{align*}
\]

For each of the positions defined above we will calculate the optimal viscosity using the minimizer of a constrained nonlinear multivariable function \(f_{\text{mincon}}\) which is incorporated in Matlab, while for the calculation of the solution of the corresponding Lyapunov equations we will use the approximation described in the last section.

Once we obtained all optimal traces (for which we will need \(\mathcal{O}(n_0^2)\) optimizations), we find that the quasi optimal position is one which belongs to the minimal trace.

Let \((i_{\text{opt}}, j_{\text{opt}})\) be the quasi optimal position. The optimal position for dampers will be obtained by viscosity optimization over the positions

\[
\begin{align*}
i_0 & \in \{i_{\text{opt}} - k_0, i_{\text{opt}} - k_0 + 1, \ldots, i_{\text{opt}} + k_0 - 1, i_{\text{opt}} + k_0\} \\
j_0 & \in \{j_{\text{opt}} - k_0, j_{\text{opt}} - k_0 + 1, \ldots, j_{\text{opt}} + k_0 - 1, j_{\text{opt}} + k_0\}.
\end{align*}
\]

One can also use the minimization process based on Newton iteration process for higher-dimensional problems which use Bartels-Stewart Lyapunov solver (Bartels and Stewart 1972) with the starting point obtained from the previous approximation of quasi optimal position.

The whole procedure costs \(\mathcal{O}(n_0^2)\mathcal{O}(1) + \mathcal{O}(k_0^2)\mathcal{O}(1)\), which can be significant improvement if \(n_0, k_0 \ll m\).

5 NUMERICAL EXAMPLES

In this section we will illustrate our new method on the mechanical system, so called \(n\)-mass oscillator, described in the Section 2, where system (2) has dimension \(n = 200\), and mass and
stiffness matrices are defined as:

\[ M = \text{diag}(1, 41, 81, 121, \ldots, (n - 1) \cdot 40 + 1), \]

\[
K = \begin{bmatrix}
  2 & -1 \\
  -1 & 2 & -1 \\
  \ddots & \ddots & \ddots \\
  -1 & 2 & -1 \\
  -1 & 2
\end{bmatrix}.
\]

This means that all springs on (Figure 1) have the same stiffness which is equal to 1, and that corresponding masses are given as \( m_1 = 1, m_2 = 41, m_3 = 81, \ldots, m_n = (n - 1) \cdot 40 + 1. \)

We consider the following optimization problem:

Find the optimal positions and optimal viscosities for the considered \( n \)-mass oscillator with 2 different dampers to insure the best calming down of the first 10 undamped frequencies \( \omega_1 < \ldots < \omega_{10}. \)

This means that we are going to perform the following procedure:

for each dampers’ position \((i, j)\) we have the following linear vibrational system:

\[ M \ddot{x} + D \dot{x} + K x = 0, \quad D \equiv C_u + C = C_u + v_i e_i e_i^T + v_j e_j e_j^T. \]

For the system above we form the corresponding Lyapunov equation

\[
A(i, j, v_i, v_j)X(i, j, v_i, v_j) + X(i, j, v_i, v_j)(A(i, j, v_i, v_j))^T = -Z, \tag{19}
\]

where

\[
A(i, j, v_i, v_j) = \begin{bmatrix}
  0 & \Omega \\
  -\Omega & -\Phi^T D \Phi
\end{bmatrix}, \quad \Phi^T K \Phi = \Omega^2, \quad \Phi^T M \Phi = I, \quad C_u = 0.02 \Omega,
\]

and

\[
Z = \begin{bmatrix}
  I_{10} & 0 & 0 & 0 \\
  0 & 0 & 0 & 0 \\
  0 & 0 & I_{10} & 0 \\
  0 & 0 & 0 & 0
\end{bmatrix}.
\]

The optimal damping (optimal positions with corresponding optimal viscosities) has the form:

\[ D_{opt} = C_u + C = C_u + v_i \text{opt} e_i \text{opt} e_i^T + v_j \text{opt} e_j \text{opt} e_j^T, \]

where \( \{i_{opt}, j_{opt}\} \) denotes the optimal dampers’ positions, and \( v_i \text{opt}, v_j \text{opt} \) denote corresponding optimal viscosities.

In other words, let \( \mathcal{P} \) be the set of all possible dampers positions defined by (18), the best damping \((i_{opt}, j_{opt}, v_i \text{opt}, v_j \text{opt})\) is defined as

\[ Tr(X_{opt}) = \min_{(i, j) \in \mathcal{P}} \min_{(v_i, v_j)} Tr(X(i, j, v_i, v_j)). \]
where $X_{i,j,v_i,v_j}$ is the solution of Lyapunov equation (19).

Note that the dimension of the phase space is $m = 400$, which means that the standard Lyapunov solvers, such as the Bartels-Stewart algorithm, has to work with matrices of dimension 400.

The new algorithm was perform on the PC with Pentium IV processor, with 1.4 GHz and 768 MB RAM. As we mentioned in the last section, first we looked the “quasi optimal position” from the positions:

\[
\begin{align*}
(i) & \quad \text{for } i = 1 : k_0 : n - 1 \quad (ii) & \quad \text{for } i = 1 : k_0 : n - 1 \\
& \quad \text{for } j = i + 1 : k_0 : n \quad & \quad \text{for } j = i + 1 : k_0 : n \\
& \quad [i_0, j_0] = [i, j]; \quad & \quad [i_0, j_0] = [i + k_0/2, j]; \\
& \quad \text{end} \quad & \quad \text{end} \\
& \quad \text{end} \quad & \quad \text{end}
\end{align*}
\]

where $k_0 = 10$.

This calculation gives the “quasi optimal position” $\{31, 162\}$. After that we look for the “optimal position” in the set of positions given by:

\[
i_0 \in \{21, 22, \ldots, 41, 42\} \text{ and } j_0 \in \{152, 153, \ldots, 172, 173\}.
\]

For the calculation of the trace for each position we use approximation $\tilde{X}_{i11}$, given as the solution of approximate Lyapunov equation (12) described in the Section 3. For the dimension of approximate Lyapunov equation we have taken $l = 60$.

Recall that the accuracy of our approximations can be examined using the error function (17). For example, for positions $i_0 = 41, j_0 = 192$, and viscosities $v_{i_0} = 55.0911$ and $v_{j_0} = 162.4839$, relative error (17) is

\[
re_{X} \equiv \frac{|Tr(X) - Tr(\tilde{X}_{i11})|}{Tr(X)} = 0.015,
\]

which means that our approximation, for that configuration, insures two exact digits in the trace.

Using the new approach, the optimal positions and corresponding optimal viscosities, obtained by the previously mentioned algorithm are:

optimal positions are : $i_{\text{opt}} = 36$ and $j_{\text{opt}} = 163$

optimal viscosities are : $v_{i_{\text{opt}}} = 59.866$ and $v_{j_{\text{opt}}} = 161.615$.

In the same time the “exact” optimal positions and corresponding viscosities were calculated on the DELL PowerEdge 1800, with Intel Xeon 3.2 GHz and 2 GB RAM. For the viscosity optimization we again use Matlab function $\text{fmincon}$, while for the Lyapunov solver (for calculating the solution of Lyapunov equation (19)) we used function $\text{lrlyapchol}$. We performed optimization over all possible positions (18) and we obtained the following result:

optimal positions are : $i_{\text{opt}} = 36$ and $j_{\text{opt}} = 163$

optimal viscosities are : $v_{i_{\text{opt}}} = 59.857$ and $v_{j_{\text{opt}}} = 161.712$.

It is obvious that the new method for this particular system finds the proper position and very good approximation for optimal viscosities.
Here it is important to emphasize that the whole optimization process with our new algorithm on PC computer did last less then 40 minutes, while the whole optimization process performed on the DELL PowerEdge 1800 did last more then 25 days. The time ratio is more than 900 times in our benefit.

Thus, we can conclude that the new algorithm which was proposed in this paper, can be used for calculating the optimal viscosities in mechanical systems presented in (Figure 1). We also assume that the performance will be more efficient and more accurate for the systems with a bigger dimension (for example for \( n > 1000 \)). This assumption has to be examined using more powerful computers.

6 CONCLUSION

The efficient algorithm for dampers’ and viscosity optimization in mechanical systems was presented. Our algorithm optimize simultaneously the dampers’ positions and their viscosities. For a criterion for optimization we used minimization of the trace of the solution of the corresponding Lyapunov equation. It is known that the problem of positions’ optimization is very hard discrete optimization problem, which has many points close, in numerical sense, to optimal one. In numerical sense this means that one have to deal with many local optima. Efficiency of the algorithm is obtained by new heuristics for finding the optimal dampers’ positions and for the approximation of the trace of the solution of the Lyapunov equation.

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REFERENCES


