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Optimal Damping of Selected Eigenfrequencies Using Dimension Reduction

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Abstract

We consider a mathematical model of a linear vibrational system described by the second-order differential equation $M\ddot{x} + D\dot{x} + Kx = 0$, where M and K are positive definite matrices, representing mass and stiffness, respectively. The damping matrix D is positive semidefinite.

We are interested in finding an optimal damping matrix which will damp a certain (critical) part of the undamped eigenfrequencies. For this we use an optimization criterion based on minimization of the average total energy of the system. This is equivalent to the minimization of the trace of the solution of the corresponding Lyapunov equation $AX + XA^T = -GG^T$, where A is the matrix obtained from linearizing the second-order differential equation and G depends on the critical part of the eigenfrequencies to be damped.

The main result is the efficient approximation and corresponding error bound for the trace of the solution of the Lyapunov equation obtained by dimension reduction, which includes the influence of the right-hand side GG^T and allows us to control the accuracy of the trace approximation. This trace approximation yields a much accelerated optimization algorithm for determining the optimal damping.

Keywords: Vibrating system, Lyapunov equation, energy minimization, dimension reduction, error bound, partial spectra
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1 Introduction

The main aim of this paper is the determination of optimal damping for the following linear vibrational system:

$$M\ddot{x} + D\dot{x} + Kx = 0, \tag{1}$$

where M and K (called mass and stiffness, respectively) are real, symmetric positive definite matrices of order n. The damping matrix is defined as $D = C_u + C_{ext}$, where the external viscous damping $C_{ext} = v_1C_1 + v_2C_2 + \cdots + v_kC_k$ is a semidefinite matrix with viscosities v_i and C_i encoding the corresponding positions of the dampers. The internal damping C_u is usually taken to be a small multiple of the critical damping, that is, $C_u = \alpha C_{crit}$, where one possibility for critical damping C_{crit} is (see, e.g., [22])

$$C_{crit} = 2M^{1/2}\sqrt{M^{-1/2}KM^{-1/2}}M^{1/2}. \tag{2}$$

Equation (1) can be transformed to phase space which yields a system of first order linear differential equations. For that purpose let Φ be a matrix that simultaneously diagonalizes M and K, i.e.,

$$\Phi^T K\Phi = \Omega^2 = \mathrm{diag}(\omega_1^2, \ldots, \omega_n^2) \quad \text{and} \quad \Phi^T M\Phi = I. \tag{3}$$

For the internal damping defined in (2) it holds that $\Phi^T C_u \Phi = \alpha\Omega$.

Then we can write the differential equation (1) in phase space as

$$\frac{d}{dt}\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} 0 & \Omega \\ -\Omega & -\Phi^T D\Phi \end{bmatrix}\begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$$

$$\text{or} \qquad \dot{y} = Ay,$$

where

$$A = \begin{bmatrix} 0 & \Omega \\ -\Omega & -\Phi^T D\Phi \end{bmatrix}, \quad y = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}, \tag{4}$$

for more details see [8, 16, 21, 22]. This system of first order differential equations has the solution $y = e^{At}y_0$, where y_0 contains the initial data. It has been shown in [16, 18] that A from (4) is a stable matrix, that is, the eigenvalues of A are in the left half of the complex plane.

As already mentioned above, our aim is to determine the "best" damping matrix D which will insure optimal evanescence of each component of y. This optimization problem has been intensively considered in the last decade in [7, 8, 16, 19, 20, 21, 22, 23]. All the mentioned papers use the criterion of minimization of the total energy of the system, that is

$$\int_0^\infty E(t)dt \quad \to \quad \min, \tag{5}$$

where $E(t)$ is the total energy of the system at a given time t, as a sum of kinetic and potential energy.

In [16] it is shown that by taking the average over all initial states of the unit total energy and a given frequency range, the minimization criterion (5) is equivalent to

$$\operatorname{trace} X \quad \rightarrow \quad \min, \tag{6}$$

where X is the solution of the Lyapunov equation

$$AX + XA^T = -GG^T, \tag{7}$$

with A as in (4). The matrix G depends on the eigenfrequencies which have to be damped.

If we are interested in damping of just the first s eigenfrequencies of the undamped system, the matrix G has the following form

$$G = \begin{bmatrix} I_s & 0 \\ 0 & 0 \\ 0 & I_s \\ 0 & 0 \end{bmatrix}. \tag{8}$$

More about the structure of G can be found in [16, 19].

The main idea of this paper is similar to the approach used in [7], where the optimization problem (6) has been considered for $G = I$ in (7), which is equivalent to damping the whole undamped spectrum.

If $s \ll n$, where $2s$ is the rank of G from (8), we can use the structure of the system more efficiently. Hence, the goal of this paper is to derive a new error bound for the trace of the solution of the Lyapunov equation (7) and to construct a corresponding efficient numerical algorithm for the trace approximation. The trace approximation can then be applied for determination of the optimal viscosity.

As in [7] our approach is based on dimension reduction of second-order systems. Dimension reduction was also considered in several other papers such as [3, 6, 10, 13, 14]. A review of several methods for dimension reduction can be found in [1, 2, 5].

Throughout the paper we will use the following notation. The symbol $\|\cdot\|$ denotes the standard 2-norm, while $\|\cdot\|_F$ stands for the Frobenius matrix norm. If p and q are integer vectors, the notation $A(p, q)$ is taken from MATLAB® and denotes the submatrix of A obtained by intersection of rows determined by the elements of p and columns determined by the elements of q. Similarly, $i : j$ is MATLAB notation and denotes the vector of integers from i to j. I_s stands for the s-dimensional identity matrix.

The paper is organized as follows. An algorithm for the approximation of the trace of the solution of the Lyapunov equation (7) is presented in Section 2. The error bound for the trace approximation of the solution of the Lyapunov equation

(7) is given in Section 3. In Section 4 we discuss an algorithm for the approximation of the optimal viscosities. A dimension reduction approach can also be applied for determination of the area that contains the optimal damping positions, which is presented in Section 5. A comparison of the new algorithm with the current standard algorithm for viscosity optimization is given in a numerical example in Section 6, where we have also presented an example which illustrates the efficiency of the algorithm from Section 5 which determines the area that contains the optimal dampers positions. Conclusions are given in Section 7.

2 Trace approximation for the solution of the structured Lyapunov equation

In this section we will present an algorithm which calculates an approximation of the trace of the solution of the structured Lyapunov equation (7). For that purpose we apply the perfect shuffle permutation matrix P, which yields the Lyapunov equation

$$A_P X_P + X_P A_P^T = -P^T G G^T P, \tag{9}$$

with $A_P = P^T A P$ and $X_P = P^T X P$, where A is given in (4) and G is given in (8). The given damper positions and corresponding viscosities are included in the matrix $C = \Phi^T C_{ext} \Phi$, where Φ is given in (3).

Our approach is based on dimension reduction of the Lyapunov equation (9). For that purpose we construct an approximation of the Lyapunov equation (9) as follows:

$$\widetilde{A}_P \widetilde{X} + \widetilde{X} \widetilde{A}_P^T = -G_P G_P^T, \tag{10}$$

where $G_P = P^T G$ and A_P is approximated by

$$\widetilde{A}_P = \begin{bmatrix} \widetilde{A}_{11} & 0 \\ 0 & \widetilde{A}_{22} \end{bmatrix}.$$

Now we will describe the construction of the matrix $\widetilde{A}_P$, i.e. of $\widetilde{A}_{11}$, $\widetilde{A}_{22}$ in more details. For this, we will need the additional permutation matrix $\widehat{P}$ which will bring up the dominant part of the damping matrix D to the upper block diagonal part.

Let the vectors $p \in \mathbb{N}^r$ and $\overline{p} \in \mathbb{N}^{n-r}$ be chosen such that the following conditions hold:

i) $p \cup \overline{p} = \{1, 2, \ldots, n\}$.

ii) p is the vector of indices of dimension $s + \widehat{s}$, where the first s correspond to the eigenfrequencies which have to be damped, and the $\widehat{s}$ closest indices.

iii) $\overline{p}$ and p are index vectors such that $\max_{ij} |C(\overline{p}(i), p(j))| \leq tol$ for a given tolerance tol.

The vectors $p \in \mathbb{N}^r$ and $\overline{p} \in \mathbb{N}^{n-r}$ should be chosen such that r is as small as possible for given parameters s, $\widehat{s}$ and tol. A strategy for determining $p, \overline{p}$ will be discussed below.

Now we define a vector $w \in \mathbb{N}^n$ by $w(i) = p(i)$ for $i = 1, \ldots, r$ and $w(i) = \overline{p}(i-r)$ for $i = r+1, \ldots, n$. Instead of the Lyapunov equation (7), we then solve the permuted Lyapunov equation

$$\widehat{P}^T P^T A P \widehat{P} \widehat{X} + \widehat{X} \widehat{P}^T P^T A^T P \widehat{P} = -\widehat{G}\widehat{G}^T, \tag{11}$$

where $\widehat{G} = \widehat{P}^T P^T G$, the matrix P is the perfect shuffle permutation matrix and $\widehat{P} = I(:, w) \otimes I_2$. Note that for these permutations it holds

$$\widehat{A}_P = \begin{bmatrix} 0 & \omega_{w(1)} & \cdots & 0 & 0 \\ -\omega_{w(1)} & -\alpha\omega_{w(1)} - c_{w(1)w(1)} & \cdots & 0 & -c_{w(1)w(n)} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & \omega_{w(n)} \\ 0 & -c_{w(1)w(n)} & \cdots & -\omega_{w(n)} & -\alpha\omega_{w(n)} - c_{w(n)w(n)} \end{bmatrix}, \tag{12}$$

where $\widehat{A}_P = \widehat{P}^T P^T A P \widehat{P}$. Since $\widehat{P}$ and P are permutation matrices, it holds that $\operatorname{trace} \widehat{X} = \operatorname{trace} X$, where X is the solution of (7).

We are interested in dimension reduction which will allow us to solve the approximated Lyapunov equation of smaller dimension instead of solving the Lyapunov equation (11). The approximated Lyapunov equation will have the following form:

$$\widetilde{A}_P \widetilde{X} + \widetilde{X} \widetilde{A}_P^T = -\widehat{G}\widehat{G}^T,$$

where

$$\widetilde{A}_P = \begin{bmatrix} \widetilde{A}_{11} & 0 \\ 0 & \widetilde{A}_{22} \end{bmatrix}$$

with

$$\widetilde{A}_{11} = \widehat{A}_P(1:2r, 1:2r) \quad \text{and} \quad \widetilde{A}_{22} = \widehat{A}_P(2r+1:2n, 2r+1:2n)$$

for $\widehat{A}_P$ as given in (12). Because of the block structure of $\widetilde{A}_P$ and $\widehat{G} = \begin{bmatrix} I_{2s} & 0 \end{bmatrix}^T$, the approximation of the solution of the Lyapunov equation (11) is given by

$$\widetilde{X} = \begin{bmatrix} \widetilde{X}_{11} & 0 \\ 0 & 0 \end{bmatrix},$$

where $\widetilde{X}_{11}$ is the solution of the Lyapunov equation

$$\widetilde{A}_{11}\widetilde{X}_{11} + \widetilde{X}_{11}\widetilde{A}_{11}^T = -\widetilde{G}\widetilde{G}^T \tag{13}$$

with $\widetilde{G} = \widehat{G}(1:2r,:)$.

Since $\widetilde{X}$ is the approximation of the solution X of the Lyapunov equation (11) we can use it for the trace approximation. This approximation will be obtained by solving a Lyapunov equation of dimension $2r \times 2r$, thus we will call the parameter r the reduced dimension.

Note that the more elements of C are small by magnitude, the smaller the reduced dimension will be. The elements of the matrix C are obtained from the corresponding rows of the matrix Φ from (3), that is, the structure of the matrix Φ is related to the magnitude of the elements of the matrix C (for more details, see e.g. [7, 12]).

Now we will present Algorithm 1 for the construction of the vectors p and $\overline{p}$.

Algorithm 1 (construction of p and $\overline{p}$)

Input: tol;

v_i, C_i, $i = 1, \ldots, k$ – viscosity and position matrix for the ith damper;

$u_1, u_2, \ldots, u_s$, – s indices of those eigenfrequencies which have to be damped;

$\widehat{u}_1, \widehat{u}_2, \ldots, \widehat{u}_{\widehat{s}}$, – $\widehat{s}$ indices of the eigenfrequencies closest to the eigenfrequencies which have to be damped.

Output: p, $\overline{p}$

1: $p = [\, u_1, u_2, \ldots, u_s, \widehat{u}_1, \widehat{u}_2, \ldots, \widehat{u}_{\widehat{s}}\,]$
2: Determine vector $\overline{p}$ such that $p \cup \overline{p} = \{1, 2, \ldots, n\}$.
3: T=1
4: $C = \Phi^T(v_1C_1 + v_2C_2 + \cdots + v_kC_k)\Phi$
5: **while** T=1 **do**
6: $\quad \widehat{C} = C(p, \overline{p})$ and $M = \max_{ij} |\widehat{C}_{i,j}|$.
7: $\quad$ **if** $M > tol$ **then**
8: $\quad\quad$ Determine indices i_0, j_0 such that $M = |C_{i_0,j_0}|$ (ensuring that j_0 is not used before and i_0 is index which is element of p).
9: $\quad\quad$ $p = [\, p,\, j_0\,]$.
10: $\quad\quad$ Determine the vector $\overline{p}$ such that $p \cup \overline{p} = \{1, 2, \ldots, n\}$.
11: $\quad$ **else**
12: $\quad\quad$ $T = 0$
13: $\quad$ **end if**
14: **end while**

We like to emphasize that in Algorithm 1, the indices $\widehat{u}_1, \widehat{u}_2, \ldots, \widehat{u}_{\widehat{s}}$ are included in vector p in order to obtain a better approximation of the eigenfrequencies which

are closest to the eigenfrequencies to be damped. Numerical experiments suggest that $\widehat{s}$ should be around $1-4\%$ of the dimension n. Besides the indices determining the eigenfrequencies to be damped and their closest neighbors, the algorithm chooses the indices to be included in the approximation (10) of the Lyapunov equation (9) by ensuring that all elements of the matrix C to be omitted are smaller than the chosen tolerance. Note that we use the term $\max_{i,j} |C(p(i), \overline{p}(j))| < tol$ since our error bound will be given in the absolute terms. On the other hand, one can use the relative terms, e.g. $|C(p(i), \overline{p}(j))| < tol||C||_\infty$, then error bound should be also in the relative terms.

Once we have obtained the vectors p and $\overline{p}$, we can introduce Algorithm 2 for calculating the trace approximation.

Algorithm 2 (Trace approximation for the solution of the Lyapunov equation (7))

Input: α, Φ – such that $\Phi^T K \Phi = \Omega^2 = \mathrm{diag}(\omega_1^2, \ldots, \omega_n^2)$ and $\Phi^T M \Phi = I$;
v_i, C_i, $i = 1, \ldots, k$ – viscosity and position matrix for the ith damper;
$p(1), p(2), \ldots, p(s)$ – indices of the eigenfrequencies which have to be damped;
$\widehat{s}$ – number of additional indices which are needed for Algorithm 1;
tol – tolerance needed for Algorithm 1.

Output: $\mathrm{trace}\, X(C)$

1: Determine vectors $\overline{p} \in \mathbb{N}^{n-r}$ and $p \in \mathbb{N}^r$ using Algorithm 1.
2: $\Omega_r = \mathrm{diag}(\omega_{p(1)}, \omega_{p(2)}, \ldots, \omega_{p(r)})$
3: $C = \Phi^T (v_1 C_1 + v_2 C_2 + \cdots + v_k C_k) \Phi$
4: Calculate $\widetilde{X}_{11}$, where

$$A_{11}\widetilde{X}_{11} + \widetilde{X}_{11}A_{11}^T = -\widetilde{G}\widetilde{G}^T, \quad A_{11} = P^T \begin{bmatrix} 0 & \Omega_r \\ -\Omega_r & -\alpha\Omega_r - C(p,p) \end{bmatrix} P,$$
$$\widetilde{G} = \begin{bmatrix} I_{2s} & 0 \end{bmatrix}^T, \quad P = \text{ perfect shuffle permutation.}$$

5: $\mathrm{trace}\, X(C) \approx \mathrm{trace}\, \widetilde{X}_{11}$.

REMARK **2.1.** *Similarly to the approach presented in [7] we can consider our approximation of the trace using the truncation matrix Q_r. That is, we construct the truncation matrix Q_r, with column rank equal to r, such that system (1) can be approximated by the system with matrices M_r, D_r and K_r of dimension r. This system is*

$$M_r \ddot{y} + D_r \dot{y} + K_r y = 0, \tag{14}$$

where $M_r = Q_r^T M Q_r$, $D_r = Q_r^T D Q_r$, $K_r = Q_r^T K Q_r$; Q_r is an $n \times r$ matrix.

The trace approximation for the Lyapunov equation corresponds to the trace of the solution of the Lyapunov equation which is obtained from the reduced system

(14) using the truncation matrix $Q_r = \Phi(:, p)$, where Φ is defined in (3) and p is calculated with Algorithm 1.

Note that in our approximation algorithm we have assumed that internal damping is defined as in (2) but this algorithm can be easily extended to the case where the internal damping is any matrix which is diagonalized by the matrix Φ defined in (3). This will be the case if the identity $KM^{-1}C_u = C_uM^{-1}K$ holds, because then the matrices M, C_u and K can be simultaneously diagonalized, for more details see [15, 23].

The question is: *how accurate is* Algorithm 2? The following section provides an error bound for the approximation given by Algorithm 2.

3 Error bound

Consider the partitioned form of the permuted Lyapunov equation (11):

$$\begin{bmatrix} A_{11} & A_{12} \\ A_{12}^T & A_{22} \end{bmatrix} \begin{bmatrix} X_{11} & X_{12} \\ X_{12}^T & X_{22} \end{bmatrix} + \begin{bmatrix} X_{11} & X_{12} \\ X_{12}^T & X_{22} \end{bmatrix} \begin{bmatrix} A_{11}^T & A_{12} \\ A_{12}^T & A_{22}^T \end{bmatrix} = -\widehat{G}\widehat{G}^T, \tag{15}$$

where

$$A = \begin{bmatrix} A_{11} & A_{12} \\ A_{12}^T & A_{22} \end{bmatrix}, \quad X = \begin{bmatrix} X_{11} & X_{12} \\ X_{12}^T & X_{22} \end{bmatrix}. \tag{16}$$

The matrix $\widehat{G}$ has rank $2s$ and after the permutations has the form $\widehat{G} = \begin{bmatrix} I_{2s} & 0 \end{bmatrix}^T$, where $s \leq r$ (r is the reduced dimension and according to our approximation algorithm, $\dim(A_{11}) = 2r \geq 2s$).

Equation (15) can be considered as the perturbed equation of

$$\begin{bmatrix} A_{11} & 0 \\ 0 & A_{22} \end{bmatrix} \begin{bmatrix} \widetilde{X}_{11} & \widetilde{X}_{12} \\ \widetilde{X}_{12}^T & \widetilde{X}_{22} \end{bmatrix} + \begin{bmatrix} \widetilde{X}_{11} & \widetilde{X}_{12} \\ \widetilde{X}_{12}^T & \widetilde{X}_{22} \end{bmatrix} \begin{bmatrix} A_{11}^T & 0 \\ 0 & A_{22}^T \end{bmatrix} = -\widehat{G}\widehat{G}^T. \tag{17}$$

For the sequel, we define

$$\widetilde{A} = \begin{bmatrix} A_{11} & 0 \\ 0 & A_{22} \end{bmatrix}, \quad \widetilde{X} = \begin{bmatrix} \widetilde{X}_{11} & \widetilde{X}_{12} \\ \widetilde{X}_{12}^T & \widetilde{X}_{22} \end{bmatrix}. \tag{18}$$

Note that $\widetilde{X}_{12} = 0$, $\widetilde{X}_{22} = 0$ and $\widetilde{X}_{11}$ is the solution of the equation

$$A_{11}\widetilde{X}_{11} + \widetilde{X}_{11}A_{11}^T = -\widetilde{G}\widetilde{G}^T, \tag{19}$$

where $\widetilde{G} = \widehat{G}(1 : 2r, 1 : 2s)$. Observe that in Algorithm 2 we solve equation (17) instead of (15), thus we are interested in the error bound for

$$\frac{|\operatorname{trace}(X) - \operatorname{trace}(\widetilde{X}_{11})|}{\operatorname{trace}(X)}, \tag{20}$$

where X is the solution of (15) and $\widetilde{X}_{11}$ is the solution of (19).

Our error bound will be based on the results from [9], where the authors considered the Sylvester equation

$$AX - XB = C, \tag{21}$$

perturbed such that

$$(A + \Delta A)(X + \Delta X) - (X + \Delta X)(B + \Delta B) = C + \Delta C.$$

As one can find in [9], the first order approximation for the change in a scalar function g with respect to perturbations is given by

$$|g(X(0)) - g(X(\eta))| \approx \sum_{\eta} \left|\frac{dg}{d\eta}\right| |\eta|. \tag{22}$$

First we need to say that we consider $\widetilde{X}$ as a perturbation of $X =: X(0)$ with respect to a parameter η and that we are merely interested in the scalar function $\mathrm{trace}(X)$ rather than X itself.

The approximation (22) corresponds to a first order approximation using Taylor expansion and the perturbation η determines the absolute change of the corresponding elements of the solution $X(0)$. When the perturbation η is small ($|\eta| \leq \varepsilon$, $\forall \eta$), by first-order approximation we have

$$|g(X(0)) - g(X(\eta))| \lessapprox \varepsilon \sum_{\eta} \left|\frac{dg}{d\eta}\right|. \tag{23}$$

Let $a_{ij} = (A)_{ij}$, $b_{ij} = (B)_{ij}$, $c_{ij} = (C)_{ij}$ be the entries of A, B, C. A bound for $\sum_{\eta} |\frac{dg}{d\eta}|$ from (23), which is independent of the perturbation η, is given in [9, (33)] with

$$\begin{aligned} \sum_{\eta} \left|\frac{dg}{d\eta}\right| &\leq \mathrm{trace}(|\Lambda|^T |C|) + \sum_{i,j} \Big| \sum_{k} a_{ji} \lambda_{ik} x_{jk} \Big| + \sum_{i,j} \Big| \sum_{k} b_{ij} \lambda_{kj} x_{ki} \Big| \\ &= \mathrm{trace}(|\Lambda|^T |C|) + \sum_{i,j} |a_{ji}| |(\Lambda X)_{ij}| + \sum_{i,j} |b_{ij}| |(\Lambda X)_{ji}|, \end{aligned} \tag{24}$$

where Λ is the solution of the Sylvester equation

$$A^T \Lambda - \Lambda B^T = \frac{dg}{dX}.$$

For the error bound (20), the scalar function g from (23) is $g(X) = \mathrm{trace}\, X$. Using that $\frac{d\,\mathrm{trace}(AX)}{dX} = A^T$ (see, e.g., [11]) yields

$$\frac{dg}{dX} = \frac{d\,\mathrm{trace}(X)}{dX} = I.$$

Hence, in our case we have to calculate Λ from the equation

$$\widetilde{A}^T\Lambda + \Lambda\widetilde{A} = I.$$

As one can see from (18), $\widetilde{A}$ is block diagonal, which means that Λ is a block diagonal matrix of the form

$$\Lambda = \begin{bmatrix} \Lambda_{11} & 0 \\ 0 & \Lambda_{22} \end{bmatrix}, \tag{25}$$

where the diagonal blocks of Λ are the solutions of the Lyapunov equations

$$\begin{aligned} A_{11}^T\Lambda_{11} + \Lambda_{11}A_{11} &= I, \\ A_{22}^T\Lambda_{22} + \Lambda_{22}A_{11} &= I. \end{aligned} \tag{26}$$

Further we will show that we can bound (20) using only Λ_{11}.

From (23) and (24) with $g(X) = \operatorname{trace}(X)$ it follows that

$$\begin{aligned} |\operatorname{trace}(X) - \operatorname{trace}(\widetilde{X}_{11})| \lessapprox \varepsilon\Bigg(& \operatorname{trace}(|\Lambda|^T|GG^T|) + \sum_{i,j} |\widetilde{a}_{ji}||(\Lambda\widetilde{X})_{ij}| \\ & + \sum_{i,j} |\widetilde{a}_{ji}||(\Lambda\widetilde{X})_{ji}|\Bigg), \end{aligned} \tag{27}$$

where ε satisfies $|(A_{12})_{ij}| \leq \varepsilon$, $\forall i, j$, $(\widetilde{A})_{ij} = \widetilde{a}_{ij}$ and $\widetilde{X}$ are given in (18). Note that the vector p from Algorithms 1 and 2 determines the permutations such that the elements of A_{12} are small by absolute value, which means that ε will be small, too. Now, we can state the following theorem.

THEOREM **3.1.** *Let A_{11} and the vectors p, $\overline{p}$ be determined by Algorithm 2. Furthermore, let X be the exact solution of (15) with approximation X_{11} computed by Algorithm 2, and let Λ_{11} be the solution of (26). For $\varepsilon = \max_{i,j} |C(p(i), \overline{p}(j))|$, if ε is small enough it holds*

$$\begin{aligned} \frac{|\operatorname{trace}(X) - \operatorname{trace}(\widetilde{X}_{11})|}{\operatorname{trace}(\widetilde{X}_{11})} \lessapprox & \frac{\varepsilon}{\operatorname{trace}(\widetilde{X}_{11})}\Bigg(\operatorname{trace}(|\Lambda_{11}|^T|\widetilde{G}\widetilde{G}^T|) + \sum_{i,j=1}^{2r} |\widetilde{a}_{ji}||(\Lambda_{11}\widetilde{X}_{11})_{ij}| \\ & + \sum_{i,j=1}^{2r} |\widetilde{a}_{ji}||(\Lambda_{11}\widetilde{X}_{11})_{ji}|\Bigg), \end{aligned} \tag{28}$$

where $(A_{11})_{ij} = \widetilde{a}_{ij}$.

Proof. The local first-order bound (28) is obtained directly from the local first-order bound (27), using the block structure of Λ defined in (25) and $\widetilde{X}$ defined in (18). □

Note that ε from the right-hand side of the above error bound, in the terms of Algorithms 1 and 2, is equal to $\varepsilon = \max_{i,j} |C(p(i), \overline{p}(j))|$. Also, this error bound now includes the structure of the right hand side (Λ_{11} is multiplied with the right hand side of the Lyapunov equation), and the structure of $\widetilde{A}$.

Since $\frac{|\operatorname{trace}(X) - \operatorname{trace}(\widetilde{X}_{11})|}{\operatorname{trace}(X)} \approx \frac{|\operatorname{trace}(X) - \operatorname{trace}(\widetilde{X}_{11})|}{\operatorname{trace}(\widetilde{X}_{11})}$, we will use the bound (28) as an estimate for the relative error given in (20). For calculating the upper error bound (28), we need to solve the Lyapunov equations (19) and (26) which have dimension $2r \times 2r$ which makes this bound easy to calculate.

4 An Algorithm for the approximation of the optimal viscosities

We will present an algorithm which calculates an approximation of the optimal viscosities. Using Algorithm 2 and the error bound (28), we determine the reduced dimension efficiently for a given tolerance.

In the optimization process we do not check the error bound for each approximation. When we determine a suboptimal viscosity we calculate the error estimate (28) and if the error is small enough we stop the optimization process (then our suboptimal viscosities are expected to be close to the optimal ones). On the other hand, if the error is too large we repeat the optimization process with a smaller tolerance, that is, we increase the reduced dimension r.

In Algorithm 3 we present an algorithm for calculation of the optimal viscosity at given damper positions. Each time we start a new optimization process for determination of the optimal viscosities in Step 4 of Algorithm 3, thus in order to accelerate the optimization process, our algorithm should use the information of suboptimal viscosities obtained in the previous step. This means that if we optimize with the Nelder-Mead method in Step 4 of Algorithm 3, first time we take some fixed starting viscosities $v_1^0, v_2^0, \ldots, v_k^0$, then in every other step of the optimization process the starting values are the suboptimal viscosities from the previous step. In Algorithm 3, the parameter $\mathbf{u}$ denotes the machine precision.

5 Area with the optimal dampers' positions

In this section we will present an algorithm for the efficient determination of the area where the optimal dampers are located. The a priori determination of this area is possible when the system has a special structure. Using an approach similar to the one used for the derivation of the error bound (28), we can obtain a similar bound which can be used to identify those damping positions which have a negligible impact on the overall damping of the system. With this approach we avoid the viscosity

Algorithm 3 (Computing optimal viscosities at given damping positions)

Input: α, $\kappa \geq 1$;
Φ – such that $\Phi^T K \Phi = \Omega^2 = \mathrm{diag}(\omega_1^2, \ldots, \omega_n^2)$ and $\Phi^T M \Phi = I$;
$p(1), p(s), \ldots, p(s)$ – indices of eigenfrequencies which have to be damped;
$\widehat{s}$ – number of additional indices which are needed for Algorithm 1;
ε – the tolerance for the relative error bound (28);
tol_{start} – the starting tolerance for viscosity optimization;
c_1 – a positive constant for scaling a tolerance ($c_1 < 1$);
$v_1^0, v_2^0, \ldots, v_k^0$ – starting viscosities for the optimization process;
$d_1, d_2, \ldots, d_k$ – damping positions at which viscosity should be optimized.

Output: Optimal viscosities $\widehat{v}_1, \ldots, \widehat{v}_k$

1: $tol = tol_{start}$
2: $\widehat{v}_1 = v_1^0, \widehat{v}_2 = v_2^0, \ldots, \widehat{v}_k = v_k^0$
3: **while** $tol > \kappa \cdot \mathbf{u}$ **do**
4: Calculate new suboptimal viscosities using an optimization algorithm (e.g. Nelder-Mead), based on Algorithm 2 for calculation of the trace $X(v_1, \ldots, v_k)$ with starting points $\widehat{v}_1, \widehat{v}_2, \ldots, \widehat{v}_k$ and tolerance tol.
5: Calculate the right-hand side of the bounds (28) at suboptimal viscosities $\widehat{v}_1, \ldots, \widehat{v}_k$, and denote it by η.
6: **if** $\eta < \varepsilon$ **then**
7: **return** optimal viscosities $\widehat{v}_1, \ldots, \widehat{v}_k$
8: **break**
9: **else**
10: $tol = c_1 \cdot tol$
11: **end if**
12: **end while**

optimization at some damping positions which can remarkably improve the efficiency of the optimization algorithm.

Denote the trace of the solution of the Lyapunov equation (15), for the case with external damping $C_{ext} = 0$, by τ_0. It is easy to show that $\tau_0 = (\frac{2}{\alpha} + \frac{\alpha}{2}) \sum_{i=1}^{s} \frac{1}{\omega_{p(i)}}$, where the vector p contains the indices of the s eigenfrequencies to be damped. The aim of our algorithm is to determine damping positions such that trace $X(C)$ is very close to τ_0. Thus, damping at these positions would have no or only negligible impact on the system's energy.

Recall that the external damping matrix is equal to $C_{ext} = v_1 C_1 + v_2 C_2 + \cdots + v_k C_k$, where k is the number of dampers and C_i, $i = 1, \ldots, k$, encodes the position of the ith damper with corresponding viscosity v_i (then $C = \Phi^T C_{ext} \Phi$). Now, for given damping positions, we consider the Lyapunov equation (15) with damping C such that $(\widehat{P}^T P^T C P \widehat{P})_{i,j} = 0$, for $i = 1, \ldots, 2s$, $j = 1, \ldots, 2n$, where P is again the perfect shuffle permutation matrix and $\widehat{P}$ is obtained using the vectors p and $\overline{p}$ (here p contains just the indices of the eigenfrequencies which have to be damped). In this case $\text{trace}(X(C)) = \tau_0$.

The basic idea is to determine a damping matrix C with $\max\limits_{\substack{1 \le i \le 2s \\ 1 \le j \le 2n}} |(\widehat{P}^T P^T C P \widehat{P})_{ij}| \le \varepsilon$ for some small ε, such that $\text{trace}(X(C)) \approx \tau_0$. If $\text{trace}(X(C))$ is very close to τ_0, this means that the damper positions used to construct C do not damp the system significantly. In order to efficiently determine positions where $\text{trace}(X(C))$ is close to τ_0, we will derive a bound from which we can determine such positions.

We will consider equation (15) as the perturbation of

$$\begin{bmatrix} \widetilde{A}_{11} & 0 \\ 0 & A_{22} \end{bmatrix} \begin{bmatrix} \widetilde{X}_{11} & \widetilde{X}_{12} \\ \widetilde{X}_{12}^T & \widetilde{X}_{22} \end{bmatrix} + \begin{bmatrix} \widetilde{X}_{11} & \widetilde{X}_{12} \\ \widetilde{X}_{12}^T & \widetilde{X}_{22} \end{bmatrix} \begin{bmatrix} \widetilde{A}_{11}^T & 0 \\ 0 & A_{22}^T \end{bmatrix} = -\widehat{G}\widehat{G}^T, \tag{29}$$

where

$$\begin{aligned} \widetilde{A} &= \begin{bmatrix} \widetilde{A}_{11} & 0 \\ 0 & A_{22} \end{bmatrix}, \quad \widetilde{X} = \begin{bmatrix} \widetilde{X}_{11} & \widetilde{X}_{12} \\ \widetilde{X}_{12}^T & \widetilde{X}_{22} \end{bmatrix}, \\ \widetilde{A}_{11} &= \widehat{A}_{p(1)} \oplus \widehat{A}_{p(2)} \oplus \cdots \oplus \widehat{A}_{p(s)}, \quad \widehat{A}_i = \begin{bmatrix} 0 & \omega_i \\ -\omega_i & -\alpha\omega_i \end{bmatrix}. \end{aligned} \tag{30}$$

Note that $\widetilde{X}_{12} = 0$, $\widetilde{X}_{22} = 0$ and $\widetilde{X}_{11}$ is the solution of the equation

$$A_{11}\widetilde{X}_{11} + \widetilde{X}_{11}A_{11}^T = -\widetilde{G}\widetilde{G}^T,$$

where $\widetilde{G} = \widehat{G}(1 : 2r, 1 : 2s)$. Since $\widetilde{G}\widetilde{G}^T = I$ it is easy to observe that

$$\widetilde{X}_{11} = \widehat{X}_1 \oplus \widehat{X}_2 \oplus \cdots \oplus \widehat{X}_s, \quad \widehat{X}_i = \frac{1}{\omega_i} \begin{bmatrix} \frac{2+\alpha^2}{2\alpha} & -\frac{1}{2} \\ -\frac{1}{2} & \frac{1}{\alpha} \end{bmatrix}, \quad i = 1, \ldots, s, \tag{31}$$

and that $\operatorname{trace}\widetilde{X} = \operatorname{trace}\widetilde{X}_{11} = \tau_0$.

As mentioned before, similarly as in Section 3, we can now derive the error bound for the approximation (29). Using that $\operatorname{trace}\widetilde{X}_{11} = \tau_0$ and $C = \Phi^T C_{ext}\Phi = \Phi^T(v_1C_1 + v_2C_2 + \cdots + v_kC_k)\Phi$ we have that

$$\frac{|\operatorname{trace}(X(C)) - \tau_0|}{\tau_0} \leq \max_{\substack{i=1,\ldots,2s\\ j=1,\ldots,2n}} |(v_1C_1^\Phi + v_2C_2^\Phi + \cdots + v_kC_k^\Phi)_{ij}| \cdot \xi \tag{32}$$

with

$$\xi = \frac{\left(\operatorname{trace}(|\Lambda_{11}|^T|\widetilde{G}\widetilde{G}^T|) + \sum_{i,j=1}^{2s}|\widetilde{a}_{ji}||(\Lambda_{11}\widetilde{X}_{11})_{ij}| + \sum_{i,j=1}^{2s}|\widetilde{a}_{ji}||(\Lambda_{11}\widetilde{X}_{11})_{ji}|\right)}{\tau_0},$$

where $(\widetilde{A}_{11})_{ij} = \widetilde{a}_{ij}$, $C_i^\Phi = \widehat{P}^T P^T \Phi^T C_i \Phi P \widehat{P}$, for $i = 1, \ldots, k$, and $\widetilde{A}_{11}$, $\widetilde{X}_{11}$ are given in (30), (31), respectively. Recall that $\widetilde{G}\widetilde{G}^T = I$ and that Λ_{11} is the solution of the Lyapunov equation $\widetilde{A}_{11}^T\Lambda_{11} + \Lambda_{11}\widetilde{A}_{11} = I$, thus we have that $\Lambda_{11} = -\widetilde{X}_{11}$ and it holds that

$$\xi = \left(1 + 2\frac{\sum_{i,j=1}^{2s}|\widetilde{a}_{ji}||(\widetilde{X}_{11}^2)_{ij}|}{\tau_0}\right). \tag{33}$$

In order to obtain an easily computable bound, we need to fix feasible maximal values of the dampers' viscosities for the external damping C_{ext}. (Such an upper bound is also usually needed in the viscosity optimization algorithm.) Let the maximal viscosities be $v_i^{\max}$, $i = 1, \ldots, k$, respectively. Using this to bound the right-hand side in (32), we obtain

$$\frac{|\operatorname{trace}(X(C)) - \tau_0|}{\tau_0} \leq \max_{\substack{i=1,\ldots,2s\\ j=1,\ldots,2n}} (v_1^{\max}|C_1^\Phi| + \cdots + v_k^{\max}|C_k^\Phi|)_{ij} \cdot \xi, \tag{34}$$

where ξ is given in (33). Now, in the resulting algorithm, we mark those positions for which the right-hand side of (34) is small enough, i.e., smaller than some given tolerance tol_{pos}, as such a configuration of damping positions has negligible influence in damping the system. The tolerance employed here should be of the order of the relative termination tolerance used for the viscosity optimization.

An algorithm for computing the area which contains the optimal damping positions is summarized in Algorithm 4.

REMARK **5.1.** *Algorithm 4 will be efficient if the system has some special structure, more precisely, when there are numerous positions such that the maximum term in the bound (34) is small. This can also be checked in advance by analyzing the magnitude of the elements of the matrix C. Particularly, if there exist numerous indices i such that $\|\Phi(i,p)\|$ (with p corresponding to indices of the eigenfrequencies*

which have to be damped) is small, then our algorithm will determine numerous positions which cannot be used to damp the system. Our experiments have shown that, for example, if we like to damp eigenfrequencies that are large by magnitude for mechanical systems with equal stiffnesses and increasing masses, $\|\Phi(i,p)\|$ will be small for a large number of pairs (i,p).

Algorithm 4 (Determination of possible optimal damping positions)

Input: $\widetilde{X}$ – approximate solution of the Lyapunov equation;
Φ – such that $\Phi^T K \Phi = \Omega^2 = \mathrm{diag}(\omega_1^2, \ldots, \omega_n^2)$ and $\Phi^T M \Phi = I$;
$p(1), p(2), \ldots, p(s)$ – indices of eigenfrequencies which have to be damped;
v_i^{max}, $i = 1, \ldots, k$ – maximal viscosities;
tol_{pos}.

Output: Set $\mathcal{S}$ of positions that contains the optimal damping positions.

1: Define $\mathcal{S}$ as set of all possible configurations for the damping positions.
2: Calculate $\xi = \left(1 + 2\frac{\sum_{i,j=1}^{2s} |\widetilde{a}_{ji}||(\widetilde{X}_{11}^2)_{ij}|}{\mathrm{trace}(\widetilde{X}_{11})}\right)$ as in (33).
3: **while** not all configurations of damping position have been checked **do**
4: For the given damping positions, calculate

$$\varepsilon = \max_{\substack{i=1,\ldots,2s \\ j=1,\ldots,2n}} (v_1^{max}|C_1^{\Phi}| + \cdots + v_k^{max}|C_k^{\Phi}|)_{ij}\xi.$$

5: **if** $\varepsilon < tol_{pos}$ **then**
6: exclude the corresponding configuration from $\mathcal{S}$.
7: **end if**
8: **end while**

Note that in the **while**-loop of Algorithm 4, all damping positions should be checked. Generally speaking, this means that we should check all configurations $(i_1, i_2, \ldots, i_k)$ such that $1 \le i_1 < i_2 < \cdots < i_k \le n$. Algorithm 4 is quite efficient since the main calculation cost for different positions is the calculation of the corresponding maximum in Step 3.

6 Numerical experiments

In this section we will present two examples. The first of them compares the new algorithm for viscosity optimization using dimension reduction with the same optimization method without dimension reduction. In both approaches, the Lyapunov equations are solved by the Bartels-Stewart algorithm [4] implemented in MATLAB function `lyap`. The second example illustrates the efficiency of Algorithm 4 which

determines the set of meaningful configurations for the damping positions. In these examples, computations have been carried out on an Intel(R) Core(TM) i7 CPU 920 with 12GB of RAM and 8 MB cache. Results were calculated in MATLAB (Version 7.11.0.584 (R2010b) 32-bit) installed on Linux.

Example 1. We will consider an n-mass oscillator or oscillator ladder with two dampers, shown in Figure 1, which describes the mechanical system of n masses and $n+1$ springs. For such a mechanical system the mathematical model is given by (1), where the mass and stiffness matrices are

$$M = \operatorname{diag}(m_1, m_2, \ldots, m_n),$$
$$K = \begin{pmatrix} k_1+k_2 & -k_2 & & & & \\ -k_2 & k_2+k_3 & -k_3 & & & \\ & \ddots & \ddots & \ddots & & \\ & & -k_{n-1} & k_{n-1}+k_n & -k_n & \\ & & & -k_n & k_n+k_{n+1} \end{pmatrix}.$$

The damping matrix is

$$D = C_u + C_{ext}, \quad \text{where the internal damping } C_u \text{ is defined as in (2).}$$

Since we will consider two dampers of different viscosities, the external damping is defined by $C_{ext} = v_1 e_i e_i^T + v_2 e_j e_j^T$, where $1 \le i < j \le n$.

We will consider the following configuration

$$n = 1600; \quad k_i = 4, \quad \forall i; \qquad m_i = \begin{cases} 120 - (i-1)/5, & i = 1, \ldots, 100, \\ i, & i = 101, \ldots, n. \end{cases}$$

The goal is to damp all those eigenfrequencies of the system which are smaller than 0.005 by magnitude. Thus, for the above configuration, we obtain $s = 34$. (Recall that s determines the matrix G from (8)).

The considered optimization process is extremely demanding, because it requires solving the Lyapunov equation (7) numerous times (see e.g. [7]). Thus, instead of

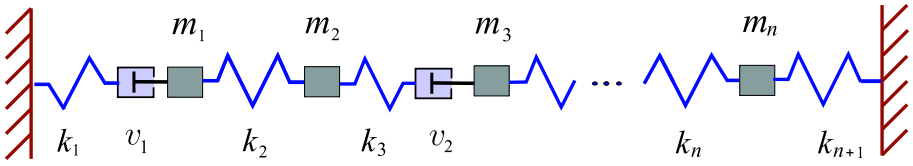

Figure 1: n-mass oscillator with two dampers

performing optimization over all damping positions (complete optimization), we will compare the new algorithm with the standard algorithm (i.e., optimization without dimension reduction) on the equidistant mesh of damping positions

$$i = 51 : 50 : n, \quad j = i + 51 : 50 : n, \tag{35}$$

yielding 465 different damping configurations. Furthermore, for the purpose of better illustration of the obtained results, we will restrict our comparison to those configurations yielding the 99 smallest traces (sorted by magnitude).

We have used the following setting in Algorithm 3 (the undamped eigenfrequencies are sorted such that $\omega_1 < \omega_2 < \cdots < \omega_n$):

$$\begin{aligned} \alpha &= 0.001; & \kappa &= 10^4; \\ p(i) &= i,\ i = 1, \ldots, s; & \widehat{s} &= 60; \\ \varepsilon &= 0.1; & tol_{start} &= 0.002; \\ c_1 &= 0.5; & v_1^0 &= v_2^0 = 50. \end{aligned}$$

For the viscosity optimization we use the Nelder-Mead algorithm [17] implemented in MATLAB function `fminsearch`. The termination tolerance for the function value is taken as 0.1 and the termination tolerance for the optimization variable (viscosity) is set to 0.001. Note that these tolerances are absolute, thus they are appropriate for our optimization since the function values is around 10^6 and the optimal viscosities vary from 10 to 1000.

With both algorithms (with and without dimension reduction), we obtain the same optimal position (with respect to the mesh given in (35)), and this is the position $(i, j) = (651, 1352)$ with optimal viscosities $(v_1, v_2) = (107.03009, 150.49333)$, while the optimal trace is $\operatorname{trace}(X(v_1, v_2)) = 993\,067.32851$.

Figures 2–4 present comparisons between results obtained with the new algorithm with dimension reduction and with the standard approach without dimension reduction. As we have mentioned above, we have plotted the data which corresponds to the 99 smallest traces sorted by magnitude. The first data point corresponds to the optimal position which is given above, i.e., the position which corresponds to the smallest trace. Figure 2 shows the relative errors for the optimal trace at the given dampers position. The relative errors for the trace are calculated using $|\operatorname{trace} X - \operatorname{trace} \widetilde{X}_{11}|/\operatorname{trace} X$, where $\operatorname{trace} X$ is the optimal trace for the given position obtained with the algorithm without dimension reduction, and $\operatorname{trace} \widetilde{X}_{11}$ is the approximation of the optimal trace calculated by Algorithm 3.

Figure 3 shows the relative errors for the optimal viscosities (first and second viscosity). The relative error for the ith viscosity ($i = 1, 2$) is calculated as $|\widetilde{v}_i^{opt} - v_i^{opt}|/v_i^{opt}$, where $\widetilde{v}_i^{opt}$ is the ith optimal viscosity obtained by Algorithm 3 and v_i^{opt} is the ith exact optimal viscosity obtained by optimization without dimension reduction. In Figure 4 we show the times required by each of the algorithms. This

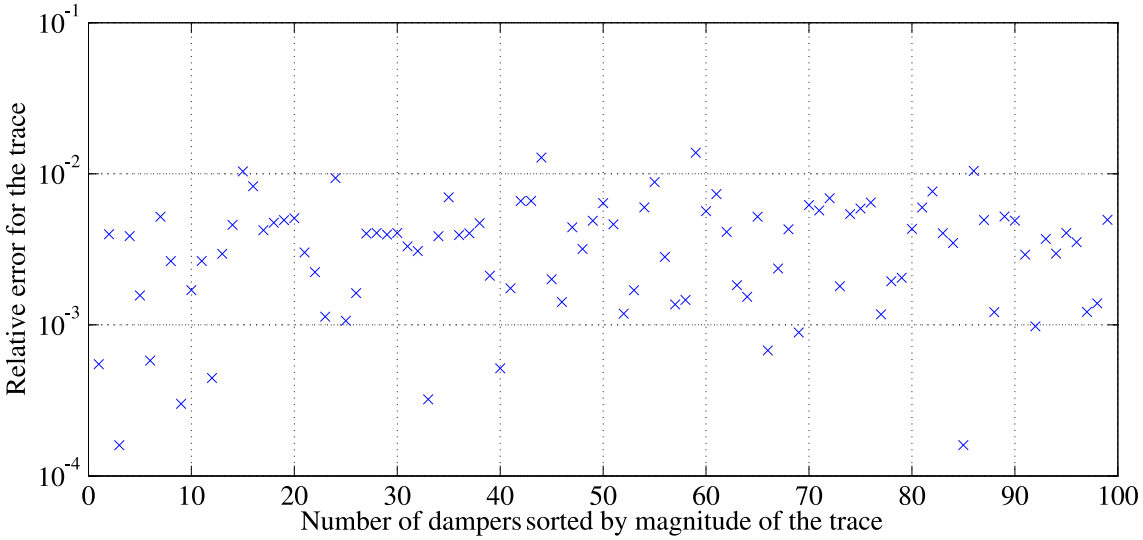

Figure 2: Example 1, relative error for trace X.

figure shows the time ratio for calculating the optimal viscosities at the given damping positions using the new algorithm and using the algorithm without dimension reduction.

For example, at the optimal damping position, the algorithm without dimension reduction needs 5.726 hours to calculate optimal viscosities while the new algorithm only needs 0.42 minutes on the same computer. This means that the new algorithm is faster by a factor of 818. From Figure 4, we can also see that there are numerous positions with similar time ratio which in global gives much shorter time for computation.

In the following example we will illustrate the efficiency of Algorithm 4.

Example 2. In this example we consider the mechanical system shown in Figure 5 with two dampers of different viscosities, $2d+1$ masses and $2d+3$ springs.

In the considered vibrational system the mass matrix is $M = \operatorname{diag}(m_1, m_2, \ldots, m_n)$, while the stiffness matrix is defined as

$$K = \begin{bmatrix} K_{11} & & -\kappa_1 \\ & K_{11} & -\kappa_1 \\ -\kappa_1^T & -\kappa_1^T & 2k_1 + k_2 \end{bmatrix},$$

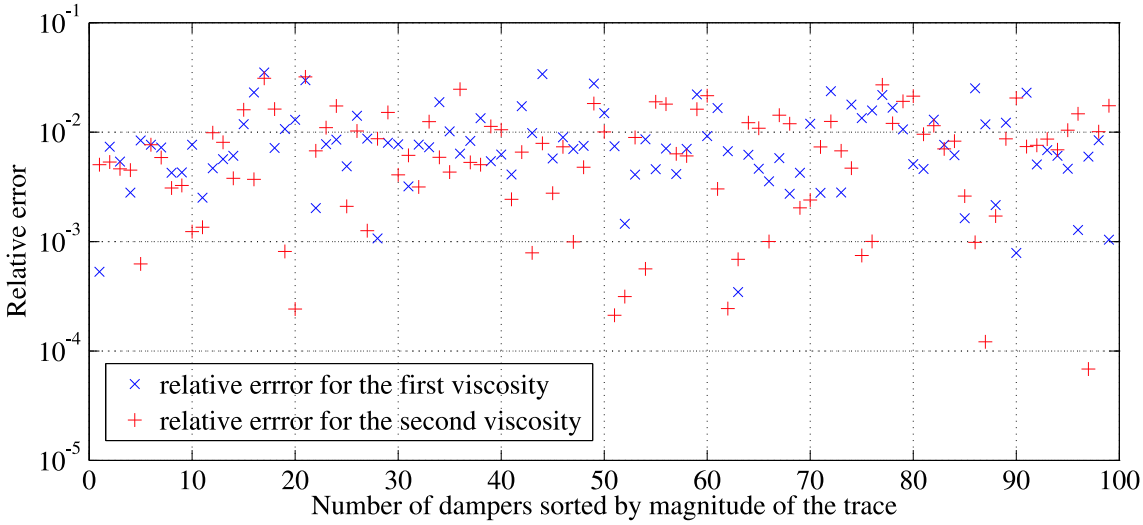

Figure 3: Example 1, relative errors for the optimal viscosities.

where

$$K_{11} = k_1 \begin{bmatrix} 2 & -1 & & & \\ -1 & 2 & -1 & & \\ & \ddots & \ddots & \ddots & \\ & & -1 & 2 & -1 \\ & & & -1 & 2 \end{bmatrix}, \quad \kappa_1 = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ k_1 \end{bmatrix}.$$

In our example we will consider the following configuration:

$$\begin{aligned} & d = 500, \quad n = 2d + 1 = 1001; \quad k_1 = 10, \quad k_2 = 20; \\ & m_i = 10 \cdot i \quad \text{for} \quad i = 1, \ldots, d/5; \\ & m_i = (12/5) \cdot d + 2 - 2i \quad \text{for} \quad i = d/5 + 1, \ldots, d; \\ & m_i = 5 \cdot (2d + 1 - i) \quad \text{for} \quad i = d + 1, \ldots, 2d; \quad m_{2d+1} = 500. \end{aligned}$$

The external and internal damping matrices are the same as in Example 2 (here we also have two dampers of different viscosities).

In this example we aim at damping all eigenfrequencies of the undamped system with magnitude larger than 1. This yields $s = 6$. We have used the following settings in Algorithm 4, with undamped eigenfrequencies sorted such that $\omega_1 > \omega_2 > \cdots > \omega_n$:

$$\alpha = 0.001; \qquad p(i) = i, \quad i = 1, \ldots, s; \qquad tol_{pos} = 10^{-8}.$$

Recall that in this example we consider two dampers of different viscosities. We determine positions i and j such that $1 \le i < j \le n$. This offers $n(n-1)/2$ different

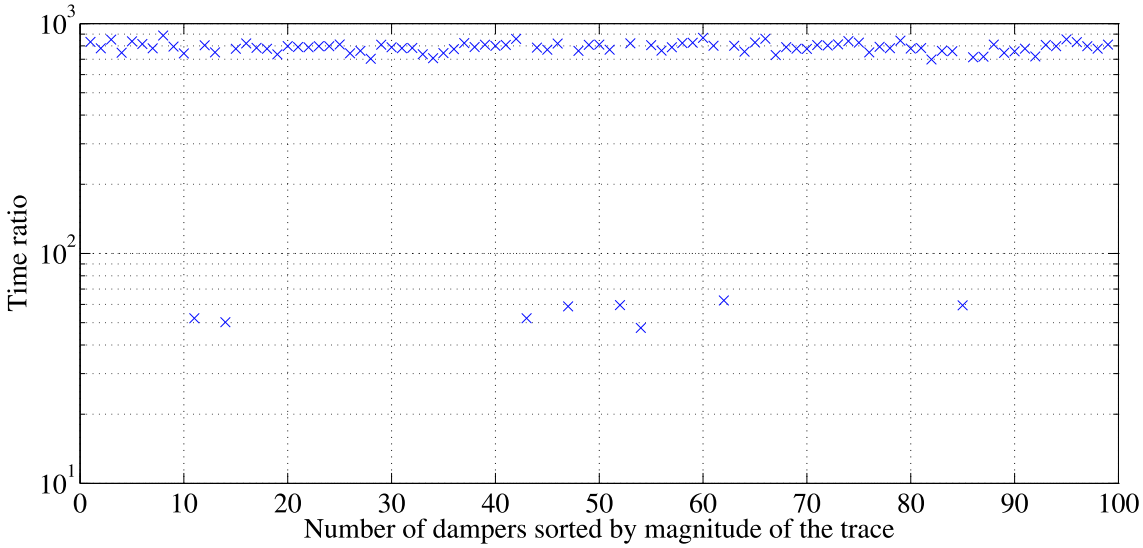

Figure 4: Example 1, timing ratio.

damping positions and for this example this means that we have 500 500 different configurations. Using Algorithm 4 with tolerance $tol_{pos} = 10^{-8}$ we obtain that 473 851 of them are not useful for damping the system. That is, set of configurations which contains the optimal position has 26 649 elements. Thus, with Algorithm 4 we have reduced the number of different damping positions to just 5.32% of the total number of possible positions. More precisely, Algorithm 4 returns that the set containing the optimal damping position (i_{opt}, j_{opt}) is equal to $\mathcal{S} = \mathcal{S}_1 \cup \mathcal{S}_2 \cup \mathcal{S}_3 \cup \mathcal{S}_4$, where

$$\begin{aligned}
\mathcal{S}_1 &= \{(i,j) : \text{for } i \leq 9;\ \ j \text{ is such that } i+1 \leq j\},\\
\mathcal{S}_2 &= \{(i,j) : \text{for } i = 499, 500;\ j \text{ is such that } i+1 \leq j\},\\
\mathcal{S}_3 &= \{(i,j) : \text{for } j = 499, 500;\ \ i \text{ is such that } i \leq j-1\},\\
\mathcal{S}_4 &= \{(i,j) : \text{for } 986 \leq j \leq 1001;\ \ i \text{ is such that } i \leq j-1\}.
\end{aligned}$$

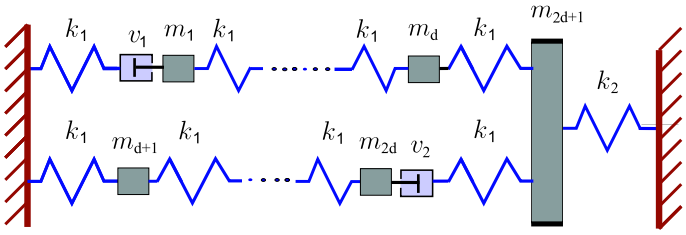

Figure 5: $2d+1$ mass oscillator

In order to test our algorithm, we calculate the minimal traces for each position of the following mesh of damping positions:

$$i = 4 : 165 : n, \quad j = i + 1 : 165 : n.$$

This mesh is constructed such that different positions from $\mathcal{S}$ are included, but also some positions which are not in $\mathcal{S}$. For this mesh, the optimal position is $(i_{opt}, j_{opt}) = (4, 995)$, the optimal viscosities at these positions are $(v_1, v_2) = (23.91853, 14.78638)$, and the corresponding trace is 1839.11344. In this example, the trace of the solution of the Lyapunov equation which corresponds to the undamped system is $\tau_0 = 4559.12291$. On the other hand, it is easy to check that for the positions which are not in $\mathcal{S}$, for example for positions $(169, 170)$, $(334, 665)$, $(664, 830)$, the minimal trace is equal to τ_0 up to $O(tol_{pos})$.

Observe that we have reduced the total number of positions to $|\mathcal{S}|$ (using the tolerance $tol_{pos} = 10^{-8}$) and that for the determination of the optimal position we only use those in $\mathcal{S}$. Thus we need to optimize the viscosities with respect to some mesh of positions contained in $\mathcal{S}$. For this optimization process we again use Algorithm 3 (now, with tolerance greater then tol_{pos}), which additionally accelerates the optimization process.

7 Conclusions

Damping optimization for a mechanical vibrating system is a very demanding problem due to the numerous Lyapunov equations which have to be solved. In this paper, we have considered the case when just a certain part of the spectrum has to be damped (the critical part). In this case, the right hand side of the considered Lyapunov equation has small rank. Thus, we propose an algorithm for the dimension reduction which uses this property and also exploits the structure of the system. An error bound obtained from perturbation theory is employed to determine which part of the matrix can be neglected in order to have a good approximation of the trace by solving a Lyapunov equation of much smaller size. Numerical experiments confirm the efficiency of the new algorithm for viscosity optimization, which considerably accelerates the optimization process, while ensuring that we still find the optima within the limits of the given tolerances. Regarding the problem of determining damping positions, in the second example we have shown that using our approach can significantly reduce the number of possible positions.

In the derivation of the method presented in this paper, we have also tried several model reduction methods based on existing approaches as described in [1]. These attempts have not proven successful, but this is an important area for future work. Future work will also include the consideration of large scale problems, for that purpose one interesting issue if avoiding the explicit calculation of the matrix Φ.

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