Perturbation of Multiple Eigenvalues

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Abstract

This paper is concerned with the perturbation of the multiple eigenvalue \( \mu \) of Hermitian matrices of the form \( A = \text{diag}(\mu I, A_{22}) \), when the matrix undergoes an off-diagonal perturbation \( E \) whose columns have widely varying magnitudes. When some of \( E \)'s columns are much smaller than the others, some copies of \( \mu \) are much insensitive than any existing bound suggests. We explain this phenomenon by showing that when \( A_{22} - \mu I \) is definite the

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ith bound scales quadratically with the norm of the ith column, and in the indefinite case the bound is necessarily proportional to the product of E’s ith column norm and E’s norm. An extension to generalized Hermitian eigenvalue problems is presented.

**Keywords:** Graded perturbation, multiple eigenvalue, generalized eigenvalue problem

**2000 MSC:** 15A22, 15A42, 65F15

1. Introduction

Consider the eigenvalue problem for a Hermitian matrix \( \tilde{A} \) of the form

\[
\tilde{A} = \begin{pmatrix} A_{11} & E^* \\ E & A_{22} \end{pmatrix}, \quad A_{11} = \mu I_m,
\]

where the superscript “\( \cdot^* \)” takes the complex conjugate transpose of a matrix or a vector, and \( I_m \) (or simply \( I \) later if its dimension is clear from the context) is the \( m \times m \) identity matrix. If \( E \) is a zero block, then \( \mu \) is a multiple eigenvalue with multiplicity \( m \). In general, if \( E \) is small then \( \tilde{A} \) has \( m \) eigenvalues close to \( \mu \). In fact more can be said qualitatively. Let \( \eta \) be the eigenvalue gap between \( A_{11} = \mu I \) and \( A_{22} \) defined as

\[
\eta = \min_{\nu \in \text{eig}(A_{22})} |\mu - \nu|,
\]

where \( \text{eig}(A_{22}) \) is the set of the eigenvalues of \( A_{22} \), and

\[
\varepsilon = \|E\|_2,
\]

where \( \|\cdot\|_2 \) is either the spectral norm of a matrix or the \( \ell_2 \)-norm of a vector. The main result in [2] says \( \tilde{A} \) has \( m \) eigenvalues \( \theta_1, \ldots, \theta_m \) such that

\[
|\mu - \theta_j| \leq \frac{2\varepsilon^2}{\eta + \sqrt{\eta^2 + 4\varepsilon^2}} \quad \text{for } 1 \leq j \leq m,
\]

The right-hand side is of second order in \( \varepsilon \) if \( \eta > 0 \) and is never larger than \( \varepsilon \). As confirmed by the 2-by-2 example in [2], in general these inequalities cannot be improved without knowing more information on \( E \) than just \( \varepsilon = \|E\|_2 \).
Suppose now that we do have additional information on $E$. For example, consider the case where the first column of $E$ is zero, which implies $\theta_j = \mu$ for some $j$. Can we derive bounds that reflect this? A possible quick answer can be given as follows: first zero out the $j$th column of $E$, and then use the well-known perturbation theorem (attributed to Lidskii, Weyl, Wielandt and Mirsky in various forms [7, pp.196-205]) to conclude that $\tilde{A}$ has an eigenvalue $\theta$ that differs from $\mu$ by no more than $\|E(:,j)\|_2$, where $E(:,j)$ denotes the $j$th column of $E$. It obviously implies that if $E$ has a zero column, then $\mu$ must be an eigenvalue of $\tilde{A}$. But there are two drawbacks to this quick answer:

1. $\|E(:,j)\|_2$ can be potentially (much) larger than the right-hand side of (1.4), making the estimate less favorable to (1.4).
2. This does not imply that $\tilde{A}$ has $m$ eigenvalues $\theta_j$ such that $|\mu - \theta_j| \leq \|E(:,j)\|_2$ because some of the $\theta$ by this argument could be the same eigenvalues of $\tilde{A}$, as mentioned in [5, Sec. 11.5].

The purpose of this article is to develop a theory that will reflect the effect of disparity in the magnitudes of the columns of $E$ on the eigenvalues of $\tilde{A}$, unlike (1.4). Our results give different bounds for the $m$ eigenvalues of $\tilde{A}$ closest to $\mu$.

For the sake of convenience, throughout this paper $\eta$ and $\varepsilon$ are always defined by (1.2) and (1.3), respectively, and set

$$\epsilon_j = \|E(:,i_j)\|_2 \quad \text{for} \quad 1 \leq j \leq m, \quad (1.5)$$

where $\{i_1, i_2, \ldots, i_m\}$ is the permutation of $\{1, 2, \ldots, m\}$ such that

$$\epsilon_1 \leq \epsilon_2 \leq \cdots \leq \epsilon_m. \quad (1.6)$$

It is well-known that $\epsilon_m \leq \varepsilon \leq \sqrt{m} \epsilon_m$. We will use $X \prec Y$ ($X \preceq Y$) for two Hermitian matrices of the same size to mean $Y - X$ is positive definite (semi-definite), and $X \succ Y$ ($X \succeq Y$) to mean $Y - X$ ($Y \preceq X$). In particular, $X \succ 0$ ($X \succeq 0$) means that $X$ is positive definite (semi-definite).

The rest of this paper is organized as follows. We first investigate specific examples in section 2, which provide insights into possible bounds that could be expected. In section 3 we give our main results, in which we separately deal with the cases where $A_{22} - \mu I$ is definite or indefinite. For the indefinite case, we give asymptotic estimates that are correct up to fourth-order terms, as well as strict bounds that are slightly larger than the asymptotic estimates. In section 4 we describe how our bounds can be extended to the generalized eigenvalue problem. Finally we summarize our conclusions in section 5.
2. Motivational Examples

The examples below will shape our expectation on possible effects of different magnitudes of the columns of $E$ on the eigenvalues of $\tilde{A}$ nearest 0.

**Example 2.1.** Consider the 4-by-4 matrix $\tilde{A}$ given by

$$
E = \begin{pmatrix} 3 \cdot 10^{-4} & -2 \cdot 10^{-2} \\ 2 \cdot 10^{-4} & -2 \end{pmatrix}, \quad A_{22} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \tilde{A} = \begin{pmatrix} 0 & E^* \\ E & A_{22} \end{pmatrix}.
$$

In this case $A_{11} = 0$, i.e., $\mu = 0$ in (1.1), and $\eta = 1$.

The two eigenvalues of $\tilde{A}$ closest to 0 are approximately

$$
1.632172864323117 \cdot 10^{-7} \text{ and } -3.00063252828267 \cdot 10^{-4}, \quad (2.1)
$$

which are about $\epsilon_1^2 = \|E(:,1)\|_2^2 = 1.3000 \cdot 10^{-7}$ and $\epsilon_2^2 = \|E(:,2)\|_2^2 = 5.0000 \cdot 10^{-4}$, respectively.

The inequality (1.4) says $\tilde{A}$ has two eigenvalues that differ from 0 by no more than $4.9978 \cdot 10^{-4}$. This estimate is very good for the second eigenvalue in (2.1) but not so for the first one which is about less than the square of the estimate. The quick answer, on the other hand, says $\tilde{A}$ has an eigenvalue that differs from 0 by no more than $\epsilon_1 = 3.6056 \cdot 10^{-4}$ and an eigenvalue from 0 by no more than $\epsilon_2 = 2.2361 \cdot 10^{-2}$, providing even worse estimates than by (1.4).

Example 2.1 may lead us to believe that there are $m$ properly ordered eigenvalues $\theta_1, \ldots, \theta_m$ of $\tilde{A}$ with each difference $|\mu - \theta_j|$ being of second order in $\epsilon_j = \|E(:,j)\|_2$ if $\eta > 0$. Later we will show this is indeed true if $A_{22} - \mu I$ is definite, but not in the general case as we can see by the next example.

**Example 2.2.** Consider the 4-by-4 matrix $\tilde{A}$ given by

$$
E = \begin{pmatrix} \delta_1 & 0 \\ 0 & \delta_2 \end{pmatrix}, \quad A_{22} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \tilde{A} = \begin{pmatrix} 0 & E^* \\ E & A_{22} \end{pmatrix},
$$

where both $\delta_i$ are real numbers and $|\delta_i| \leq 1$. The characteristic equation of $\tilde{A}$ is

$$
\lambda^4 - (\delta_1^2 + \delta_2^2 + 1)\lambda^2 + \delta_1^2\delta_2^2 = 0,
$$

whose two smallest eigenvalues in magnitude satisfy

$$
|\lambda| = \sqrt{\frac{2}{\delta_1^2 + \delta_2^2 + 1 + \sqrt{[1 + (\delta_1 + \delta_2)^2][1 + (\delta_1 - \delta_2)^2]}}} |\delta_1\delta_2|.
$$

4
Thus \(|\lambda|/|\delta_1\delta_2| = 1 + O(\delta_1^2 + \delta_2^2)|. It follows that the smaller \(|\lambda|\) can be made arbitrarily larger than \(O(\min\{\delta_1^2, \delta_2^2\})\).

\[\begin{align*}
\text{3. Main Results} \\
\text{Throughout this section, } \tilde{A} \text{ is Hermitian and given by (1.1). Without loss of generality, we assume } \\
\mu = 0, \\
\text{and thus the gap } \eta \text{ as defined by (1.2) is} \\
\eta = 1/\|A_{22}^{-1}\|_2.
\end{align*}\]

For any \(\lambda \notin \text{eig}(A_{22})\), set

\[X = \begin{pmatrix} I & -E^*(A_{22} - \lambda I)^{-1} \\ 0 & I \end{pmatrix}.\]

Then

\[X(\tilde{A} - \lambda I)X^* = \begin{pmatrix} (-\lambda)I - E^*(A_{22} - \lambda I)^{-1}E \\ A_{22} - \lambda I \end{pmatrix}, \quad (3.1)\]

and thus

\[\det(\tilde{A} - \lambda I) = \det(-E^*(A_{22} - \lambda I)^{-1}E - \lambda I) \times \det(A_{22} - \lambda I). \quad (3.2)\]

From this we see that the eigenvalues of \(\tilde{A}\) consist of two subsets: one spawned from \(m\) copies of \(\mu\) and another from \(\text{eig}(A_{22})\) upon being moved by \(E\). Suppose \(E\) is sufficiently tiny in magnitude. Then \(\tilde{A}\) has \(m\) eigenvalues very close to 0 and these \(m\) eigenvalues are the zeros of

\[\det(-E^*(A_{22} - \lambda I)^{-1}E + (-\lambda)I) \quad (3.3)\]

near 0. Note that

\[-E^*(A_{22} - \lambda I)^{-1}E + (-\lambda)I = -\sum_{j=0}^{\infty} \lambda^j E^* A_{22}^{-j-1} E + (-\lambda)I. \quad (3.4)\]

\textbf{Theorem 3.1.} \textit{Let } \tilde{A} \textit{ be a Hermitian matrix of form (1.1) with } \mu = 0. \textit{ Then}

1. Assume \(\varepsilon < \sqrt{3/4}\eta\). Then
(a) \( \tilde{A} \) has exactly \( m \) eigenvalues \( \theta_j \) in the open interval \((-\eta/2, \eta/2)\), and moreover
\[
|\theta_j| \leq \frac{2\varepsilon^2}{\eta + \sqrt{\eta^2 + 4\varepsilon^2}},
\]
for \( 1 \leq j \leq m \);
(b) The function (3.3) has exactly \( m \) zeros in \((-\eta/2, \eta/2)\) and these zeros are precisely the eigenvalues \( \theta_j \) of \( \tilde{A} \).

2. \( \tilde{A} \) has \( m \) eigenvalues \( \theta_j = \vartheta_j + O(\varepsilon^4) \), where \( \vartheta_j, j = 1, \ldots, m \) are the eigenvalues of \(-E^*A_{22}^{-1}E\).

Proof. Since \( 4t^2/(1 + \sqrt{1 + 4t^2}) < 1 \) if \( t^2 < 3/4 \), we have
\[
\frac{2\varepsilon^2}{\eta + \sqrt{\eta^2 + 4\varepsilon^2}} < \frac{\eta}{2} \text{ if } \frac{\varepsilon}{\eta} < \sqrt{\frac{3}{4}}.
\]

By the main result of [2], we conclude that \( \tilde{A} \) has exactly \( m \) eigenvalues \( \theta_j \) in the open interval \((-\eta/2, \eta/2)\) and (3.5) holds.

Item 1(b) is a consequence of Item 1(a), (3.2) and \( \det(A_{22} - \lambda I) \neq 0 \) for \( \lambda \in (-\eta/2, \eta/2) \).

The expression in (3.4) is equal to \(-E^*A_{22}^{-1}E + (-\lambda)I\), up to \( O(\varepsilon^4) \), for \( |\lambda| \leq O(\varepsilon^2) \). Since by (1.4) \( \tilde{A} \) has exactly \( m \) eigenvalues no larger than \( O(\varepsilon^2) \) in magnitude, we conclude that \( \theta_j = \vartheta_j + O(\varepsilon^4) \).

\[\square\]

Example 2.1 (revisit). The eigenvalues of \(-E^*A_{22}^{-1}E\) are
\[
1.632173307879875 \cdot 10^{-7}, \quad -3.002132173307880 \cdot 10^{-4}
\]
which are extremely close to the exact values given in (2.1).

Theorem 3.1 gives asymptotic estimates for \( \theta_j \) in terms of \( \vartheta_j \). In the subsections that follow, we will establish bounds that reflect the effect of disparity in the magnitudes of the columns of \( E \). To this end, we normalize the columns of \( E \) by their \( \ell_2 \)-norms to get
\[
E = E_0D,
\]
where
\[
D = \text{diag}(\|E(:,1)\|_2, \|E(:,2)\|_2, \ldots, \|E(:,m)\|_2), \quad (3.7a)
\]
\[
(E_0)_{(c,:)} = \begin{cases} 
E_{(c,:)}/\|E_{(c,:)}\|_2, & \text{if } E_{(c,:)} \neq 0, \\
0, & \text{if } E_{(c,:)} = 0.
\end{cases} \quad (3.7b)
\]
Lemma 3.1. Let $\tau_1, \tau_2, \ldots, \tau_m$ be the eigenvalues of $E^*E$, arranged in ascending order:
\begin{equation}
0 \leq \tau_1 \leq \tau_2 \leq \cdots \leq \tau_m,
\end{equation}
and let $\epsilon_j$ be defined as in (1.5) and (1.6). Then
\begin{equation}
\tau_j \leq \|E_0\|_2^2 \epsilon_j^2 \leq m \epsilon_j^2.
\end{equation}

Proof. Use $0 \preceq E^*E = DE_0^*E_0D \preceq \|E_0\|_2^2 \epsilon_j^2$ to get $\tau_j \leq \|E_0\|_2^2 \epsilon_j^2$. The second inequality is due to $\|E_0\|_2 \leq \sqrt{m}$. \hfill $\square$

Next, we separately consider the cases according to whether $A_{22}$ is definite or not. All bounds will be given in terms of $\tau_j$. Corresponding bounds in terms of $\epsilon_j$ can then be easily derived by using (3.9).

3.1. Positive (negative) definite $A_{22}$

Theorem 3.2. For Hermitian matrix $\tilde{A}$ as in (1.1) with $\mu = 0$, suppose $\varepsilon < \sqrt{3/4} \eta$. If $A_{22}$ is positive (negative) definite, then $\tilde{A}$ has $m$ nonpositive (nonnegative) eigenvalues $\theta_1, \ldots, \theta_m$ arranged in ascending order satisfying
\begin{align}
0 &\leq -\theta_m - j + 1 \leq \frac{2\tau_j}{\eta + \sqrt{\eta^2 + 4\tau_j}}, \quad \text{if } A_{22} > 0, \tag{3.10a} \\
0 &\leq \theta_j \leq \frac{2\tau_j}{\eta + \sqrt{\eta^2 + 4\tau_j}}, \quad \text{if } A_{22} < 0, \tag{3.10b}
\end{align}
for $1 \leq j \leq m$.

Proof. The case in which $A_{22} < 0$ can be turned into the case in which $A_{22} > 0$ by considering $-\tilde{A}$ instead. Suppose that $A_{22} > 0$, i.e., $A_{22}$ is positive definite. Set
\begin{equation}
B(t) = -E^*(A_{22} - tI)^{-1}E. \tag{3.11}
\end{equation}
By Theorem 3.1 and the assumption $\varepsilon < \sqrt{3/4} \eta$, we know $\tilde{A}$ has exactly $m$ eigenvalues in $(-\eta/2, \eta/2)$ and these $m$ eigenvalues are the zeros of $\det (B(\lambda) - \lambda I)$ in $(-\eta/2, \eta/2)$. Since for any $\lambda \in (-\eta/2, \eta/2)$, $0 \preceq A_{22} - \lambda I$ and thus $B(\lambda) \preceq 0$; so
\begin{equation}
B(\lambda) - \lambda I < 0 \quad \text{for } \lambda \in (0, \eta/2).
\end{equation}
Therefore the $m$ eigenvalues of $\tilde{A}$ are in $(-\eta/2, 0]$. Denote them by

$$-\eta/2 < \theta_1 \leq \theta_2 \leq \cdots \leq \theta_m \leq 0.$$ 

Also denote by

$$\lambda_1(t) \leq \lambda_2(t) \leq \cdots \leq \lambda_m(t) \leq 0$$

the $m$ eigenvalues of $B(t)$ for $t \in (-\eta/2, 0]$. They are continuous. The fixed points of $\lambda_i(t)$ within $(-\eta/2, 0]$ give all the $\theta_j$. In fact, we have $\lambda_j(\theta_j) = \theta_j$. This is because $\lambda_j(t)$ is a decreasing function for $t \in (-\eta/2, 0]$ and thus $\lambda_j(t) = t$ has a unique solution on $(-\eta/2, 0]$. Hence $\theta_j$ is the $j$th smallest eigenvalue of $-B(\theta_j)$. Since

$$-B(\theta_j) = E^*(A_{22} - \theta_j I)^{-1}E \leq \frac{E^*E}{\eta + |\theta_j|},$$

we have

$$|\theta_j| \leq \frac{\tau_{m-j+1}}{\eta + |\theta_j|} \Rightarrow |\theta_j| \leq \frac{2\tau_{m-j+1}}{\eta + \sqrt{\eta^2 + 4\tau_{m-j+1}}}$$

which gives (3.10a).

**Remark 3.1.** Since the right-hand sides in (3.10) are increasing as $\tau_j$ does, replacing $\tau_j$ by its upper bound in (3.9) yields bounds on $|\theta_j|$ in terms of $\epsilon_j$, the norms of $E$’s columns.

**Example 3.1.** Consider the 4-by-4 matrix $\tilde{A}$ given by

$$E = \left(\begin{array}{cc}
3 \cdot 10^{-4} & -2 \cdot 10^{-2} \\
2 \cdot 10^{-4} & 10^{-2}
\end{array}\right), \quad A_{22} = \left(\begin{array}{cc}
1 & 0 \\
0 & 2
\end{array}\right), \quad \tilde{A} = \left(\begin{array}{cc}
0 & E^* \\
E & A_{22}
\end{array}\right).$$

In this case $A_{11} = 0$, i.e., $\mu = 0$ in (1.1), and $\eta = 1$. The following table displays the eigenvalues $\theta_j$ of $\tilde{A}$ nearest to 0, the eigenvalues $\vartheta_j$ of $-E^* A_{22}^{-1} E$, and the bounds by (3.10) and the ones after $\tau_j$ replaced by $m\epsilon_j^2$.

<table>
<thead>
<tr>
<th>$\theta_j$</th>
<th>$\vartheta_j$</th>
<th>$\frac{2\tau_j}{\eta + \sqrt{\eta^2 + 4\tau_j}}$</th>
<th>$\frac{2m\epsilon_j^2}{\eta + \sqrt{\eta^2 + 4m\epsilon_j^2}}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$-4.4986 \cdot 10^{-4}$</td>
<td>$-4.5006 \cdot 10^{-4}$</td>
<td>$4.9978 \cdot 10^{-4}$</td>
<td>$9.9900 \cdot 10^{-4}$</td>
</tr>
<tr>
<td>$-5.4438 \cdot 10^{-8}$</td>
<td>$-5.4438 \cdot 10^{-8}$</td>
<td>$9.7994 \cdot 10^{-8}$</td>
<td>$2.6000 \cdot 10^{-7}$</td>
</tr>
</tbody>
</table>

The bounds are remarkably sharp.

\[\diamond\]
3.2. Indefinite $A_{22}$

Consider now that $A_{22}$ is indefinite. We will use the following result, which is a direct consequence of the proof of [4, Proof of Theorem 1].

**Lemma 3.2.** Let $W$ be an $\ell$-by-$\ell$ Hermitian matrix, and let $D = \text{diag}(\delta_1, \delta_2, \ldots, \delta_\ell)$ with $|\delta_1| \leq |\delta_2| \leq \cdots \leq |\delta_\ell|$. Denote the eigenvalues of $D^*W D$ by $\omega_1, \ldots, \omega_\ell$ arranged such that $|\omega_1| \leq |\omega_2| \leq \cdots \leq |\omega_\ell|$. Then for $i = 1 : \ell$

$$|\omega_i| \leq \min_{1 \leq j \leq \ell - i + 1} |\delta_{\ell - j + 1}\delta_{i + j - 1}| \|W\|_2 \leq |\delta_{\ell}\delta_i| \|W\|_2.$$ 

Two types of bounds will be proven for the eigenvalues of interest $\theta_j$ of $\tilde{A}$: asymptotical bounds up to $O(\varepsilon^4)$, and strict bounds at a tradeoff of being slightly larger than the asymptotic bounds with higher order terms $O(\varepsilon^4)$ ignored.

**Lemma 3.3.** Let $\vartheta_j$ ($j = 1 : m$) be the eigenvalues of $-E^*A_{22}^{-1}E$ arranged such that

$$|\vartheta_1| \leq |\vartheta_2| \leq \cdots \leq |\vartheta_m|.$$ 

Then

$$|\vartheta_j| \leq \zeta_j \overset{\text{def}}{=} \frac{1}{\eta} \min_{1 \leq k \leq m - j + 1} \sqrt{\tau_{m+1-k} \tau_{j+k-1}} \quad (3.12a)$$

$$\leq \frac{1}{\eta} \sqrt{\tau_m \tau_j}, \quad (3.12b)$$

where $\tau_i$ ($i = 1 : m$) are the eigenvalues of $E^*E$ as in Lemma 3.1.

**Proof.** Inequality (3.12b) follows from (3.12a) by simply picking $k = 1$ without the minimization.

We now prove (3.12a). Let $E = U \Sigma V^*$ be its SVD, where $U$ and $V$ are unitary. We have $E^*A_{22}^{-1}E = V \Sigma U^*A_{22}^{-1}U \Sigma V^*$ which has the same eigenvalues as $\Sigma U^*A_{22}^{-1}U \Sigma$. It can then be seen that we can write

$$\Sigma U^*A_{22}^{-1}U \Sigma = DW D, \quad D = \text{diag}(\sqrt{\tau_1}, \sqrt{\tau_2}, \ldots, \sqrt{\tau_m}), \quad \|W\|_2 \leq 1/\eta.$$ 

Apply Lemma 3.2 to complete the proof.

**Remark 3.2.** When $A_{22}$ is definite, we can get $|\vartheta_j| \leq \tau_j/\eta$ which is sharper than (3.12a) and thus (3.12b).
Theorem 3.3. For Hermitian matrix $\tilde{A}$ as in (1.1) with $\mu = 0$, suppose $\varepsilon < \sqrt{3/4} \eta$. Then $\tilde{A}$ has $m$ eigenvalues $\theta_1, \ldots, \theta_m$ arranged such that

$$|\theta_1| \leq |\theta_2| \cdots \leq |\theta_m|$$

(3.13)

satisfying

$$|\theta_j| \leq \zeta_j + O(\varepsilon^4),$$

(3.14)

where $\zeta_j$ is defined by (3.12a).

Proof. It is a consequence of Theorem 3.1 and Lemma 3.3.

Next we derive strict bounds, i.e., without the term $O(\varepsilon^4)$. One difficulty here is that $\lambda_i(t)$ is no longer monotonic. However, the fact remains true that if $\theta_i \in (-\eta/2, \eta/2)$ is an eigenvalue of $B(\theta_i) = -E^*(A_{22} - \theta_i I)^{-1}E$, then $\theta_i$ is also an eigenvalue of $\tilde{A}$.

Lemma 3.4. Let $B(t)$ be defined as in (3.11) with eigenvalues

$$\lambda_1(t) \leq \lambda_2(t) \leq \cdots \leq \lambda_m(t)$$

(3.15)

each of which are piecewise differentiable. If $\varepsilon < \eta/2$, then

$$\left\| \frac{dB(t)}{dt} \right\|_2 \leq \frac{4\varepsilon^2}{\eta^2} < 1 \quad \text{and} \quad \left| \frac{d\lambda_j(t)}{dt} \right| \leq \frac{4\varepsilon^2}{\eta^2} < 1 \quad \text{for} \quad t \in (-\eta/2, \eta/2).$$

(3.16)

Proof. We have

$$B(t) - B(t + \Delta t) = E^*(A_{22} - tI)^{-1}E - E^* [A_{22} - (t + \Delta t)I]^{-1}E$$

$$= E^* \left\{ (A_{22} - tI)^{-1} - [A_{22} - (t + \Delta t)I]^{-1} \right\} E$$

$$= E^* (A_{22} - tI)^{-1} \left\{ I - [I - \Delta t(A_{22} - tI)^{-1}]^{-1} \right\} E.$$
Therefore
\[
\left\| \frac{B(t) - B(t + \Delta t)}{\Delta t} \right\|_2 \leq \frac{\left\| E^*(A_{22} - tI)^{-1} \left\{ I - \left[ I - \Delta t(A_{22} - tI)^{-1} \right] \right\} E \right\|_2}{|\Delta t|} \leq \frac{\varepsilon \left\| (A_{22} - tI)^{-1} \right\|_2 \left\| I - \left[ I - \Delta t(A_{22} - tI)^{-1} \right] \right\|_2}{|\Delta t|} \varepsilon.
\]

Noting that for \( t \in (-\eta/2, \eta/2) \), we have
\[
\left\| (A_{22} - tI)^{-1} \right\|_2 \leq \frac{2}{\eta},
\]
\[
\left\| I - \left[ I - \Delta t(A_{22} - tI)^{-1} \right] \right\|_2 \leq \frac{1}{1 - |\Delta t| \cdot 2/\eta} - 1
\]
\[
= \frac{|\Delta t| \cdot 2/\eta}{1 - |\Delta t| \cdot 2/\eta},
\]
and thus
\[
\left\| \frac{B(t) - B(t + \Delta t)}{\Delta t} \right\|_2 \leq \frac{\varepsilon^2 \cdot 2/\eta \cdot \frac{|\Delta t| \cdot 2/\eta}{1 - |\Delta t| \cdot 2/\eta}}{4\varepsilon^2} = \frac{\eta^2(1 - |\Delta t| \cdot 2/\eta)}{4\varepsilon^2}.
\]

Let \( \Delta t \to 0 \) to get
\[
\left\| \frac{dB(t)}{dt} \right\|_2 \leq \frac{4\varepsilon^2}{\eta^2} < 1,
\]
since \( \varepsilon < \eta/2 \). Finally, we use the well-known perturbation theorem (attributed to Lidskii, Weyl, Wiedlandt and Mirsky in various forms [7, pp.196-205]) to conclude that
\[
\left| \frac{d\lambda_j(t)}{dt} \right| \leq \left\| \frac{dB(t)}{dt} \right\|_2 \leq \frac{4\varepsilon^2}{\eta^2} < 1,
\]
as expected. \( \square \)

**Theorem 3.4.** For Hermitian \( \tilde{A} \) as in (1.1), if \( \varepsilon < \eta/2 \), then \( \tilde{A} \) has \( m \) eigenvalues \( \theta_1, \ldots, \theta_m \) (arranged as in (3.13)) satisfying
\[
|\theta_j| \leq \frac{\zeta_j}{1 - 4\rho^2},
\]
for \( j = 1 : m \), where \( \rho = \varepsilon/\eta < 1/2 \) and \( \zeta_j \) is defined by (3.12a).
Proof. Instead of proving (3.17) directly, we shall prove that for any given
$1 \leq j \leq m$ there are $j$ of $\theta_i$’s satisfying $|\theta_i| \leq \zeta_j/(1-4\rho^2)$. Thus (3.17) must hold.

Adopt the notations in Lemmas 3.3 and 3.4. By (3.16), for any $t \in (-\eta/2, \eta/2)$, we have

$$|\lambda_i(t) - \lambda_i(0)| \leq \int_0^t \left| \frac{d\lambda_i(\tau)}{d\tau} \right| d\tau \leq \frac{4\varepsilon^2 |t|}{\eta^2} = 4\rho^2 |t| \quad (3.18)$$

for $i = 1 : m$. Let $\delta_j = \frac{\zeta_j}{1-4\rho^2}$. We claim that there are at least $j$ of $\lambda_i(t)$ such that

$$\lambda_i(t) \in [-\delta_j, \delta_j] \quad \text{for all } t \in [-\delta_j, \delta_j]. \quad (3.19)$$

This means that each particular $\lambda_i(t)$ of those maps the interval $[-\delta_j, \delta_j]$ into itself, so by Brouwer’s fixed point theorem, $\lambda_i(t)$ has a fixed point $t_0 \in [-\delta_j, \delta_j]$ such that $\lambda_i(t_0) = t_0$ which thus is an eigenvalue of $\tilde{A}$. Note that the second inequality in (3.16) implies that $t_0$ is a unique fixed point of $\lambda_i(t)$ in $(-\eta/2, \eta/2)$. Therefore all counted, $\tilde{A}$ has at least $j$ eigenvalues $\theta_k$ in $[-\delta_j, \delta_j]$.

It remains to show that there are at least $j$ of $\lambda_i(t)$ satisfying (3.19). To see this, we notice

$$\vartheta_k \in [-\zeta_k, \zeta_k] \subseteq [-\zeta_j, \zeta_j] \subseteq [-\delta_j, \delta_j] \quad \text{for } k = 1 : j. \quad \text{(3.20)}$$

These $\vartheta_k$ for $k = 1 : j$ are taken by $j$ different $\lambda_i(t)$ at $t = 0$, i.e., $\vartheta_k = \lambda_{\ell_k}(0)$, where $\ell_k \in \{1 : m\}$ are distinct for $k = 1 : j$. We now prove that $\lambda_{\ell_k}(t)$ for $k = 1 : j$ are the $j$ of $\lambda_i(t)$ satisfying (3.19). In fact, for $t \in [-\delta_j, \delta_j]$ and $k = 1 : j$

$$|\lambda_{\ell_k}(t)| \leq |\lambda_{\ell_k}(0)| + |\lambda_{\ell_k}(t) - \lambda_{\ell_k}(0)|$$

$$= |\vartheta_k| + |\lambda_{\ell_k}(t) - \lambda_{\ell_k}(0)|$$

$$\leq \zeta_j + 4\rho^2 \delta_j$$

$$= \delta_j,$$

as expected. \hfill \Box

Remark 3.3. Compared with (3.14) and (3.17) in Theorem 3.3, the bound in (3.17) removes the term $O(\varepsilon^4)$ at the expense of the factor $(1 - 4\rho^2)^{-1}$. 

12
Example 2.1 (revisit). The following table displays the eigenvalues $\theta_j$ of $\tilde{A}$ nearest to 0, the eigenvalues $\vartheta_j$ of $-E^*A_{22}^{-1}E$, and the bounds by (3.17) and the ones after $\tau_j$ replaced by $m\epsilon_j^2$.

<table>
<thead>
<tr>
<th>$\theta_j$</th>
<th>$\vartheta_j$</th>
<th>$\frac{\zeta_j}{1-4\rho^2}$</th>
<th>$\frac{m\epsilon_j\tau_m}{1-4\rho^2}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$-3.0006 \cdot 10^{-4}$</td>
<td>$-3.002 \cdot 10^{-4}$</td>
<td>$5.0103 \cdot 10^{-4}$</td>
<td>$1.0020 \cdot 10^{-3}$</td>
</tr>
<tr>
<td>$1.6322 \cdot 10^{-7}$</td>
<td>$1.6322 \cdot 10^{-7}$</td>
<td>$7.0140 \cdot 10^{-6}$</td>
<td>$1.6157 \cdot 10^{-5}$</td>
</tr>
</tbody>
</table>

The bounds are rather sharp.

4. Possible extensions to the generalized eigenvalue problem

So far we have focused on the Hermitian eigenvalue problem (1.1). We now consider the following Hermitian definite generalized eigenvalue problem

$$\tilde{A} = \left( \mu B_{11} E^* \right), \quad \tilde{B} = \left( \begin{array}{cc} B_{11} F^* \\ E_{22} \end{array} \right),$$

(4.1)

where $B_{ii} > 0$, and $\|F\|_2$ is sufficiently small\(^5\) so that $\tilde{B} > 0$ also.

If $E = F = 0$, then $\mu$ is an eigenvalue of the pencil $A - \lambda B$ of multiplicity $m$. In this section we outline how to develop perturbation bounds using what we have gotten in section 3.

4.1. Special Case: $B_{ii} = I$ and $\mu = 0$

In this case,

$$\tilde{A} = \left( \begin{array}{cc} 0 & E^* \\ E & A_{22} \end{array} \right), \quad \tilde{B} = \left( \begin{array}{cc} I_m & F^* \\ F & I_n \end{array} \right).$$

(4.2)

Assuming that $\|F\|_2 < 1$, we can apply a similar approach to the one in [3, section 2.1]. We first let

$$X = \left( \begin{array}{cc} I_m & -F^* \\ 0 & I_n \end{array} \right), \quad W = \left( \begin{array}{cc} I_m & 0 \\ 0 & [I - FF^*]^{1/2} \end{array} \right),$$

(4.3)

and then let

$$\hat{B} \overset{\text{def}}{=} X^*\tilde{B}X = \left( \begin{array}{cc} I_m & 0 \\ 0 & I - FF^* \end{array} \right) = W^2,$$

(4.4a)

\(^5\)For example, $\|F\|_2 < \min_i\{\sigma_{\text{min}}(B_{ii})\}$ which guarantees $\hat{B} > 0$, where $\sigma_{\text{min}}(B_{ii})$ is the smallest singular value of $B_{ii}$. 

13
\[ \tilde{A} \overset{\text{def}}{=} X^* \tilde{A} X = \begin{pmatrix} 0 & E^* \\ E & \tilde{A}_{22} \end{pmatrix}, \] 

(4.4b)

where \( W \) is the unique Hermitian definite square root \([1, \text{Ch. 6}]\) of \( \tilde{B} \), and \( \tilde{A}_{22} = A_{22} - EF^* - FE^* \).

\( \tilde{A} - \lambda \tilde{B} \) has the same eigenvalues as \( W^{-1} \tilde{A} W^{-1} - \lambda N \). Since \( W^{-1} \tilde{A} W^{-1} \) takes the form of (1.1), our theory in section 3 applies to \( W^{-1} \tilde{A} W^{-1} \), leading to various bounds.

4.2. General Case

Now we consider the general case (4.1). Assume \( \mu = 0 \); otherwise we shall consider 

\[ (\tilde{A} - \mu \tilde{B}) - \lambda \tilde{B} \]

instead. Suppose

\[ \tilde{A} = \begin{pmatrix} 0 & E^* \\ E & A_{22} \end{pmatrix}, \quad \tilde{B} = \begin{pmatrix} B_{11} & F^* \\ F & B_{22} \end{pmatrix}. \] 

(4.5)

Set \( Y = \text{diag}(B_{11}^{-1/2}, B_{22}^{-1/2}) \) to get

\[ Y^* \tilde{A} Y = \begin{pmatrix} 0 & \hat{E}^* \\ \hat{E} & \hat{A}_{22} \end{pmatrix}, \quad Y^* \tilde{B} Y = \begin{pmatrix} I_m & \hat{F}^* \\ \hat{F} & I_n \end{pmatrix}, \] 

(4.6)

returning to the case in subsection 4.1, where

\[ \hat{A}_{22} = B_{22}^{-1/2} A_{22} B_{22}^{-1/2}, \quad \hat{F} = B_{22}^{-1/2} F B_{11}^{-1/2}, \quad \hat{E} = B_{22}^{-1/2} E B_{11}^{-1/2}. \] 

(4.7)

5. Conclusion

We presented perturbation bounds on the changes of the multiple eigenvalue \( \mu \) of a Hermitian matrix under a perturbation in the off-diagonal block:

\[ A = \begin{pmatrix} \mu I_m & 0 \\ 0 & A_{22} \end{pmatrix} \text{ perturbed to } \tilde{A} = \begin{pmatrix} \mu I_m & E^* \\ E & A_{22} \end{pmatrix} \]

with an emphasis on the case when the magnitudes of \( E \)'s columns varying widely. We show that \( \tilde{A} \) has \( m \) eigenvalues \( \theta_i \ (i = 1 : m) \) such that the \( i \)th difference \( |\theta_i - \mu| \) is bounded by a quantity that is proportional to the square of the norm of \( E \)'s \( i \)th column when the \( A_{22} - \mu I \) is definite, but in the indefinite case the quantity is proportional to the product of the \( i \)th column norm and the norm of \( E \). We also outline an extension to the Hermitian definite generalized eigenvalue problem.
References


