Spectral representation of transition density of Fisher - Snedecor diffusion
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Abstract. Spectral properties of an ergodic diffusion with the Fisher-Snedecor invariant distribution are analyzed. We compute the spectral representation of the corresponding transition density. The spectral representation is given in terms of a sum involving finitely many eigenvalues and eigenfunctions (Fisher-Snedecor orthogonal polynomials) and an integral over the absolutely continuous spectrum of the corresponding Sturm-Liouville operator. This result enables the computation of the two-dimensional density of the Fisher-Snedecor diffusion as well as calculation of moments of the form $E\left[ X_{m+s}^{X_{n+s}} \right]$, where $m$ and $n$ are at most equal to the number of Fisher-Snedecor polynomials, which is particularly important for explicit calculations associated with this process.

Key words. Diffusion process, Fisher-Snedecor polynomials, Hypergeometric function, Infinitesimal generator, Sturm-Liouville equation, Transition density.

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1 Introduction

Historical roots. The study of diffusion processes with invariant distributions from the Pearson family started in the 1930’s, when Kolmogorov (26; 44) studied the Fokker-Planck or forward Kolmogorov equation

$$ \frac{\partial p}{\partial t} = -\frac{\partial}{\partial x} \left( \left( a_1 x + a_0 \right) p \right) + \frac{\partial^2}{\partial x^2} \left( \left( b_2 x^2 + b_1 x + b_0 \right) p \right), \quad p = p(x, t), \quad x \in \mathbb{R}, \quad t \geq 0, $$

with a linear drift and a quadratic squared diffusion. He also noticed that the corresponding invariant density $p(\cdot)$ satisfies the differential equation

$$ \frac{p'(x)}{p(x)} = \frac{\left( a_1 - 2b_2 \right) x + \left( a_0 - b_1 \right)}{b_2 x^2 + b_1 x + b_0} = \frac{c_1 x + c_0}{b_2 x^2 + b_1 x + b_0} = \frac{q(x)}{s(x)}, \quad x \in \mathbb{R}, \quad (1) $$

introduced by K. Pearson (41) in 1914 (and of illustrious history - see Diaconis and Zabel (15)), in order to unify some of the most important statistical distributions.

It seems appropriate to call this important class of processes Kolmogorov-Pearson (KP) diffusions, or Gauss-hypergeometric diffusions, due to the appearance of the Gauss $2F_1$ function (and its limiting confluent forms) in various explicit formulas.

For a long period of time after that, KP diffusions were neglected, with some notable exceptions like Wong (50), who in 1964 reemphasized the importance of this class of models as a most natural extension of the “first order statistical description characterized by $p(x)$” to a time dependent model, and computed
Mathematical finance motivations. Recently, the interest in these processes was reawakened in the context of financial modeling. The most famous case is the Merton-Black-Scholes SDE

$$dX_t = \theta (\mu - X_t) \, dt + \sigma_1 \, dW_{1}^{(1)} + \sigma_2 \, X_t \, dW_{1}^{(2)}, \quad t \geq 0, \quad \Rightarrow$$

$$dX_t = \theta (\mu - X_t) \, dt + \sqrt{\frac{\theta}{\nu - 1}} \left( (X_t - \mu)^2 + \delta^2 \right) \, dB_t, \quad t \geq 0,$$

where $k = \sigma_2^2$, $W_{1}^{(1)}$ and $W_{1}^{(2)}$, $t \geq 0$, are standard Brownian motions with correlation $\rho$, and $\{W_t, \, t \geq 0\}$ is a standard Brownian motion resulting from combining the two.

Notes: 1) The second formulation, to be called the skew Student parametrization, may be used for all KP diffusions (by restricting if necessary to the range of values of $x$ where the square root makes sense). In the first formulation, one implicitly assumes $|\rho| \leq 1, \, k \geq 0$, which characterize the Student subclass of KP diffusions – see below.

2) The linearity of the drift and the quadratic variance ensure the existence of polynomial eigenfunctions.

The Student parametrization of Kolmogorov-Pearson (KP) diffusions. Following the modern formulations of (43) and (18), consider the SDE

$$dX_t = \theta (\mu - X_t) \, dt + \sigma_1 \, dW_{1}^{(1)} + \sigma_2 \, X_t \, dW_{1}^{(2)}, \quad t \geq 0, \quad \Rightarrow$$

$$dX_t = \theta (\mu - X_t) \, dt + \sqrt{\frac{\theta}{\nu - 1}} \left( (X_t - \mu)^2 + \delta^2 \right) \, dB_t, \quad t \geq 0,$$

where $\delta^2 = (1 - \rho^2)(\frac{\sigma_1}{\sigma_2})^2$. This parametrization makes sense for the whole KP family (by allowing $\delta^2 \leq 0$), but it will only produce diffusions living on $(-\infty, \infty)$ when $\delta \in \mathbb{R}, \, k > 0$.

A classification of Kolmogorov-Pearson diffusions in six basic subfamilies may be achieved by using the criteria based on the degree $\deg(s)$ of the polynomial $s(x)$ from the denominator of the Pearson equation (1), the sign of its leading coefficient $b_2$ and the sign of its discriminant $\Delta(s)$ in the quadratic case. The classification is given in the following table.

<table>
<thead>
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<th>Pearson diffusion</th>
<th>Characteristic property</th>
<th>Invariant density</th>
<th>Parameter space</th>
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<tr>
<td>Ornstein-Uhlenbeck</td>
<td>$\deg(s) = 0$</td>
<td>$\frac{1}{\sqrt{\pi \sigma^2}} e^{-\frac{(x-\mu)^2}{2\sigma^2}}$</td>
<td>$\mu \in \mathbb{R}, , \sigma &gt; 0$</td>
</tr>
<tr>
<td>Gamma or CIR</td>
<td>$\deg(s) = 1$</td>
<td>$\frac{\alpha}{\Gamma(\alpha)} x^{\alpha-1} e^{-\alpha x} I_{(0, \infty)}(x)$</td>
<td>$\alpha &gt; 0, , \beta &gt; 0$</td>
</tr>
<tr>
<td>Beta or Jacobi</td>
<td>$b_2 &lt; 0, , \deg(s) = 2, , \Delta(s) &gt; 0$</td>
<td>$\frac{\beta}{\Gamma(\alpha + \beta)} x^{\alpha-1} (1 - x)^{\beta-1} I_{(0, 1)}(x)$</td>
<td>$\alpha &gt; 0, , \beta &gt; 0$</td>
</tr>
<tr>
<td>Student (see (6))</td>
<td>$b_2 &gt; 0, , \deg(s) = 2, , \Delta(s) &lt; 0$</td>
<td>$c_{\mu + \nu', \alpha, \delta} \exp \left( \frac{\delta(x + \mu')}{\mu' + \nu'} \right) \left[ 1 + \frac{\delta(x + \mu')}{\mu' + \nu'} \right]^{\frac{\nu}{\mu' + \nu'}}$</td>
<td>$\mu, \mu' \in \mathbb{R}, , \alpha &gt; 0, , \delta &gt; 0$</td>
</tr>
<tr>
<td>Reciprocal gamma</td>
<td>$b_2 &gt; 0, , \deg(s) = 2, , \Delta(s) = 0$</td>
<td>$\frac{\alpha}{\Gamma(\alpha)} x^{\alpha-1} e^{-\frac{\alpha}{\beta}} I_{(0, \infty)}(x)$</td>
<td>$\alpha &gt; 0, , \beta &gt; 1$</td>
</tr>
<tr>
<td>Fisher-Snedecor</td>
<td>$b_2 &gt; 0, , \deg(s) = 2, , \Delta(s) &gt; 0$</td>
<td>$\frac{\alpha}{\Gamma(\alpha + \beta)} x^{\alpha-1} (1 + \frac{\alpha + \beta}{\beta} x)^{-\frac{\alpha}{\beta}} I_{(0, \infty)}(x)$</td>
<td>$\alpha &gt; 2, , \beta &gt; 2$</td>
</tr>
</tbody>
</table>
Table 1. Classification of ergodic stationary Pearson diffusions.

Example: In the important Student case, the scale and invariant speed densities are:

\[ g(x) = \left(\delta(x^2 + 1)\right)^{\frac{1}{2}} e^{-\frac{\mu - \mu'}{\delta} \arctg \left(\frac{x}{\delta}\right)}, \quad m(x) = \frac{e^{\frac{\mu - \mu'}{\delta} \arctg \left(\frac{x}{\delta}\right)}}{\left(x^2 + 1\right)^{\frac{1}{2} + 1}}, x \in \mathbb{R}, \]

where we put \( \tilde{x} := (x - \mu')/\delta, 2\alpha = 1 - \frac{\nu}{\nu - 2} \).

When \( \nu > 1 \Leftrightarrow a > 0 \), the KP diffusions will have Student stationary distribution with \( \nu \) degrees of freedom, since the speed density may be normalized, arriving thus to diffusions with the invariant density:

\[ f(x) = c(\mu, \mu', a, \delta) \frac{\exp \left\{ \frac{\mu - \mu'}{\delta} \arctg \left(\frac{x-\mu'}{\delta}\right) \right\}}{\left[1 + \left(\frac{x-\mu'}{\delta}\right)^2\right]^{\frac{1}{2} + 1}}, \quad x \in \mathbb{R}, \tag{5} \]

where (see Avram et al. (7))

\[ c(\mu, \mu', a, \delta) = \frac{\Gamma \left(1 + \frac{1}{\nu} \right)}{\delta \sqrt{\pi} \Gamma \left(\frac{\nu}{2} + \frac{\nu}{2\nu - 2}\right)} \prod_{k=0}^{\infty} \left[1 + \left(\frac{\mu - \mu'}{2\delta} + k\right)^2\right]^{-1}. \tag{6} \]

The Ornstein-Uhlenbeck process, CIR process and Jacobi diffusion are well studied and widely applied. However, the first results concernig spectral and statistical analysis of reciprocal gamma, Student and Fisher-Snedecor diffusion, which all have heavy-tailed invariant distributions, are quite new (see (18; 29; 30)). Their study involves the analysis of the spectrum of the corresponding infinitesimal generators: Hermite, Laguerre and Jacobi polynomials, respectively. In the case of Pearson diffusions with heavy-tailed invariant distributions, the spectrum of the infinitesimal generator consist of two disjoint parts: the discrete spectrum (consisting of finitely many simple eigenvalues) and the essential spectrum. Furthermore, in all these cases corresponding eigenfunctions are less known finite systems of orthogonal polynomials: Bessel polynomials for reciprocal gamma diffusion (see (29)), Routh-Romanovski polynomials for Student diffusion (see (30)) and polynomials related to Fisher-Snedecor invariant distribution which have no common name (we will refer to them as to Fisher-Snedecor polynomials).

In this paper we focus on the analysis of spectral properties of the ergodic stationary Fisher-Snedecor diffusion, by which we mean solutions of the non-linear stochastic differential equation

\[ dX_t = -\theta (X_t - \mu) \ dt + \sqrt{2\theta X_t \left(\frac{X_t}{\beta/2 - 1} + \frac{\mu}{\alpha/2}\right)} \ dW_t, \quad t \geq 0, \tag{7} \]

with speed measure/invariant distribution proportional to:

\[ \frac{(\alpha x)^{\frac{\mu}{\beta}} - 1}{(\alpha x + \mu)\beta^{-2} + \frac{\mu}{\beta}}} I_{(0,\infty)}(x) \tag{8} \]

By taking \( \mu \) as the mean of the Fisher-Snedecor distribution \( \mu = \frac{\beta}{\beta - 2} \) (see Section 2), which may be always achieved by scaling, and by assuming \( \alpha > 0 \) and \( \beta > 2 \) (ensuring thus the existence of the first moment), we arrive to the classical Fisher-Snedecor distribution \( FS(\alpha, \beta) \), with \( (\alpha, \beta) \) degrees of freedom, as invariant distribution.

For various applications of stochastic processes it is often important to know its transition density, which for the most of diffusion models is not known in the explicit form. However, since for the Fisher-Snedecor diffusion the structure of the spectrum of the infinitesimal generator can be fully determined, it implies that it is possible to compute the spectral representation of its transition density. Therefore we present in Section 4 a closed expression for the spectral representation of the transition density of Fisher-Snedecor diffusion given in form of the finite sum related to discrete spectrum of the infinitesimal generator (i.e. eigenvalues and eigenfunctions - Fisher-Snedecor polynomials) and the integral which is taken over the absolutely continuous spectrum of the infinitesimal generator. The complicated form of this expression
surely has significant impact on possible applications of this diffusion. However, orthogonality of Fisher-
Snedecor polynomials and hypergeometric functions appearing in the continuous part of the spectral
representation makes this result applicable in the statistical analysis (see Remark 4.6) - it basically en-
ables calculation of moments of the form \( E[X^n_m] \), where \( m \) and \( n \) are at most equal to the number of
Fisher-Snedecor polynomials. For example, this result could be used for computation of the elements of the
covariance matrix and asymptotic confidence intervals for estimators of unknown parameters. This
suggests the importance of Fisher-Snedecor polynomials which have been ignored in standard mathemat-
ical books, for instance in Abramowitz and Stegun (1), Chihiara (13), Erdely (17), Nikiforov and Uvarov
(40) and Szegö (47). Significant progress in their research is recently made by Masjed-Jamei (35; 36) and
by Koepp and Masjed-Jamei (25).

Contents and main results. The purpose of the Introduction is to place the observed problem in the
historical as well as in the corresponding contemporary framework. In Section 2 well known information on
Fisher-Snedecor distribution and its moments are presented. In Section 3 the Fisher-Snedecor diffusion is
studied in detail. In Appendix A we present classification of boundaries of the state space of Fisher-Snedecor diffusion, while Appendix B contains well known facts about Gauss hypergeometric functions.

2 General information about the Fisher-Snedecor distribution

Random variable \( X \) has Fisher-Snedecor distribution with \( \alpha > 0 \) and \( \beta > 0 \) degrees of freedom, \( X \sim \mathcal{F}(\alpha, \beta) \), if its probability density function is given by

\[
\mathfrak{f}_x(x) = \frac{\beta x^{\frac{\alpha}{2} - 1}}{B\left(\frac{\alpha}{2}, \frac{\beta}{2}\right)} \alpha I_{(0, \infty)}(x) = \frac{(\frac{\alpha}{\alpha + \beta})^{\frac{\alpha}{2}} \beta x^{\frac{\beta}{2}}}{xB\left(\frac{\alpha}{2}, \frac{\beta}{2}\right)} I_{(0, \infty)}(x),
\]

(9)

where \( B(\cdot, \cdot) \) is the standard Beta function.

Remark 2.1. While a more general density

\[
\frac{w^{\frac{\alpha}{2}}}{B\left(\frac{\alpha}{2}, \frac{\beta}{2}\right)} \frac{(\alpha x)^{\frac{\alpha}{2} - 1}}{(\alpha + w)^{\frac{\alpha}{2} + \frac{\beta}{2}}} \alpha I_{(0, \infty)}(x) = \frac{(\frac{\alpha}{\alpha + w})^{\frac{\alpha}{2}} \beta w^{\frac{\beta}{2}}}{xB\left(\frac{\alpha}{2}, \frac{\beta}{2}\right)} I_{(0, \infty)}(x), \quad w > 0,
\]

might be of interest as well, in this paper we study the Fisher-Snedecor distribution with \( w = \beta \).

This distribution belongs to the Pearson family of continuous distributions and is also known as the
Pearson type VI distribution (see Pearson (41)). In particular, it follows that the tail of the Fisher-
Snedecor distribution with the density (9) decrease like \( x^{-(1+\beta/2)} \), and so this distribution is heavy-tailed.

Moment of the \( n \)-th order of \( \mathcal{F}(\alpha, \beta) \) distribution is given by the expression

\[
E[X^n] = \frac{\beta^n}{\alpha^{n-1}} \prod_{k=1}^{n} (\alpha + 2k) \frac{\Gamma\left(\frac{\beta}{2} + n\right)}{\Gamma\left(\frac{\beta}{2}\right)} \frac{\Gamma\left(\frac{\alpha}{2} + n\right)}{\Gamma\left(\frac{\alpha}{2}\right)}, \quad \beta > 2n, \quad n \in \mathbb{N}.
\]

Furthermore, moments could be calculated according to the recurrence relation

\[
\left(\frac{2(k+2)}{\beta + 2} - 1\right) E[X^{k+1}] = \frac{\beta(\alpha - 2) + 2\beta(k+1)}{\alpha(\beta + 2)} E[X^k], \quad \beta > 2(k+1), \quad k \in \mathbb{N}.
\]

In particular, expectation and variance are:

\[
E[X] = \frac{\beta}{\beta - 2}, \quad \beta > 2, \quad \text{Var}(X) = \frac{2\beta^2(\alpha - 2) + 2\beta(\alpha + 2)}{\alpha(\beta - 2)^2}, \quad \beta > 4.
\]

Remark 2.2. Fisher-Snedecor distribution with positive integer degrees of freedom is frequently used
in statistics (e.g. in analysis of variance). In particular, if \( \chi^2_n \) and \( \chi^2_m \) are independent chi-square
random variables with \( n \in \mathbb{N} \) and \( m \in \mathbb{N} \) degrees of freedom, respectively, then the random variable
\( \left[\frac{(\chi^2_n)^n}{n}/\chi^2_m/m\right] \) has Fisher-Snedecor \( \mathcal{F}(n, m) \) distribution.
3 Fisher-Snedecor diffusion

Definition. The Fisher-Snedecor diffusion is a solution of the non-linear stochastic differential equation

\[ dX_t = -\theta \left( X_t - \frac{\beta}{\beta - 2} \right) dt + \sqrt{\frac{4\theta}{\alpha(\beta - 2)}} X_t(\alpha X_t + \beta) dW_t, \quad t \geq 0, \]  
(10)

(which is equation (7) for \( \mu = \beta/(\beta - 2) \)). The infinitesimal parameters, i.e. the drift parameter \( \mu(x) \) and the diffusion parameter \( \sigma(x) \), of the Fisher-Snedecor diffusion (10) are respectively given by

\[ \mu(x) = -\theta \left( x - \frac{\beta}{\beta - 2} \right), \quad \sigma(x) = \sqrt{\frac{4\theta}{\alpha(\beta - 2)}} x(\alpha x + \beta), \quad \beta > 2, \]  
(11)

where the restriction \( \beta > 2 \) ensures simultaneously that \( \mu = \beta/(\beta - 2) \) is positive and that \( \theta > 0 \), so that \( \mu \) is the stationary mean.

Scale and speed densities. For \( x > 0 \) the corresponding scale density is

\[ s(x) = \exp \left( -\int_0^x \frac{2\mu(u)}{\sigma^2(u)} \, du \right) = x^{-\frac{\alpha}{2}} (\alpha x + \beta)^{\frac{\beta}{2} + \frac{\alpha}{2} - 1}, \]  
(12)

while the speed density is

\[ m(x) = \frac{2}{\sigma^2(x)s(x)} = \frac{\alpha(\beta - 2)}{2\theta} x^{\frac{\beta}{2} - 1} (\alpha x + \beta)^{-\frac{\beta}{2} - \frac{\alpha}{2}}. \]  
(13)

For any positive \( \alpha \) and \( \beta \) the speed density (13) is integrable on the diffusion state space, i.e.

\[ \int_0^\infty m(x) \, dx = \frac{\alpha(\beta - 2)}{2\theta} \alpha^{-\frac{\beta}{4}} \beta^{-\frac{\alpha}{4}} B \left( \frac{\alpha}{2}, \frac{\beta}{2} \right) = M < \infty. \]  
(14)

However, the scale density \( s(x) \) has the following two properties:

\[ \int_0^\infty s(x) \, dx = \infty, \quad \int_0^x s(x) \, dx = \infty \quad \Leftrightarrow \quad \alpha \geq 2, \]

where \( x_0 \) is an arbitrary point from the interior of the diffusion state space \((0, \infty)\). The first property ensures that starting from the arbitrary point \( x_0 \) the boundary \( \infty \) almost surely cannot be attained (see Aït-Sahalia (2)). A similar statement holds for the boundary 0, i.e. provided that diffusion starts from the arbitrary point \( x_0 \), the boundary 0 almost surely cannot be attained if and only if \( \alpha \geq 2 \). We restrict ourselves to this case \( \alpha \geq 2 \) (but note that in the opposite case a stationary diffusion may also be constructed, subject to instantaneous reflection at 0).

For any positive \( \alpha \) and \( \beta \) the stochastic differential equation (10) admits a unique strong Markovian solution \( \{X_t, 0 \leq t \leq T\} \) with the time-homogenous transition densities, since it satisfies the following sufficient conditions given by Aït-Sahalia (2):

(C1) the drift coefficient \( \mu(x) \) and the diffusion coefficient \( \sigma(x) \) given by expressions (11) are continuously differentiable in \( x \) on \((0, \infty)\) and \( \sigma^2(x) \) is strictly positive on \((0, \infty)\),

(C2) the speed density (13) has the property given by (14).

According to Aït-Sahalia (2), these conditions are considerably less restrictive than the global Lipschitz and the linear growth conditions which are usually imposed on drift and diffusion coefficients to obtain existence and uniqueness of a strong solution (see Mikosch (39)).

Ergodicity and stationarity. For any \( \alpha \geq 2 \) and \( \beta > 2 \) the Fisher-Snedecor diffusion is ergodic (see for example Genon-Catalot et al. (20), or Sørensen (46)). If furthermore \( X_0 \sim FS(\alpha, \beta) \), then the Fisher-Snedecor diffusion is strictly stationary. For \( \beta > 2 \) the conditional expectation satisfies

\[ E[X_{s+t}|X_s = x] = xe^{-\theta t} + \frac{\beta}{\beta - 2} (1 - e^{-\theta t}), \]
and if $\beta > 4$, i.e. if the invariant distribution has finite variance, the autocorrelation function is given by
\[
\rho(t) = \text{Corr}(X_{s+t}, X_s) = e^{-\delta t}, \quad t \geq 0, \quad s \geq 0
\]
(see Bibby et al. (9), Theorem 2.3.(iii)).

**$\alpha$-mixing property.** According to the general result by Genon-Catalot et al. (see (20), Corollary 2.1), the Fisher-Snedecor diffusion is an $\alpha$-mixing process with an exponentially decaying rate, i.e.
\[
\alpha_X(t) = \sup_{s \geq 0} \alpha(F_s, F^{s+t}) \leq \frac{1}{4} e^{-\delta t}, \quad \delta > 0,
\]
where
\[
\alpha(F_s, F^{s+t}) = \sup_{A \in F_s, B \in F^{s+t}} |P(A \cap B) - P(A)P(B)|,
\]
\[
A \in F_s = \sigma\{X_u, u \leq s\}, \quad B \in F^{s+t} = \sigma\{X_u, u \geq s + t\}.
\]
Indeed, Fisher-Snedecor diffusion has the sufficient properties given by Genon-Catalot et al.:

(i) the drift coefficient $\mu(x)$ and the squared diffusion coefficient $\sigma^2(x)$, given by expressions (11), satisfy condition (C1). Furthermore, there exists a strictly positive constant
\[
K = \frac{\theta(\beta + 2\sqrt{\alpha})^2}{\alpha(\beta - 2)}
\]
such that
\[
|\mu(x)| \leq K (1 + |x|) \quad \text{and} \quad \sigma^2(x) \leq K (1 + x^2).
\]

(ii) The speed density $m(x)$ given by (13) is integrable on the diffusion state space (see expression (14)). For $\alpha \geq 2$ the scale density $s(x)$ given by (12) is non-integrable in the neighborhood of boundary points 0 and $\infty$.

(iii) The random variable $X_0$ has (Fisher-Snedecor) density which is proportional to the speed density $m(x)$, i.e.
\[
f_0(x) = \frac{m(x)}{M} 1_{(0,\infty)}(x).
\]

(iv) The product of the diffusion coefficient $\sigma(x)$ and the speed density $m(x)$ converge to 0 as $x \to 0$ and $x \to \infty$, i.e.
\[
\lim_{x \to 0} \sigma(x)m(x) = \lim_{x \to \infty} \sigma(x)m(x) = 0.
\]

(v) If we define
\[
\gamma(x) = \sigma'(x) - \frac{2\mu(x)}{\sigma(x)} = \frac{\beta [\alpha(x - 1) + 1]}{x(\alpha x + \beta)} \sqrt{\frac{\theta x(\alpha x + \beta)}{\alpha(\beta - 2)}}, \quad x \in (0, \infty),
\]
then it follows
\[
\lim_{x \to 0} \frac{1}{\gamma(x)} = \sqrt{\frac{\alpha(\beta - 2)}{\beta^2 \theta}} \lim_{x \to 0} \frac{\sqrt{x(\alpha x + \beta)}}{\alpha(x - 1) + 1} = 0 < \infty,
\]
\[
\lim_{x \to \infty} \frac{1}{\gamma(x)} = \sqrt{\frac{\alpha(\beta - 2)}{\beta^2 \theta}} \lim_{x \to \infty} \frac{\sqrt{x(\alpha x + \beta)}}{\alpha(x - 1) + 1} = \frac{1}{\beta} \sqrt{\frac{\beta - 2}{\theta}} < \infty.
\]

Detailed exposition of the theory of mixing processes and additional information on corresponding central limit theorems are given by Doukhan (16).
Remark 3.1. The autonomous stochastic differential equation (10) is a special case of the generalized non-linear mean reversion Aït-Sahalia model

\[ dX_t = (c_{-1}X_t^{-1} + c_0 + c_1X_t + c_2X_t^2) \, dt + \sqrt{\sigma_0^2 + \sigma_1^2X_t + \sigma_2^2X_t^2} \, dW_t \]

for following parameter values:

\[
\begin{align*}
  c_{-1} &= 0, & \sigma_0^2 &= 0, \\
  c_0 &= \frac{\theta\beta}{\beta - 2} > 0, & \sigma_1^2 &= \frac{4\theta}{\alpha(\beta - 2)} > 0, \\
  c_1 &= -\theta < 0, & \sigma_2^2 &= \frac{4\theta}{\alpha(\beta - 2)} > 0, \\
  c_2 &= 0, & \delta &= 2.
\end{align*}
\]

The generalized non-linear mean reversion Aït-Sahalia model is frequently used for interest rates modeling. For more details about this model see (2) or (22).

Remark 3.2. The Lamperti transform \( \{Y_t, t \geq 0\} \) of the Fisher-Snedecor diffusion is a solution of the stochastic differential equation

\[ dY_t = \frac{\alpha - \beta - 1}{2} \sqrt{\frac{\theta}{\beta - 2}} \tan \left( Y_t \sqrt{\frac{\beta\theta}{\beta - 2}} \right) \, dt + dW_t, \quad t \geq 0 \]

with unit diffusion parameter. According to Iacus (22), Lamperti transforms of diffusion processes are recommended for use in simulation studies.

4 Spectral representation of the transition density of Fisher-Snedecor diffusion

4.1 Fokker-Planck equation

In this section we present the spectral representation of the transition density

\[ p = p(x, t) = p(x; x_0, t) = \frac{d}{dx} P(X_t \leq x \mid X_0 = x_0), \quad x > 0, \quad t \geq 0, \quad (15) \]

of the Fisher-Snedecor diffusion in terms of solutions of the corresponding Sturm-Liouville equation. Note that the corresponding Sturm-Liouville operator is closely related to the infinitesimal generator of the Fisher-Snedecor diffusion.

According to Karlin and Taylor (24), the transition density (15) is the principal solution of the Fokker-Planck or the forward Kolmogorov equation

\[ \frac{\partial p}{\partial t} = -\frac{\partial}{\partial x} \left( -\theta \left( x - \frac{\beta}{\beta - 2} \right) p \right) + \frac{1}{2} \frac{\partial^2}{\partial x^2} \left( \frac{4\theta}{\alpha(\beta - 2)} x(\alpha x + \beta) p \right), \quad x > 0, \quad t \geq 0, \]

and its Laplace transform is known explicitly (see (36)).

Inverting the Laplace transform of the transition density, also known as the resolvent kernel or Green’s function, yields its spectral decomposition - see for example the classical books by Titchmarsh (48), Karlin and Taylor (24) and Itô and McKean (23), as well as the paper by McKean (37) (who call this "eigendifferential expansion").

The discrete part of the spectrum can be treated in a unified manner for all Pearson diffusions - see Wong (50) and Forman and Sorensen (18). The exact formulas for the continuous part of the spectrum require however computations which need to be performed on a case by case basis. Some of such computations for special cases of Pearson diffusions were performed recently, motivated by applications in finance, by Linetsky (31; 32; 33), Davydov and Linetsky (14) and Carr et al. (38). We treated here the Fisher-Snedecor case, and we chose, for self containedness, to include a detailed account of the inversion of the Laplace transform of the corresponding transition density.
4.2 Infinitesimal generator and Sturm-Liouville equation

In order to calculate the explicit form of the Laplace transform of the transition density we observe the infinitesimal generator of the Fisher-Snedecor diffusion:

\[ G f(x) = \frac{2\theta}{\alpha(\beta - 2)} x^{1-\frac{2}{\beta}} (ax + \beta)^{\frac{2}{\beta} + \frac{2}{\alpha}} \frac{d}{dx} \left( x^\frac{2}{\beta} (ax + \beta)^{1-\frac{2}{\beta} - \frac{2}{\alpha}} f'(x) \right) = \]

\[ = \frac{2\theta}{\alpha(\beta - 2)} x(ax + \beta)f''(x) - \theta \left( x - \frac{\beta}{\beta - 2} \right) f'(x), \quad x > 0. \]  

(16)

Infinitesimal generator \( G \) is a closed, generally unbounded, negative semidefinite, self-adjoint operator acting on twice continuously differentiable functions from \( L^2((0, \infty), m) \) which satisfy boundary conditions

\[ \lim_{x \to 0} f'(x) = \lim_{x \to \infty} \frac{f(x)}{\sigma(x)} = 0. \]

In order to obtain the explicit form of the Laplace transform of the transition density (15) we first observe the equation \( Gf(x) = sf(x) \) for \( s > 0 \), i.e. the equation

\[ \frac{2\theta}{\alpha(\beta - 2)} x(ax + \beta)f''(x) - \theta \left( x - \frac{\beta}{\beta - 2} \right) f'(x) - sf(x) = 0, \quad s > 0. \]  

(17)

According to Itô and McKeon (see (23), page 128) or Borodin and Salminen (see (10), page 18), for any \( s > 0 \) equation of this type admits two positive, linearly independent and increasing/decreasing solutions with Wronskian proportional to the scale density, called fundamental solutions. They intervene in the expression of Green’s function, as well as in several first-passage problems (6).

Substitution

\[ f(x) = v(y(x)) = v \left( \frac{\alpha}{\beta} x \right), \]

transforms the equation (17) into the equation

\[ y(y + 1)v''(y) + \left( \frac{\alpha}{2} - \frac{\beta - 2}{2} \right) v'(y) - s^* v(y) = 0, \]

(18)

where \( s^* = s(\beta - 2)/2\theta \). Equation (18) is known as the Gauss or the hypergeometric equation and its detailed study could be found in Titchmarsh (48), Slater (45) and Luke (34).

Denote the roots of the quadratic equation \( z^2 + \frac{2}{\beta} z - s^* = 0 \) by

\[ z_\pm = z_{\pm, s} = -\frac{\beta}{4} \pm \Delta_s, \quad \Delta_s = \sqrt{\frac{\beta^2}{16} + \frac{s(\beta - 2)}{2\theta}}. \]  

(19)

According to Titchmarsh (see (48), Example 4.18., page 100), two linearly independent solutions of this equation for \( |y| < 1 \) are Gauss hypergeometric functions

\[ v_1(y) = \binom{\frac{\alpha}{2}}{\frac{\beta}{2}}_2 F_1 \left( z_+, z_-; \frac{\alpha}{2}; -y \right), \]

\[ v_2(y) = y^{1-\frac{2}{\beta}} \binom{\alpha}{\beta - 2} \binom{\alpha}{\beta} F_1 \left( u_+, u_-; 2 - \frac{\alpha}{2}; -y \right), \quad \alpha > 2, \quad \alpha \notin \{2(m + 1), m \in \mathbb{N}\}, \]

where \( u_\pm = u_{\pm, s} = 1 - \frac{\alpha}{2} \pm z_\pm \) and where the condition \( \alpha \notin \{2(m + 1), m \in \mathbb{N}\} \) ensures that the function \( v_2(y) \) is well defined. However, if \( \alpha > 2 \) is an even integer, then the corresponding solution is given by much more complicated expression in terms of the digamma function \( \psi(\cdot) \) which is hardly evaluated in explicit calculations (see Abramowitz and Stegun (1)).

Furthermore, two linearly independent solutions for \( |y| > 1 \) are

\[ v_{3, 4}(y) = y^{-z_\mp} \binom{\alpha}{1 + 2\Delta_s} \binom{\alpha}{2} F_1 \left( z_{\mp}, u_{\mp}; 1 + 2\Delta_s; -\frac{1}{y} \right), \]

where the upper sign refers to the solution \( v_3(y) \) and the lower sign refers to the solution \( v_4(y) \).
According to Luke (see (34), Section 3.9., Expression (1); see also Slater (45), Section 1.8.1., Expression 1.8.11.), the solution \( v_1(y) \) valid in the region \(|y| < 1\) can be analytically continued to the whole complex plane cut along the interval \((-\infty, -1]\) (see Appendix B). This continuation is provided by the following expression which represents the solution \( v_1(y) \) as the linear combination of the solutions \( v_3(y) \) and \( v_4(y) \)

\[
v_1(y) = B_s v_3(y) + A_s v_4(y),
\]

where

\[
B_s = \frac{\Gamma \left( \frac{\alpha}{2} \right) \Gamma (2\Delta_s)}{\Gamma (z_+, s) \Gamma (1 - u_-, s)}, \quad A_s = \frac{\Gamma \left( \frac{\beta}{2} \right) \Gamma (-2\Delta_s)}{\Gamma (z_+, s) \Gamma (1 - u_+, s)}.
\]

Similarly, solutions \( v_2(y), v_3(y) \) and \( v_4(y) \) can also be analytically continued (see Luke (34)). Changing back variables, solutions of the equation (17) for \( x \in (0, \beta/\alpha) \) are

\[
f_1(x) = f_1(x, s) = 2F_1 \left( \frac{\alpha}{2}; -\frac{x}{\beta}; \frac{\alpha}{2} \right),
\]

\[
f_2(x) = f_2(x, s) = \left( \frac{\alpha}{\beta} x \right)^{1 - \frac{\alpha}{2}} 2F_1 \left( u_+ + u_-; 2 - \frac{\alpha}{2}; -\frac{x}{\beta} \right),
\]

\[
\alpha > 2, \quad \alpha \notin \{2(m + 1), m \in \mathbb{N}\},
\]

while solutions for \( x \in (\beta/\alpha, \infty) \) are

\[
f_{3, 4}(x) = f_{3, 4}(x, s) = \left( \frac{\alpha}{\beta} x \right)^{-z_x} 2F_1 \left( z_+, u_+ + 2\Delta_s; -\frac{\beta}{\alpha x} \right),
\]

where the upper sign refers to the solution \( f_3(x, s) \) and the lower sign refers to the solution \( f_4(x, s) \).

According to the analytic continuation of the hypergeometric functions, solutions \( f_1(x, s), f_3(x, s) \) and \( f_4(x, s) \) are all analytic on the state space of the Fisher-Snedecor diffusion. Therefore, fundamental solutions of equation (17) on \((0, \infty)\) could be identified with the increasing solution \( f_1(x, s) \) and the decreasing solution \( f_4(x, s) \), see Figure 1.

**Figure 1.** Graphs of fundamental solutions of equation (17) for \( \alpha = 5, \beta = 20 \) and \( \theta = 0.05 \).

**Remark 4.1.** According to McKean (37), Buchholz (11) and Borodin and Salminen (10), Wronskian of linearly independent solutions \( f_1(x, s) \) and \( f_4(x, s) \) of the equation (17) is proportional to the scale density (12), i.e.

\[
W(f_4, f_1) = W_s(f_4, f_1)(x) = W_s(f_4, f_1)(x_0) s(x),
\]

where the value \( x_0 > 0 \) could be chosen so that \( f_1(x_0, s), f'_1(x_0, s), f_4(x_0, s) \) and \( f'_4(x_0, s) \) or the limit \( \lim_{x \to -2} W_s(f_4(x, s), f_1(x, s)) \) are easy to calculate (see Titchmarsh (48)). According to relation (20) it follows that

\[
W_s(f_4, f_1)(x, s) = B_s W_s(f_3, f_4)(x, s).
\]

Furthermore, according to Luke (see (34), page 85, expression (34)), Wronskian of solutions \( f_3(x, s) \) and \( f_4(x, s) \) is given by

\[
W_s(f_3, f_4)(x, s) = (-1)^{2 - \frac{\alpha}{2} \left( -1 - z_+ - z_- \right) \left( -\frac{\alpha}{\beta} x - 1 \right)} \left( -\frac{\alpha}{\beta} x \right)^{\frac{\alpha}{2} z_x - z_x - 1},
\]
where \( z_+ \) and \( z_- \) are given in (19). From here it follows that \( W_s(f_4(x, s), f_1(x, s)) \) is given by

\[
W_s(f_4, f_1) = 2a(x)B_s\alpha^{1-\frac{2}{\beta}}\beta^{-\frac{2}{\beta}}\sqrt{\frac{\beta^2}{16} + \frac{s(\beta - 2)}{2\theta}} = 2a(x)\alpha^{1-\frac{2}{\beta}}\beta^{-\frac{2}{\beta}}B_s\Delta_s,
\]

where \( B_s \) is given by (21) and \( \Delta_s \) by (19).

**Remark 4.2.** The negative \((-G)\) of the infinitesimal generator \( G \) is the Sturm-Liouville operator. Therefore, we could also consider the Sturm-Liouville equation \((-G) \psi(x) = \lambda \psi(x)\) for \( \lambda \geq 0 \), i.e. the equation

\[
\frac{2\theta}{\alpha(\beta - 2)} x(\alpha x + \beta)\psi''(x) - \theta \left( x - \frac{\beta}{\beta - 2}\right) \psi'(x) + \lambda \psi(x) = 0, \quad \lambda \geq 0.
\]

Solutions of the equation (26) are:

- \( f_1(x, -\lambda) \) and \( f_2(x, -\lambda) \) for \( x \in (0, \beta/\alpha) \),
- \( f_3(x, -\lambda) \) and \( f_4(x, -\lambda) \) for \( x \in (\beta/\alpha, \infty) \).

### 4.3 Spectral category

Explicit form of the spectral representation of the transition density of one-dimensional diffusion process is closely related to the structure of the spectrum of the corresponding Sturm-Liouville operator \((-G)\) (see Linetsky (31)). According to Linetsky’s boundary classification scheme for one-dimensional diffusions introduced in (31), Fisher-Snedecor diffusion belongs to the spectral category II. In particular, \( e_1 = 0 \) is non-oscillatory boundary (regular for \( \alpha < 2 \) and entrance for \( \alpha > 2 \), \( \alpha \notin \{2m, m \in \mathbb{N}\} \)), while \( e_2 = \infty \) is oscillatory/non-oscillatory singular boundary (natural for all \( \alpha > 0 \), \( \alpha \notin \{2(m+1), m \in \mathbb{N}\} \)) with unique cutoff

\[
\Lambda = \frac{\theta\beta^2}{8(\beta - 2)}, \quad \beta > 2.
\]

For details on boundary classification of boundaries of the state space of Fisher-Snedecor diffusion see Appendix A.

Since \( e_2 = \infty \) is NO for \( \lambda = \Lambda \), the Sturm-Liouville operator \((-G)\) has a finite set of simple eigenvalues in \([0, \Lambda]\) and an essential spectrum \( \sigma_e(-G) = [\Lambda, \infty) \). Hence, the operator \((-G)\) has a discrete spectrum \( \sigma_d(-G) \) in \([0, \Lambda]\), i.e. \( \sigma_d(-G) \subset [0, \Lambda] \), and a purely absolutely continuous spectrum \( \sigma_{ac}(-G) \) of multiplicity one in

\[
[\Lambda, \infty) = \left\{ \frac{\theta\beta^2}{8(\beta - 2)}, \infty \right\}.
\]

#### 4.3.1 Discrete part of the spectrum

In order to calculate eigenvalues (elements of \( \sigma_d(-G) \)), and corresponding eigenfunctions of the Sturm-Liouville operator \((-G)\), we observe the Sturm-Liouville differential equation

\[
\frac{2\theta}{\alpha(\beta - 2)} x(\alpha x + \beta)\psi''(x) - \theta \left( x - \frac{\beta}{\beta - 2}\right) \psi'(x) + \lambda \psi(x) = 0,
\]

in the self-adjoint form

\[
-\frac{1}{2} \frac{d}{dx} \left( \sigma^2(x) f_n(x) \right) = \lambda_n f_n(x) \psi(x),
\]

where \( \lambda_n \) is the real spectral parameter representing the simple eigenvalues and \( f_n(x) \) is the corresponding polynomial eigenfunction (see Nikiforov and Uvarov (40)). Eigenfunctions \( f_n(x) \) are supposed to be normalized and orthogonal with respect to the invariant density of Fisher-Snedecor diffusion. Multiplying both sides of the equation (28) by \( f_n(x) \) and taking the integral from 0 to \( \infty \) yields

\[
-\frac{1}{2} \int_0^\infty f_n(x) \frac{d}{dx} \left( \sigma^2(x) \psi(x) f_n'(x) \right) dx = \lambda_n \int_0^\infty f_n^2(x) \psi(x) dx.
\]
Now according to orthogonality of the normalized eigenfunctions \( f_n(x) \) with respect to \( \mathbb{fs}(x) \) it follows that

\[
\lambda_n = -\frac{1}{2} \int_0^\infty f_n(x) \frac{d}{dx} \left( \sigma^2(x) \mathbb{fs}(x)f_n'(x) \right) \, dx.
\]

Since \( f_n(x) \) is the polynomial of the \( n \)-th degree, \( f_n'(x) \) is the polynomial of the degree \( 2n \) and therefore the finiteness of the number of simple eigenvalues follows from the fact that there exists the number \( N \in \mathbb{N}_0 \) such that

\[
\int_0^{2n} \mathbb{fs}(x) \, dx < \infty, \quad n = 0, 1, \ldots, N \quad \text{and} \quad \int_0^{\infty} \mathbb{fs}(x) \, dx = \infty, \quad n = N + 1, N + 2 \ldots
\]

In particular, for the Fisher-Snedecor diffusion \( N = [\beta/4], \beta > 2 \), since the moment of the order \( 2n \) of the Fisher-Snedecor distribution exists under the condition \( \beta > 4n \). Therefore, the discrete part of the spectrum of the operator \(-G\) is of the form \( \sigma_\ell(-G) = \{\lambda_n, n = 0, 1, \ldots, [\beta/4]\} \). According to the general theory from Nikiforov and Uvarov (40), simple eigenvalues \( \lambda_n \) are given by the explicit expression

\[
\lambda_n = \frac{\theta}{\beta - 2} n(\beta - 2n), \quad n \in \left\{ 0, 1, \ldots, \left\lfloor \frac{\beta}{4} \right\rfloor \right\}, \quad \beta > 2,
\]

while corresponding eigenfunctions, i.e. polynomial solutions of the equation (28), are given by the Rodrigues formula

\[
\tilde{F}_n(x) = x^{1-\frac{n}{2}} (\alpha x + \beta)^{\frac{\alpha}{2}} \frac{d^n}{dx^n} \left\{ 2^n x^{\frac{n-1}{2}} (\alpha x + \beta)^{n-\frac{1}{2}} \right\}, \quad n \in \left\{ 0, 1, \ldots, \left\lfloor \frac{\beta}{4} \right\rfloor \right\}, \quad \beta > 2.
\]

Polynomials \( \tilde{F}_n(x) \) are called here Fisher-Snedecor polynomials. They form the finite system \( \{\tilde{F}_n(x), n = 0, 1, \ldots, [\beta/4]\} \) of polynomials orthogonal with respect to the Fisher-Snedecor density, i.e.

\[
\int_0^\infty \tilde{F}_m(x) \tilde{F}_n(x) \mathbb{fs}(x) \, dx = 0, \quad m, n \in \left\{ 0, 1, \ldots, \left\lfloor \frac{\beta}{4} \right\rfloor \right\}, \quad m \neq n, \quad \beta > 4.
\]

In particular, the first six non-normalized Fisher-Snedecor polynomials are

\[
\begin{align*}
\tilde{F}_0(x) &= 1, \\
\tilde{F}_1(x) &= -\alpha(\beta - 2)x + \alpha\beta, \\
\tilde{F}_2(x) &= \alpha^2(\beta - 4)(\beta - 6)x^2 - 2\alpha\beta(\alpha + 2)(\beta - 4)x + \alpha\beta^2(\alpha + 2), \\
\tilde{F}_3(x) &= -\alpha^3(\beta - 6)(\beta - 8)(\beta - 10)x^3 + 3\alpha^2\beta(\alpha + 4)(\beta - 6)(\beta - 8)x^2 - \\
&\quad -3\alpha\beta^2(\alpha + 2)(\beta - 6)x + \alpha\beta^3(\alpha + 2)(\alpha + 4), \\
\tilde{F}_4(x) &= \alpha^4(\beta - 8)(\beta - 10)(\beta - 12)(\beta - 14)x^4 - 4\alpha^3\beta(\alpha + 6)(\beta - 8)(\beta - 10)(\beta - 12)x^3 + \\
&\quad + 6\alpha^2\beta^2(\alpha + 4)(\alpha + 6)(\beta - 8)(\beta - 10)x^2 - 4\alpha\beta^3(\alpha + 2)(\alpha + 4)(\alpha + 6)(\beta - 8)x + \\
&\quad +\alpha\beta^4(\alpha + 2)(\alpha + 4)(\alpha + 6), \\
\tilde{F}_5(x) &= -\alpha^5(\beta - 10)(\beta - 12)(\beta - 14)(\beta - 16)(\beta - 18)x^5 + \\
&\quad + 5\alpha^4\beta(\alpha + 8)(\beta - 10)(\beta - 12)(\beta - 14)(\beta - 16)x^4 - \\
&\quad -10\alpha^3\beta^2(\alpha + 6)(\alpha + 8)(\beta - 10)(\beta - 12)(\beta - 14)x^3 + \\
&\quad + 10\alpha^2\beta^3(\alpha + 4)(\alpha + 6)(\alpha + 8)(\beta - 10)(\beta - 12)x^2 - \\
&\quad -5\alpha\beta^4(\alpha + 2)(\alpha + 4)(\alpha + 6)(\alpha + 8)(\beta - 10)x + \\
&\quad + \alpha\beta^5(\alpha + 2)(\alpha + 4)(\alpha + 6)(\alpha + 8).
\end{align*}
\]

For normalization of Fisher-Snedecor polynomials with respect to the density (9), we must multiply each of them by the normalizing constant

\[
K_n = \frac{(-1)^n}{\sqrt{(-1)^n n! d_n I_n}}, \quad n \in \left\{ 0, 1, \ldots, \left\lfloor \frac{\beta}{4} \right\rfloor \right\},
\]

where

\[
d_n = 2^n \alpha^n \frac{\Gamma \left( 2n - \frac{\beta}{2} \right)}{\Gamma \left( n - \frac{\beta}{2} \right)}, \quad I_n = \left( \frac{2\beta^2}{\alpha} \right)^n \frac{B \left( \frac{n + \frac{\beta}{2}}{2}, -2n + \frac{\beta}{2} \right)}{B \left( \frac{\beta}{2}, \frac{\beta}{2} \right)}.
\]
Remark 4.3. Comparison of the solution \( F_n(x, -\lambda) \) of the Sturm-Liouville equation (26) with the explicit definitions of Fisher-Snedecor polynomials generated by the Rodrigues formula (31) yields the hypergeometric representation of Fisher-Snedecor polynomials:

\[
F_n(x) = K_n \binom{\beta}{2}^n (\alpha x + \beta)^{\frac{\beta}{2} + \frac{n}{2}} _2F_1 \left( \frac{\beta}{2} + n, \frac{\beta}{2}; \frac{\beta}{2} + \beta, -x \right), \quad n \in \left\{ 0, 1, \ldots, \left\lfloor \frac{\beta}{4} \right\rfloor \right\}.
\]

(32)

Remark 4.4. Orthogonality relation for the normalized Fisher-Snedecor polynomials, i.e. the relation

\[
\int_{0}^{\infty} F_m(x) F_n(x) \, d\mu(x) = \delta_{mn}, \quad m, n \in \left\{ 0, 1, \ldots, \left\lfloor \frac{\beta}{4} \right\rfloor \right\}, \quad \beta > 4,
\]

implies interesting properties of the random variables \( F_n(X_t) \), where \( X_t \) is random variable from the Fisher-Snedecor diffusion. In particular, random variables \( F_n(X_t) \) are orthonormal, i.e.

\[
E[F_m(X_t) F_n(X_t)] = \int_{0}^{\infty} F_m(x) F_n(x) \, d\mu(x) = \delta_{mn}, \quad m, n \in \left\{ 0, 1, \ldots, \left\lfloor \frac{\beta}{4} \right\rfloor \right\}.
\]

Since \( F_0(x) = 1 \), for \( m = 0 \) and \( n \neq 0 \) the previous expression takes the form

\[
E[F_m(X_t)] = 0, \quad n \in \left\{ 1, \ldots, \left\lfloor \frac{\beta}{4} \right\rfloor \right\}, \quad \beta > 4,
\]

i.e. \( \{F_n(X_t), n = 1, \ldots, \lfloor \beta/4 \rfloor \} \) is the orthonormal system of centered random variables.

Remark 4.5. Fisher-Snedecor polynomials satisfy the following Favard-Jacobi recurrent relation:

\[
F_{n+1}(x) = \frac{1}{a_n} ((b_n - x) F_n(x) + a_{n-1} F_{n-1}(x)), \quad n \in \left\{ 1, \ldots, \left\lfloor \frac{\beta}{4} \right\rfloor \right\},
\]

\[
F_0(x) = F_{-1}(x) = 1,
\]

where

\[
a_n = \frac{2(\beta - 2n)}{\alpha(4n - \beta)(2 + 4n - \beta)} \sqrt{\frac{(n + 1)(2n + \alpha)(\alpha + \beta - 2n - 2)}{(2 + 4n - \beta)(4 + 4n - \beta)}},
\]

\[
b_n = \frac{\beta n (2n + \alpha - 2) - \alpha (n + 1)(2n + \alpha)}{\alpha(4n - \beta - 2)}, \quad \mu_n = \frac{\beta (2n + \alpha - 2)}{\alpha(4n - \beta - 2)}.
\]

Here \( a = \frac{\beta(\alpha - 2)}{\alpha(\beta + 2)}, \quad c_0 = 0, \quad c_1 = \frac{2\alpha}{\alpha(\beta + 2)} \) and \( c_2 = \frac{2}{\beta + 2} \) are coefficients from the numerator and the denominator of the Pearson equation for Fisher-Snedecor distribution:

\[
\frac{\mu'(x)}{\mu(x)} = \frac{\beta(\alpha - 2) - \alpha(\beta + 2)}{2x(\alpha x + \beta)} = \frac{-(x + a)}{c_2 x^2 + c_1 x + c_0}.
\]
4.3.2 Essential part of the spectrum

Essential spectrum of the Sturm-Liouville operator \((- \mathcal{G})\) is \(\sigma_e(- \mathcal{G}) = [\Lambda, \infty)\). Moreover, operator \((- \mathcal{G})\) has purely absolutely continuous spectrum of multiplicity one in \((\Lambda, \infty)\), where
\[
\Lambda = \frac{\theta \beta^2}{8(\beta - 2)}, \quad \beta > 2,
\]
is the unique positive cutoff between the discrete and the absolutely continuous part of the spectrum (see Linetsky (31), Theorem 4). Therefore, elements of the absolutely continuous spectrum can be parameterized by
\[
\lambda = \Lambda + \frac{2\theta k^2}{\beta - 2} - \frac{2\theta}{\beta - 2} \left( \frac{\beta^2}{16} + k^2 \right), \quad \beta > 2, \quad k > 0.
\]

**Remark 4.6.** Fisher-Snedecor polynomials \(F_n(x), \ n = 0, \ldots, \lfloor \beta/4 \rfloor\), and functions \(f_1(x, -\lambda)\) for \(\lambda > \Lambda\) belong to orthogonal subspaces of the Hilbert space \(H = L^2((0, \infty), \mathfrak{s}(\cdot))\), i.e.
\[
F_n(x) \in \mathcal{H}_{pp}, \quad n \in \{0, 1, \ldots, \lfloor \beta/4 \rfloor\}, \quad f_1(x, -\lambda) \in \mathcal{H}_{ac}, \quad \forall \lambda > \Lambda.
\]
Here \(\mathcal{H}_{pp}\) denotes the subspace of the Hilbert space \(L^2((0, \infty), \mathfrak{s}(\cdot))\) containing functions having only the pure point spectral measure, while \(\mathcal{H}_{ac}\) denotes the subspace of the Hilbert space \(L^2((0, \infty), \mathfrak{s}(\cdot))\) containing functions having only the spectral measure which is absolutely continuous with respect to the Lebesgue measure (see Reed and Simon (42) or Linetsky (31)). From here it follows that Fisher-Snedecor polynomials and functions \(f_1(x, -\lambda)\) for \(\lambda > \Lambda\) are orthogonal with respect to the invariant density of Fisher-Snedecor diffusion, i.e.
\[
\int_0^\infty F_n(x) f_1(x, -\lambda) \mathfrak{s}(x) \, dx = 0.
\]
This orthogonality relation has a key role in explicit calculations related to this diffusion process.

4.4 Spectral representation of transition density

As well-known (see e.g. Borodin and Salminen (10), page 19), the Laplace transform of transition density \(p(x; x_0, t)\), also known as the resolvent kernel or Green’s function
\[
G_s(x_0, x) = \int_0^\infty e^{-st} p(x; x_0, t) \, dt,
\]
adopts the explicit representation
\[
G_s(x_0, x) = \frac{m(x)}{w_s(\varphi_s, \psi_s)} \psi_s(x_0 \wedge x) \varphi_s(x_0 \vee x),
\]
where \(m(\cdot)\) is the speed density, \(\psi_s(\cdot)\) and \(\varphi_s(\cdot)\) are the fundamental solutions of the equation (17) (increasing and decreasing, respectively), and \(w_s(\varphi_s, \psi_s)\) is their Wronskian with respect to the scale density \(\mathfrak{s}(x)\), i.e.
\[
w_s(\varphi_s, \psi_s) = \frac{1}{\mathfrak{s}(x)} (\psi'_s(x) \varphi_s(x) - \psi_s(x) \varphi'_s(x)) = \frac{W_s(\varphi_s, \psi_s)}{\mathfrak{s}(x)}.
\]
Therefore, if it is possible to determine the explicit form of the fundamental solutions, then the transition density can be obtained by the Laplace inversion formula
\[
p(x; x_0, t) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} e^{st} G_s(x_0, x) \, ds, \quad t > 0,
\]
where \(s > 0\) is the spectral parameter from equation (17) and where integration is performed along a line \(\text{Re}(s) = c, \ c > 0\), leaving all the singularities of the Green’s function to its left.
According to Itô and McKean (23) (see also Linetsky (31)), the general form of the spectral representation of transition density (15) for diffusions belonging to spectral category II is of the following form:

\[ p(x; x_0, t) = m(x) \sum_{n=0}^{N} e^{-\lambda_n t} \varphi_n(x_0) \varphi_n(x) + m(x) \int_{\Lambda} e^{-\lambda t} \varphi(x_0, -\lambda) \varphi(x, -\lambda) d\lambda, \]

where \( \lambda_n, \ n \in \{0, \ldots, N\} \), are elements of the discrete spectrum \( \sigma_d(-\mathcal{G}) \) and \( \varphi_n(\cdot) \) are corresponding eigenfunctions normalized with respect to the speed density \( m(x) \), while \( \lambda > \Lambda \) are elements of the absolutely continuous spectrum \( \sigma_{ac}(-\mathcal{G}) \) and \( \varphi(\cdot, -\lambda) \) are solutions of the equation (26) also normalized with respect to the speed density \( m(x) \). Now we proceed to compute the spectral representation of the transition density for our process of interest following the general procedure from Itô and McKean (23) and Linetsky (31) and using the results of the analysis of the spectrum of the corresponding Sturm-Liouville operator presented in the Subsection 4.3.

**Theorem 4.1.** Spectral representation of the transition density of ergodic stationary diffusion with marginal Fisher-Snedecor distribution with parameters \( \alpha > 2, \ \alpha \notin \{2(m+1), \ m \in \mathbb{N}\} \), and \( \beta > 2 \) is of the form

\[ p(x; x_0, t) = p_d(x; x_0, t) + p_c(x; x_0, t). \]  

The discrete part of the spectral representation

\[ p_d(x; x_0, t) = \mathfrak{f}_s(x) \sum_{n=0}^{\lfloor \frac{\beta}{2} \rfloor} e^{-\lambda_n t} F_n(x_0) F_n(x) \]  

is given in terms of the eigenvalues \( \lambda_n \) given by (29) and the normalized Fisher-Snedecor polynomials \( F_n(\cdot) \) given by (32). The continuous part of the spectral representation

\[ p_c(x; x_0, t) = \mathfrak{f}_s(x) \frac{1}{\pi} \int_{\frac{\beta}{2}}^{\infty} e^{-\lambda t} k(\lambda) \times \left| \frac{B^\frac{1}{2} \left( \frac{\alpha}{2}, \frac{\beta}{2} \right) \Gamma \left( -\frac{\beta}{2} + i k(\lambda) \right) \Gamma \left( \frac{\beta}{2} + \frac{i k(\lambda)}{2} \right)}{\Gamma \left( \frac{\beta}{2} \right) \Gamma \left( 1 + 2 i k(\lambda) \right)} \right|^2 f_1(x_0, -\lambda) f_1(x, -\lambda) d\lambda \]  

is given in terms of the elements \( \lambda \) of the absolutely continuous spectrum of the operator \( -\mathcal{G} \) given by (34), solution \( f_1(\cdot, -\lambda) \) of the Sturm-Liouville equation (26) and parameter \( k(\lambda) = -i \Delta_{-\lambda} \), where \( \Delta_{-\lambda} \) is given in (19).

**Proof.** Since Wronskian of fundamental solutions \( f_1(x, s) \) and \( f_4(x, s) \) of equation (17) is given by expression (25), it is possible to obtain the explicit form of the Green’s function for Fisher-Snedecor diffusion. In particular,

\[ G_s(x_0, x) = \frac{m(x)}{2^{\alpha-1} \beta^\frac{1}{2} - \frac{\beta}{2} B_s \Delta_s} f_1(x, x_0, s) f_4(x \vee x_0, s), \]

where \( m(\cdot) \) is the speed density given by (13), \( f_1(\cdot, s) \) and \( f_4(\cdot, s) \) are non-normalized (with respect to the speed density) linearly independent solutions of the differential equation (17) and

\[ W_s(f_4, f_1) = 2 B_s \alpha^\frac{1}{2} \beta^\frac{1}{2} - \frac{\beta}{2} \sqrt{\frac{\beta^2}{16} + \frac{s(\beta - 2)}{2\theta}} = 2^\alpha - \frac{1}{2} \frac{\beta^2}{4} \beta^\frac{1}{2} B_s \Delta_s \]

is their Wronskian with respect to the scale density (12), where \( B_s \) is given by (21) and \( \Delta_s \) is given by (19). Observe now the Green’s function (41) as the function of the complex variable \( s \). The Gamma function

\[ \Gamma(z+s) = \Gamma \left( -\frac{\beta}{4} + \sqrt{\frac{\beta^2}{16} + \frac{s(\beta - 2)}{2\theta}} \right) \]
from $B_s$ in expression (41) has simple poles in

$$s = -\lambda_n = -\frac{\theta}{\beta} - 2(n+1), \quad n = 0, \ldots, \left\lfloor \frac{1}{\beta} \right\rfloor,$$

since for that value of $s$ we have $\Gamma(z_{+a}) = \Gamma(-n)$, $n = 0, \ldots, \left\lfloor \beta/4 \right\rfloor$. Note that these simple poles coincide with the negative simple eigenvalues of the Sturm-Liouville operator $(-G)$ and that, since $\alpha > 2$, $\alpha \notin \{2(m+1), m \in \mathbb{N}\}$, and $\beta > 2$, these are the only poles of the Green’s function (41).

According to Abramowitz and Stegun (1), $f_4(x,s) = C_s f_1(x, s) + D_s f_2(x, s)$, where

$$C_s = \frac{\Gamma(1 - \frac{\alpha}{2}) \Gamma(1 + 2\Delta_n)}{\Gamma(u_{+s}) \Gamma(1-z_{-s})} \quad \text{and} \quad D_s = \frac{\Gamma \left( \frac{\alpha}{2} - 1 \right) \Gamma(1 + 2\Delta_n)}{\Gamma(z_{+s}) \Gamma(1 - u_{-s})}.$$

Therefore, regarding the fact that Green’s function has simple poles at $s = -\lambda_n$, expression (41) can be written in terms of solutions $f_1(\cdot, s)$ and $f_2(\cdot, s)$, i.e.

$$G_{s=-\lambda_n}(x_0, x) = m(x) \left\{ \frac{\Gamma(-n) \Gamma \left( \frac{\alpha}{2} + \frac{\beta}{2} - n \right)}{\alpha^{1 - \frac{\alpha}{2} - \beta - \frac{\beta}{2}} \Gamma \left( \frac{\alpha}{2} \right) \Gamma \left( 1 + \frac{\beta}{2} - n \right)} (-1)^n \prod_{j=0}^{n-1} \left( \frac{\alpha}{2} + j \right) f_1(x_0, s) f_1(x, s) + \right.$$

$$\left. + \frac{\Gamma \left( \frac{\alpha}{2} - 1 \right) \Gamma \left( 1 + \frac{\beta}{2} - n \right)}{\alpha^{1 - \frac{\alpha}{2} - \beta - \frac{\beta}{2}} \Gamma \left( \frac{\alpha}{2} \right)} f_1(x \land x_0, s) f_2(x \lor x_0, s) \right\}. \quad (42)$$

Due to supposed values of parameter $\alpha$, the second term in expression (42) has no poles and therefore does not contribute to the residues of the Green’s function. However, due to simple poles at $s = -\lambda_n$ and according to Titchmarsh (see (48), Chapter IV, Example 4.19.), the residues of Green’s function at these poles are given by

$$\text{Res}_{s=-\lambda_n} G_s(x_0, x) = m(x) \left\{ (-1)^{2n} \frac{2n}{n!} \left( \frac{\alpha}{2} + \frac{\beta}{2} - n \right) \prod_{j=0}^{n-1} \left( \frac{\alpha}{2} + j \right) \times \right.$$

$$\left. \times 2F_1 \left( -n, n - \frac{\beta}{2} \frac{\alpha}{2} ; -\frac{\alpha}{\beta} x_0 \right) 2F_1 \left( -n, n - \frac{\beta}{2} \frac{\alpha}{2} ; -\frac{\alpha}{\beta} x \right) \right\}. \quad (43)$$

In expression (43) we can recognize non-normalized (with respect to the speed density) Fisher-Snedecor polynomials given by formula (32).

Furthermore, Green’s function (41) has a branch point at $s = -\Lambda = -\frac{\theta \beta^2}{2(\beta - \theta)}$, since for $s < -\Lambda$ the gamma function $\Gamma(z_{+s})$ has the argument with non-zero imaginary part. Note that this branch point coincides with the negative cutoff between the discrete and the continuous part of the spectrum of the Sturm-Liouville operator $(-G)$. The branch cut of discontinuity is placed from $-\Lambda$ to $-\infty$ on the negative part of the real axis and is parameterized as

$$s = -\lambda = -\frac{2\theta}{\beta} - 2 \left( \frac{\beta^2}{16} + 1 \right), \quad k > 0.$$ 

After this analysis we can start with the evaluation of the inverse Laplace transform of Green’s function (41) using the inversion formula (37). This procedure results in the spectral representation of transition density of Fisher-Snedecor diffusion. First we observe the Bromwich contour $C$ as shown in the following figure.
Since simple poles \((-\lambda_n)\) of the Green’s function are placed inside the contour \(C\), according to the Cauchy Residue Theorem it follows that

\[
\frac{1}{2\pi i} \oint_C e^{st} G_s(x_0, x) \, ds = \sum_{n=0}^{\left\lfloor \frac{\pi}{t} \right\rfloor} \text{Res}_{s=-\lambda_n} e^{st} G_s(x_0, x). \tag{44}
\]

On the other hand, the integral around the contour \(C\) is equal to the sum of the integral along the line \(AB\), the integrals along the arcs \(BCD\) and \(HIA\), the integrals along the lines \(DE\) and \(GH\) on each side of the branch cut, and the integral along the arc \(EFG\) around the branch point \(s = -\Lambda\).

Asymptotic behavior of the Green’s function (41) as the radius \(r\) of the arc \(EFG\) around the branch point tends to zero implies that the integral along this arc vanishes. To verify this, first substitute \(s = re^{i\gamma} - \Lambda\), \(\gamma \in (-\pi, \pi)\). Then the integral along the arc \(EFG\) has the following form:

\[
\int_{EFG} e^{st} G_s(x_0, x) \, ds = -ie^{-\Lambda t} \int_{-\pi}^{\pi} e^{re^{i\gamma} t} G_{re^{i\gamma} - \Lambda}(x_0, x) re^{i\gamma} \, d\gamma, \tag{45}
\]

where

\[
G_{re^{i\gamma} - \Lambda}(x_0, x) = \frac{m(x)}{2\alpha^{\frac{1}{2}} - \frac{\beta - \frac{\pi}{2}}{2} B_{re^{i\gamma} - \Lambda} \Delta_{re^{i\gamma} - \Lambda}} f_1(x_0 \wedge x, re^{i\gamma} - \Lambda) f_4(x_0 \vee x, re^{i\gamma} - \Lambda). \tag{46}
\]

The expression under the integral vanishes as radius \(r \to 0\) if

\[
\lim_{r \to 0} G_{re^{i\gamma} - \Lambda}(x_0, x) < \infty,
\]

which can be verified by observing the power series expansion of \(G_{re^{i\gamma} - \Lambda}(x_0, x)\) around the point zero:

\[
G_{re^{i\gamma} - \Lambda}(x_0, x) = c^{(1)}_{\alpha, \beta}(x_0 \wedge x, r) + c^{(2)}_{\alpha, \beta}(x_0 \wedge x, r) e^{\frac{\pi}{4} \sqrt{r}} + c^{(3)}_{\alpha, \beta}(x_0 \wedge x, r) e^{\gamma \sqrt{r}} + O(r), \tag{47}
\]

where quantities \(c^{(i)}_{\alpha, \beta}(x_0 \wedge x, r), i \in \{1, 2, 3\}\), are independent of \(r\). Expression (47) implies that \(\lim_{r \to 0} G_{re^{i\gamma} - \Lambda}(x_0, x) < \infty\). Therefore, the integral (45) vanishes as \(r \to 0\).

Furthermore, asymptotic behavior of the Green’s function (41) as the radius \(R\) of the arcs \(BCD\) and \(HIA\) tends to infinity implies that the integrals along these arcs also vanish. Since these integrals are treated similarly, we explain the procedure regarding the arc \(BCD\). Substitution \(s = Re^{i\gamma} - \Lambda\), where \(\gamma \in (\pi/2, \pi)\), transforms the integral along the arc \(BCD\) in the following form:

\[
\int_{BCD} e^{st} G_s(x_0, x) \, ds = ie^{-\Lambda t} \int_{-\pi}^{\pi} Re^{re^{i\gamma} t} G_{Re^{i\gamma} - \Lambda}(x_0, x) e^{i\gamma} \, d\gamma, \tag{48}
\]
where
\[ G_{R \gamma - \Lambda}(x_0, x) = m(x) \frac{\Gamma \left( \frac{-\beta}{2} + \sqrt{\frac{R(\beta - 2)}{2\theta}} e^{i\frac{\beta}{2}} i \gamma \right) \Gamma \left( \frac{\alpha}{2} + \frac{\beta}{2} + \frac{\sqrt{R(\beta - 2)}}{2\theta} e^{i\frac{\beta}{2}} \right)}{\alpha^{1 - \frac{\beta}{2}} \beta^{-\frac{\beta}{2}} \Gamma \left( \frac{\alpha}{2} \right) \Gamma \left( 1 + 2 \sqrt{\frac{R(\beta - 2)}{2\theta}} e^{i\frac{\beta}{2}} \right)} \times f_1(x_0 \wedge x, R e^{i\gamma} - \Lambda) f_2(x_0 \lor x, R e^{i\gamma} - \Lambda). \]

The expression under the integral on the right side of the expression (48) vanishes as the radius \( R \to \infty \) if
\[
\lim_{R \to \infty} R e^{i\gamma} G_{R \gamma - \Lambda}(x_0, x) = 0.
\]

According to the formulas for asymptotic behavior of the Gauss hypergeometric function \( _2F_1(a, b; c; z) \) as two or more of its parameters tend to infinity (see Erdely (17), page 77, expressions (16) and (17)) it follows that
\[
f_1(x_0 \wedge x, R e^{i\gamma} - \Lambda) f_2(x_0 \lor x, R e^{i\gamma} - \Lambda) =
\]
\[
= \frac{\Gamma \left( \frac{-\beta}{2} + \sqrt{\frac{R(\beta - 2)}{2\theta}} e^{i\frac{\beta}{2}} i \gamma \right)}{2\pi^{1+1} \sqrt{\pi}} \Gamma \left( 1 + \frac{\beta}{2} \right) \Gamma \left( 1 + \frac{\beta}{2} + \sqrt{\frac{R(\beta - 2)}{2\theta}} e^{i\frac{\beta}{2}} \right) \times
\]
\[
\times \frac{1}{\alpha^{1 - \frac{\beta}{2}} \beta^{-\frac{\beta}{2}} \Gamma \left( \frac{\alpha}{2} \right) \Gamma \left( 1 + 2 \sqrt{\frac{R(\beta - 2)}{2\theta}} e^{i\frac{\beta}{2}} \right)} \times
\]
\[
\times \sqrt{\frac{2\theta}{R(\beta - 2)}} e^{-i\frac{\beta}{2}} k_{\alpha, \beta}(x_0 \wedge x) e^{i\frac{\beta}{2}} \left[ k_{\alpha, \beta}(x_0 \wedge x) e^{-i\frac{\beta}{2}} \right] \left[ 1 + O \left( \frac{1}{Re^{i\gamma}} \right) \right],
\]
for large values of \( R \), where
\[
k_{\alpha, \beta}(x_0 \wedge x) = 1 + \frac{2\alpha}{\beta} (x_0 \wedge x) + \sqrt{4\alpha/\beta} (x_0 \wedge x) (1 + (x_0 \wedge x)) > 1,
\]
\[
k_{\alpha, \beta}(x_0 \lor x) = 1 + \frac{2\alpha}{\beta} (x_0 \lor x) + \sqrt{4\alpha/\beta} (x_0 \lor x) (1 + (x_0 \lor x)) > 1
\]
and \( k_{\alpha, \beta}(x_0 \wedge x) < k_{\alpha, \beta}(x_0 \lor x) \). From expression (50) it follows that
\[
Re^{i\gamma} G_{R \gamma - \Lambda}(x_0, x) =
\]
\[
= \frac{\sqrt{\frac{2\theta}{R(\beta - 2)}} e^{-i\frac{\beta}{2}} k_{\alpha, \beta}(x_0 \wedge x)}{2\pi^{1+1} \sqrt{\pi}} \left[ k_{\alpha, \beta}(x_0 \wedge x) e^{-i\frac{\beta}{2}} \right] \left[ 1 + O \left( \frac{1}{Re^{i\gamma}} \right) \right]
\]
for large values of \( R \). Since \( k_{\alpha, \beta}(x_0 \wedge x) < k_{\alpha, \beta}(x_0 \lor x) \) and since for \( \pi/2 < \gamma < \pi \) we know that \( \cos \frac{\gamma}{2} > 0 \) and \( \cos \gamma < 0 \) it follows that \( Re^{i\gamma} G_{R \gamma - \Lambda}(x_0, x) \) tends to zero as \( R \to \infty \). Therefore, the integral (48) vanishes as \( R \to \infty \). The same procedure is used for verifying that the integral along the arc \( HIA \) also vanishes as \( R \to \infty \). Problem observed in a recent paper by Carr, Linetsky and Mendoza (38) is treated with the similar technique.
Finally, it follows that the inversion formula (37) for the Green’s function (41) reduces to the sum of the integral along the line \( \text{Re}(x) = c, \ c > 0 \), and the integral of the jump across the branch cut, i.e.

\[
\frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} e^{st}G_s(x_0, x) ds = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} e^{st}G_s(x_0, x) ds + \frac{1}{2\pi i} \int_{-\infty}^{-\Lambda} e^{st}(G_s(f_1, f_4) - \overline{G_s}(f_1, f_4)) ds, \tag{51}
\]

where \( \overline{G_s}(f_1, f_4) \) is the complex conjugate of the Green’s function (41). From expressions (44) and (51) it follows:

\[
\frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} e^{st}G_s(x_0, x) ds = \left[ \sum_{n=0}^{\left\lfloor \frac{c}{\beta} \right\rfloor} \text{Res}_{s=-\lambda_n} e^{st}G_s(x_0, x) + \frac{1}{2\pi i} \int_{-\infty}^{-\Lambda} e^{st}(\overline{G_s}(f_1, f_4) - G_s(f_1, f_4)) ds \right]. \tag{52}
\]

Now it remains to determine the explicit form of the expression \( \overline{G_s}(f_1, f_4) - G_s(f_1, f_4) \) for the jump across the branch cut. Since \( \overline{f}_1(\cdot, s) = f_1(\cdot, s) \), \( \overline{f}_4(\cdot, s) = f_3(\cdot, s) \) and \( f_1(\cdot, s) = Bf_3(\cdot, s) + Af_4(\cdot, s) \) (see Remark 20.), for \( s = -\lambda \) the jump across the branch cut is given by expression

\[
\overline{G}_{-\lambda}(x_0, x) - G_{-\lambda}(x_0, x) = \tag{53}
\]

\[
m(x) 2i k(\lambda) \left| \frac{\alpha^{2\alpha} \beta^2 \Gamma \left( -\frac{\beta}{4} + ik(\lambda) \right) \Gamma \left( \frac{\alpha}{4} + \frac{\beta}{4} + ik(\lambda) \right)}{\Gamma \left( \frac{\alpha}{4} \right) \Gamma \left( 1 + 2ik(\lambda) \right)} \right|^2 \overline{G}(x_0) f_1(x).
\]

Substitution of the expressions (43) and (53) in the relation (52) results in the spectral representation of the transition density of Fisher-Snedecor diffusion. However, Fisher-Snedecor polynomials and solutions of the Sturm-Liouville equation (26) for \( \lambda > \Lambda \) are not normalized with respect to the Fisher-Snedecor density (9). Now the uniqueness of solutions of the Sturm-Liouville equation up to a constant factor ensures that instead of the solution \( f_1(x, -\lambda) \) we can use the solution \( \overline{f}_1(x, -\lambda) = \sqrt{\frac{2\alpha}{\beta-2}} f_1(x, -\lambda) \). This procedure results in the expression for the spectral representation of transition density in terms of functions normalized with respect to the Fisher-Snedecor density, i.e. the result is exactly the expression \( p(x; x_0, t) = p_d(x; x_0, t) + p_c(x; x_0, t) \), where \( p_d(x; x_0, t) \) is given by (39) and \( p_c(x; x_0, t) \) by (40).

**Remark 4.7.** The two-dimensional density of the ergodic diffusion with invariant Fisher-Snedecor distribution with parameters \( \alpha > 2, \alpha \notin \{2(m+1), m \in \mathbb{N}\} \), and \( \beta > 2 \) is given by the expression

\[
f(x, y, t) = \frac{\partial^2}{\partial x \partial y} P(X_{s+t} \leq x, X_s \leq y) = \Phi_\alpha(y) p(x; y, t) = \Phi_\alpha(y) (p_d(x; y, t) + p_c(x; y, t)), \tag{54}
\]

where \( p_d(x; y, t) \) is given by (39) while \( p_c(x; y, t) \) is given by (40).

The statement of the following proposition follows from the general result for the spectral representation of transition density for one-dimensional diffusions (see Itô and McKean (23)). This special case for the Fisher-Snedecor diffusion is treated here in details to demonstrate the use of the spectral representation of transition density (see Theorem 4.1) in possible explicit calculations.

**Proposition 4.1.** Fix \( i \) and \( j \) such that \( i, j \in \{1, \ldots, \lfloor \beta/4 \rfloor \} \) and suppose that \( \beta > 2(i + j), \alpha > 2, \alpha \notin \{2(m+1), m \in \mathbb{N}\} \) and \( \theta > 0 \). If \( (X_{s+t}, X_s) \) is a two-dimensional random vector with the density (54) and \( F_n(x), n \in \{1, \ldots, \lfloor \beta/4 \rfloor \}, \) are orthonormal Fisher-Snedecor polynomials given by (32), then

\[
E[F_i(X_{s+t}) F_j(X_s)] = e^{-\lambda_{ij} t} \delta_{ij},
\]

where \( \lambda_j \) is the eigenvalue given by (29) and \( \delta_{ij} \) the standard Kronecker symbol.

**Proof.** The two-dimensional density \( p(x, y, t) \) given by (54) could be written in the following form:

\[
p(x, y, t) = \sum_{n=0}^{\left\lfloor \frac{\beta}{4} \right\rfloor} e^{-\lambda_n t} \Phi_n(x) \Phi_n(y) + \frac{1}{2\pi} \int_{\frac{\beta}{4} - \frac{\alpha}{2}}^{\infty} e^{-\lambda t} \nu^2(\lambda) \Psi(x, \lambda) \Psi(y, \lambda) d\lambda, \tag{55}
\]
where

\[ \nu^2(\lambda) = k(\lambda) \left[ \frac{B^\frac{1}{4} \left( \frac{\alpha}{2}, \frac{\beta}{2} \right)}{\Gamma \left( \frac{3}{2} \right)} \frac{\Gamma \left( -\frac{\beta}{2} + ik(\lambda) \right) \Gamma \left( \frac{\alpha}{2} + \frac{\beta}{2} + ik(\lambda) \right)}{\Gamma \left( \frac{1}{2} \right) \Gamma \left( 1 + 2ik(\lambda) \right)} \right]^2, \]

and where

\[ \Phi_n(\cdot) = \frac{f_s(\cdot)}{F_n(\cdot)}, \]
\[ \Psi(\cdot) = \frac{f_s(\cdot)}{f_1(\cdot, -\lambda)}. \]

Here \( F_n(\cdot), n \in \{1, \ldots, \lfloor \beta/4 \rfloor \}, \) are Fisher-Snedecor polynomials and \( \sqrt{\nu^2(\lambda)} f_1(\cdot, -\lambda) \) are solutions of the Sturm-Liouville equation (26) for \( \lambda > \Lambda, \) both normalized with respect to the invariant density of the Fisher-Snedecor diffusion. Orthogonality of Fisher-Snedecor polynomials \( F_n(\cdot), n \in \{1, \ldots, \lfloor \beta/4 \rfloor \}, \) and functions \( f_1(\cdot, -\lambda), \lambda > \Lambda, \) (see the Remark 4.6) together with the representation (55) of two-dimensional density (54) implies the following:

\[ E[F_1(X_{s+t}) F_j(X_s)] = \int_0^\infty \int_0^\infty F_1(x) F_j(y) p(x, y, t) \, dx \, dy = \]

\[ = \sum_{n=0}^{\left\lfloor \frac{\beta}{4} \right\rfloor} e^{-\lambda_n t} \left( \int_0^\infty F_1(x) F_n(x) f_s(x) \, dx \right) \left( \int_0^\infty F_j(y) F_n(y) f_s(y) \, dy \right) + \]
\[ + \frac{1}{\pi} \int_{\frac{\beta}{2} / \pi}^\infty \nu^2(\lambda) e^{-\lambda t} \left( \int_0^\infty F_1(x) f_1(x, \lambda) f_s(x) \, dx \right) \left( \int_0^\infty F_j(y) f_1(y, \lambda) f_s(y) \, dy \right) \, d\lambda = \]
\[ = e^{-\lambda t} \delta_{ij} = \exp \left\{ -\frac{\beta - 2j}{\beta - 2} t \right\} \delta_{ij}, \]

where, according to the Remark 4.6, the term related to the continuous part of the spectral representation (38) disappears. In order to apply the Fubini’s theorem in the above computation we used the property that the solution \( f_1(z, -\lambda) = 2 F_1(a, b; c; -z) \) of the hypergeometric equation can be written in the form

\[ f_1(z, -\lambda) = C_1 (z)^{-a} + C_2 (z)^{-b} + O((-z)^{-a-1}) + O((-z)^{-b-1}), \]

where \( C_1 \) and \( C_2 \) are constants and \( (a - b) \) is not an integer. \( \square \)

The result of the Proposition 4.1 is highly applicable in statistical analysis of Fisher-Snedecor diffusion. It enables explicit calculations of, for example, covariances of method of moments estimators of unknown parameters. For direct application of this and similar results regarding other heavy-tailed Pearson diffusions we refer to Avram et al. (5) and Leonenko and Suvaš (29; 30).

**Remark 4.8.** The general approach to the spectral theory of one-dimensional diffusions is treated in a number of papers by Linetsky (see (31; 32; 33; 38)). Beside the concise and informative introduction to basics of the spectral theory, he gives the procedure for calculation of the spectral representation of the transition density for one-dimensional diffusions (see (33; 38)). We already used his approach for transition densities of reciprocal gamma and Student diffusion (see (29; 30; 8)). Since these processes, together with the Fisher-Snedecor diffusion, make a class of heavy-tailed Pearson diffusions, it is natural that spectral representations of their transition densities have similar forms. Namely, for all three heavy-tailed Pearson diffusions spectral representations of transition densities are given in terms of the finite sum of residues and an integral of the jump across the branch cut of the corresponding Green’s function. However, the differences between mentioned spectral representations are generated by different types of the corresponding Sturm-Liouville equations. In particular, Sturm-Liouville equation related to reciprocal gamma diffusion can be transformed to the Whittaker equation, while the Sturm-Liouville equations for Student and Fisher-Snedecor diffusions can be transformed to the hypergeometric equations. According to the qualitative nature of the spectrum of the corresponding Sturm-Liouville operator \((-\mathcal{G})\), the continuous part of the spectral representation of transition density of reciprocal gamma diffusion is written in terms of Whittaker functions or hypergeometric function \( _2F_0(a; b; \cdot) \) (see (32; 33; 29; 50)), while the continuous part of the spectral representation for Student and Fisher-Snedecor diffusions is written in terms of the hypergeometric function \( _2F_1(a, b; c; \cdot) \), see Appendix B.
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Appendix A

Feller’s and Weyl’s classification of boundaries of the state space of Fisher-Snedecor diffusion

In this section we present classical Feller’s, Weyl’s limit-point/limit-circle (LP/LC) and oscillatory/non-oscillatory (O/NO) classification of boundaries of the diffusion state space. For general information on Feller’s classification scheme we refer to Karlin and Taylor (24) and Linetsky (31), while for Weyl’s LP/LC and O/NO classification we refer to Fulton et al. (19)) and Linetsky (31). The nature of each boundary is determined according to the behavior of the Sturm-Liouville equation (26) near it.

Theorem 4.2. Boundaries of the state space of Fisher-Snedecor diffusion with parameters $\alpha > 0$, $\alpha \notin \{2m, m \in \mathbb{N}\}$, and $\beta > 2$ are classified as follows:

(i) Boundary $e_1 = 0$ is regular for $\alpha < 2$ and entrance otherwise, while $e_2 = \infty$ is natural for $\alpha > 0$, $\alpha \notin \{2m, m \in \mathbb{N}\}$.

(ii) For $\alpha > 0$, $\alpha \notin \{2m, m \in \mathbb{N}\}$, boundary $e_1 = 0$ is non-oscillatory, while $e_2 = \infty$ is oscillatory/non-oscillatory with unique positive cutoff

$$\Lambda = \frac{\theta \beta^2}{8(\beta - 2)}.$$

Boundary $e_2 = \infty$ is non-oscillatory for $\lambda \leq \Lambda$ and oscillatory for $\lambda > \Lambda$.

(iii) Boundary $e_1 = 0$ is of limit-circle type for $\alpha < 4$ and of limit-point type otherwise, while boundary $e_2 = \infty$ is of limit-point type for $\alpha > 0$, $\alpha \notin \{2m, m \in \mathbb{N}\}$.

Proof. The proof consists of three parts, as we observe three boundary classification schemes.

(i) Feller’s boundary classification is based on the behavior of the scale function

$$S[x, y] = \int_x^y s(z) \, dz$$

near both boundaries, where $s(x)$ is the scale density defined by (12). In case of the Fisher-Snedecor diffusion, this non-standard integral is evaluated using Mathematica:

$$S[x, y] = \frac{2 \beta^2 + \frac{\alpha}{2} - 1}{2 - \alpha} \left[ y^{1 - \frac{\alpha}{2}} \frac{2}{1 - \alpha} \right] _{\alpha 2, 1 - \alpha 2 \beta 2, 2 - \alpha 2 \beta y} - x^{1 - \frac{\alpha}{2}} \frac{2}{1 - \alpha 2, 1 - \alpha 2 \beta 2, 2 - \alpha 2 \beta y}.$$ 

Since $\lim_{x \to 0} \frac{2}{1 - \alpha 2, 1 - \alpha 2 \beta 2, 2 - \alpha 2 \beta y}$ does not converge as $y \to \infty$, for $\alpha \notin \{2m, m \in \mathbb{N}\}$ the function $S[x, y]$ has the following properties:

$$S[0, y] = \lim_{x \to 0} S[x, y] < \infty, \quad \alpha < 2,$$

$$S[0, y] = \lim_{x \to 0} S[x, y] = \infty, \quad \alpha > 2,$$

$$S[x, \infty] = \lim_{y \to \infty} S[x, y] = \infty, \quad \alpha > 0.$$
These results imply that for $\alpha \notin \{2m, m \in \mathbb{N}\}$ and arbitrary $\varepsilon > 0$ we have:

$$
\int_0^\varepsilon S(0, y)m(y)\,dy < \infty, \alpha < 2, \quad \int_0^\varepsilon S(0, y)m(y)\,dy = \infty, \alpha > 2, \quad \int_0^\infty S[x, \infty)m(x)\,dx = \infty, \alpha > 0,
$$

$$
\int_0^\varepsilon S[\varepsilon, y)m(y)\,dy = \infty, \alpha > 0, \quad \int_0^\varepsilon S[x, \varepsilon)m(x)\,dx < \infty, \alpha > 0.
$$

Therefore, according to the standard Feller’s boundary classification scheme, $e_1 = 0$ is regular boundary for $\alpha < 2$ and entrance boundary for $\alpha > 2$, while $e_2 = \infty$ is natural boundary for all positive values of $\alpha, \alpha \notin \{2m, m \in \mathbb{N}\}$.

(ii) Since $e_1 = 0$ is either regular or entrance boundary, it is necessary NO (see Linetsky (31)). For O/NO classification of the natural boundary $e_2 = \infty$ the standard procedure is transformation of the Sturm-Liouville equation (26) to the Liouville normal form (see Fulton et al. (19)). If we observe the Sturm-Liouville equation (26) written in the form

$$
-x^2(\alpha x + \beta)^{1-\frac{2}{\alpha}} f'(x') = \lambda \frac{\alpha(\beta - 2)}{2\theta} x^{\frac{2}{\alpha}-1}(\alpha x + \beta)^{-\frac{2}{\alpha}} f(x),
$$

then

$$
u(x) = \sqrt{\frac{2(\beta - 2)}{\theta}} \arcsinh \left(\sqrt{\frac{\alpha}{\beta}} x\right), \quad x(u) = \frac{\beta}{\alpha} \sinh^2 \left(u \sqrt{\frac{\theta}{2(\beta - 2)}}\right),$$

$$h(u) = \sqrt[4]{\frac{\alpha(\beta - 2)}{2\theta} \left[ \beta \cosh^2 \left(u \sqrt{\frac{\theta}{2(\beta - 2)}}\right) \right]^{1-\alpha - \beta} \left[ \frac{\beta}{\alpha} \sinh^2 \left(u \sqrt{\frac{\theta}{2(\beta - 2)}}\right) \right]^{\alpha - 1}}$$

and

$$Q(u) = \frac{\theta}{8(\beta - 2)} \left( \beta^2 + (\alpha - 1)(\alpha - 3) \cosh^2 \left(u \sqrt{\frac{\theta}{2(\beta - 2)}}\right) - [(\alpha + \beta)^2 - 1] \sinh^2 \left(u \sqrt{\frac{\theta}{2(\beta - 2)}}\right) \right).$$

Therefore, the Liouville normal form of the Sturm-Liouville equation (26) is

$$-u''(u) + (Q(u) - \lambda) u(u) = 0. \quad (56)$$

In this particular case the natural boundary $e_2 = \infty$ remains unchanged under the Liouville transformation, i.e. the corresponding boundary of the equation (56) is $e_2^* = u(e_2) = \infty$. O/NO classification of the boundary $e_2^* = \infty$ depends on the behavior of the potential function $Q(u)$ near that endpoint. Since

$$\lim_{u \to \infty} \cosh^2 \left(u \sqrt{\frac{\theta}{2(\beta - 2)}}\right) = 0 \quad \text{and} \quad \lim_{u \to \infty} \sinh^2 \left(u \sqrt{\frac{\theta}{2(\beta - 2)}}\right) = 0,$$

it follows that

$$\lim_{u \to \infty} Q(u) = \frac{\theta \beta^2}{8(\beta - 2)}, \quad \beta > 2.$$

Since $\lim_{u \to \infty} Q(u) < \infty$, according to Fulton et al. (see (19), Theorem 6) $e_2^* = \infty$ is O/NO boundary of the equation (56) with the unique positive cutoff

$$\Lambda = \frac{\theta \beta^2}{8(\beta - 2)}, \quad \beta > 2,$$

i.e. it is NO for $\lambda < \Lambda$ and O for $\lambda > \Lambda$. According to Linetsky (see (31), Theorem 3), the boundary $e_2^* = \infty$ is non-oscillatory for $\lambda = \Lambda$. This classification of boundaries remains invariant under the
For LP/LC classification of the boundary $e_1 = e_2 = \infty$ is O/NO boundary of the Sturm-Liouville equation (26) with the unique positive cutoff $\Lambda$. Furthermore, it is NO for $\lambda \leq \Lambda$ and O for $\lambda > \Lambda$.

(iii) Since $e_1 = 0$ is the regular boundary for $\alpha < 2$, it is necessary of the LC type. For LP/LC classification of the boundary $e_1 = 0$ for $\alpha > 2$, $\alpha \notin \{2m, m \in \mathbb{N}\}$, we observe its Liouville trasformation $e_1^* = u(e_1) = 0$ and $\liminf_{u \to e_1^*} u^2 Q(u) = \limsup_{u \to e_1^*} u^2 Q(u)$ (see Fulton (19), Theorem 5). Since

$$\lim_{u \to 0} u^2 \text{csch}^2 \left( u \sqrt{\frac{\theta}{2(\beta - 2)}} \right) = \frac{2(\beta - 2)}{\theta}$$

it follows that

$$\lim_{u \to 0} u^2 Q(u) = \frac{(\alpha - 1)(\alpha - 3)}{4}$$

and

$$\liminf_{u \to 0} u^2 Q(u) = \limsup_{u \to 0} u^2 Q(u) = \frac{(\alpha - 1)(\alpha - 3)}{4}.$$ 

Now it follows that

$$\liminf_{u \to 0} u^2 Q(u) < \frac{3}{4}$$

for $\alpha \in (2, 4)$, while

$$\limsup_{u \to 0} u^2 Q(u) \geq \frac{3}{4}$$

for $\alpha > 4$, $\alpha \notin \{2m, m \in \mathbb{N}\}$. According to Fulton et al., $e_1^* = 0$ is the boundary of LC type for $\alpha \in (2, 4)$ and of LP type for $\alpha > 4$, $\alpha \notin \{2m, m \in \mathbb{N}\}$.

For LP/LC classification of the boundary $e_2 = \infty$ we observe LP/LC classification of its Liouville transformation $e_2^* = \infty$ which depends on the behavior of the function $Q(u)/u^2$ near that endpoint. Since

$$\lim_{u \to \infty} u^{-2} \text{csch}^2 \left( u \sqrt{\frac{\theta}{2(\beta - 2)}} \right) = 0$$

and

$$\lim_{u \to \infty} u^{-2} \text{sech}^2 \left( u \sqrt{\frac{\theta}{2(\beta - 2)}} \right) = 0,$$

it follows that

$$\lim_{u \to \infty} \frac{Q(u)}{u^2} = 0.$$

According to Fulton et al. (see (19), Theorem 7) $e_2^* = \infty$ is the boundary of LP type for all $\alpha > 0$, $\alpha \notin \{2m, m \in \mathbb{N}\}$. This classification of boundaries remains invariant under the Liouville transformation (see (19), Lemma 1), i.e. the boundary $e_1 = 0$ is of LC type for $\alpha < 4$ and of LP type for $\alpha > 4$, while the boundary $e_2 = \infty$ is of LP type for all $\alpha > 0$, $\alpha \notin \{2m, m \in \mathbb{N}\}$.

Appendix B

Gauss Hypergeometric Functions

Gauss hypergeometric series is the power series

$$\text{$_2F_1(a, b; c; z) = \sum_{k=0}^{\infty} \frac{(a)_k (b)_k}{(c)_k k!} z^k = 1 + \frac{ab}{c} z + \frac{a(a+1) b(b+1)}{c(c+1)} \frac{z^2}{2} + \ldots, \quad (57)}$$

where $z$ is a complex variable, $a$, $b$ and $c$ are real or complex parameters and $(a)_k$ is the Pochhammer symbol which denotes the quantity

$$(a)_0 = 1, \quad (a)_k = \frac{\Gamma(a + k)}{\Gamma(a)} = a(a+1) \cdot \ldots \cdot (a+k-1), \quad k \in \mathbb{N}.$$ 

The series is not defined for $c = -m$, $m \in \mathbb{N}_0$, provided that $a$ or $b$ is not the negative integer $n$ such that $n < m$. Furthermore, if the series (57) is defined but $a$ or $b$ is equal to $(-n)$, $n \in \mathbb{N}_0$, then it terminates in a finite number of terms and its sum is then the polynomial of degree $n$ in variable $z$. Except for this case, in which the series is absolutely convergent for $|z| < \infty$, the radius of absolute convergence of the
series (57) is the unit circle, i.e. $|z| < 1$. In this case it is said that the series (57) defines the Gauss or hypergeometric function
\[ g(z) = {}_2F_1(a, b; c; z). \] (58)

It can be verified (see Slater (45), Section 1.2., page 6) that the function $g(z)$ is the solution of the second order differential equation
\[ z(1-z)g''(z) + (c - (a + b + 1)z)g'(z) - abg(z) = 0, \] (59)
in the region $|z| < 1$. However, the function (58) can be analytically continued to the other parts of the complex plane, i.e. solutions of the equation (59) are also defined outside the unit circle. These solutions are provided by following substitutions in the equation (59):
- substitution $z = 1 - y$ yields solutions valid in the region $|1-z| < 1$,
- substitution $z = 1/y$ yields solutions valid in the region $|z| > 1$.

Furthermore, solutions in these three regions of the complex plane yield solutions valid in the regions $|1-z| > 1$, $\text{Re}(z) > 1/2$ and $\text{Re}(z) < 1/2$. The number of solutions in these six regions is 24 and they are known as Kummer’s system of solutions. This system of solutions is very important since it provides analytic continuation of the hypergeometric function to the whole complex plane with an exception of the branch cut along the interval $[1, \infty)$. Hence, the hypergeometric function $g(z)$ is a solution of the equation (59) which provides analytic continuation of the hypergeometric function, can be found in Luke (34) (see Section 3.9., page 69).

\begin{align*}
\frac{\Gamma(c)}{2\pi i \Gamma(a) \Gamma(b)} \int_{-i\infty}^{i\infty} \frac{\Gamma(a+s) \Gamma(b+s) \Gamma(-s)}{\Gamma(c+s)} (-z)^s \, ds.
\end{align*}
(60)

Namely, Mellin-Barnes integral defines the analytic function on the complex plane with the branch cut along the positive real axis, i.e. along the interval $(0, \infty)$. Analytic function (58) defined by the Gauss hypergeometric series and the analytic function defined by the Mellin-Barnes integral coincide on the region $\{ z \in \mathbb{C} : |z| < 1, z \notin \{0, 1\} \}$. Therefore, the Mellin-Barnes integral provides the analytic continuation of the function (58) to the region $|z| > 1$ cutted along the interval $(1, \infty)$. Hence, the hypergeometric function $g(z)$ is an analytic function defined on the complex plane with the branch cut situated along the interval $(1, \infty)$. Similarly, the hypergeometric function $\text{Kummer's system of solutions}$ is very important since it provides analytic continuation of the hypergeometric function, can be found in Luke (34) (see Section 3.9., page 69).

References


