On spectral analysis of heavy-tailed Kolmogorov-Pearson diffusions

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Abstract  The self-adjointness of the semigroup generator of one dimensional diffusions implies a spectral representation (McKean, 1956; Kac & Krein, 1974) which has found many useful applications, for example for the prediction of second order stationary sequences (Dym & McKeen, 1976) and in mathematical finance (Linetsky, 2007).

However, on non-compact state spaces, the spectrum of the generator will typically include both a discrete and a continuous part, with the latter starting at a spectral cutoff point related to the nonexistence of stationary moments, and the significance of this fact for statistical estimation is not fully understood.

We consider here the problem of spectral representation of transition density for an interesting class of examples: the hypergeometric diffusions with heavy-tailed Pearson type invariant distribution of a) Reciprocal (inverse) gamma, b) Fisher-Snedecor, or c) skew-Student type. As opposed to the "classic" hypergeometric diffusions (Ornstein-Uhlebeck, Gamma/CIR, Beta/Jacobi), these diffusions have a continuum spectrum, whose spectral cutoff and transition density we present.

Keywords  Diffusion process · Infinitesimal generator · Kolmogorov-Pearson diffusion · Sturm-Liouville equation · Transition density

Mathematics Subject Classification (2010) 33C47 60G10 60J25 60J60 62M15

1 Introduction

Motivation. The investigation of the connections between the analytic and probabilistic aspects of the theory of diffusions has been initiated by classic papers of Kolmogorov and Feller (Kolmogoroff, 1931; Feller, 1952). This line of research achieved its maturity in the works of McKeen and Itô (McKean, 1956; Itô & McKeen, 1963, 1974), culminating with Itô’s excursion theory (Itô, 1971) (see (Pitman & Yor, 2007).
for a friendly introduction) and with the Dym-McKean prediction theory for second order stationary sequences (Dym & McKean, 1976). However, many "exercises" left to the reader in these monumental works are yet to be solved, and nowadays research is still clarifying the interplay of various diffusion parameters - scale, speed, Green function, Laplace exponent, Krein string representation, spectral measure ((Pitman & Yor, 2003; Steinsaltz & Evans, 2007; Kolb & Steinsaltz, 2010; Comtet & Tourigny, 2011)). This motivated us to investigate below the example of heavy-tailed Kolmogorov-Pearson (KP) diffusions, a family which seems quite interesting both as a case study for diffusion theory, and for statistical modeling.

**Historical roots of the problem.** The study of diffusion processes with invariant distributions from the Pearson family started in the celebrated paper (Kolmogoroff, 1931) in which Kolmogorov studied the Fokker-Planck (now called forward Kolmogorov) equation for diffusions with linear drift and quadratic diffusion coefficient

\[
\frac{\partial p}{\partial t} = -\frac{\partial}{\partial y} \left( (b_1 y + b_0) p \right) + \frac{\partial^2}{\partial y^2} \left( (a_2 y^2 + a_1 y + a_0) p \right), \quad p = p(x, y, t), \quad y \in \mathbb{R}, \quad t \geq 0,
\]

and introduced the backward Kolmogorov equation

\[
\frac{\partial p}{\partial t} = (b_1 x + b_0) \frac{\partial}{\partial x} p + (a_2 x^2 + a_1 x + a_0) \frac{\partial^2}{\partial x^2} p, \quad p = p(x, y, t), \quad x \in \mathbb{R}, \quad t \geq 0.
\]

In particular, Kolmogorov observed that the differential equation for the invariant density \( m(\cdot) \)

\[
\frac{m'(x)}{m(x)} = \frac{b(x) - a'(x)}{a(x)} = \frac{(b_1 - 2a_2)x + (b_0 - a_1)}{a_2 x^2 + a_1 x + a_0}, \quad x \in \mathbb{R},
\]

becomes, in the case of a linear drift \( b(x) = b_1 x + b_0 \) and a quadratic diffusion rate \( a(x) = a_2 x^2 + a_1 x + a_0 \), the famous Pearson equation introduced in (Pearson, 1895) (and of illustrious history - see Diaconis and Zabell (Diaconis & Zabell, 1991)) in order to unify some of the most important statistical distributions

A second motivation for studying this class is a celebrated result of S. Bochner (Bochner, 1929): if an infinite sequence of polynomials \( P_n(x) \) satisfies a second order eigenvalue equation of the form

\[
a(x)P_n(x)'' + b(x)P_n(x)' + c(x)P_n(x) = \lambda_n P_n(x), \quad n = 0, 1, 2, \ldots
\]

then \( a(x), b(x) \) and \( c(x) \) must be polynomials of degree 2, 1 and 0, respectively. If in addition \( P_n(x) \) are the infinite sequence of orthogonal eigenfunctions of a diffusion, then they must be, up to an affine transformation of \( x \), one of the classical orthogonal polynomial systems of Jacobi, Laguerre or Hermite, all computable from the "Rodrigues formula"

\[
P_j(x) = \frac{1}{m(x)} D^j(m(x)a(x))'j.
\]

\[1\] The first appearance of the Pearson equation and of the important family of densities it defines was in the work of K. Pearson (Pearson, 1895), who noted that the densities of several of the most important distributions in statistics satisfy (1). The motivation of Pearson was to generalize an identity of Alexander Cuming and De Moivre concerning the first absolute centered moment of the binomial distribution to continuous distributions. In doing so, he provided a useful parametric model which includes some of the most important distributions in probability and statistics: Gaussian, exponential, Gamma, Beta, etc. The corresponding class of diffusions having these distributions as stationary measures includes also some of the most useful diffusion models, for example Ornstein-Uhlenbeck, the CIR/Feller process, (quite popular in the modeling of interest rates) and the Black-Scholes model.
In cases of other Kolmogorov-Pearson diffusions corresponding eigenfunctions are less known finite systems of orthogonal polynomials: Fisher-Snedecor polynomials for Fisher-Snedecor diffusion (see (Avram et al., 2010, Appendix A)), Bessel polynomials for Reciprocal gamma diffusion (see (Leonenko & Šuvak, 2010a, Appendix A)) and Routh-Romanovski polynomials for Student diffusion (for symmetric Student case see (Leonenko & Šuvak, 2010b, subsection 4.4)). It seems appropriate to call this important class of processes Kolmogorov-Pearson (KP) or hypergeometric diffusions, due to the appearance of the Gauss $2F_1$ function and its limiting confluent forms in various explicit formulas.

For a long period of time after (Kolmogoroff, 1931), Kolmogorov-Pearson diffusions were neglected in the probability literature, with some notable exceptions like Karlin and McGregor (Karlin & McGregor, 1960) and Wong (Wong, 1963). The first have checked on the Kolmogorov-Pearson class the total positivity of diffusion transition kernels (which is equivalent to the continuity of diffusion paths). The latter reemphasized the importance of this class of models as a most natural extension of the "first order statistical description characterized by $m(x)$" to a time dependent model, and computed spectral representations of the transition density in some cases (but omitted the Fisher-Snedecor case).

**Mathematical finance revival.** Recently, the interest in these processes was reawakened in the context of financial modeling. The most famous case is the Merton-Black-Scholes diffusion

$$dX_t = rX_t \, dt + \sigma X_t \, dW_t \Leftrightarrow X_t = X_0 e^{(r - \sigma^2/2)t + \sigma W_t}.$$  

This has had a huge impact in mathematical finance due to its tractability, which lead to a large variety of explicit formulas for the transition and first passage probabilities necessary for the pricing of options. The Ornstein-Uhlenbeck (Vasicek), Cox-Ingersoll-Ross (CIR) and constant elasticity of variance (CEV) models have also been used intensively.

In the search for more flexible models, financial mathematicians have investigated recently the family of all diffusions which are equivalent up to stochastic transformations to the Kolmogorov-Pearson class. Some notable contributions are due to Albanese, Campolieti, Carr and Lipton (Albanese et al., 2001; Campolieti & Makarov, 2009), Linetsky (Linetsky, 2004), Kuznetsov (Kuznetsov, 2004; Albanese & Kuznetsov, 2007), and Mendoza-Arriaga, Carr and Linetsky (Mendoza-Arriaga et al., 2010). In parallel, the statistical analysis of these processes was developed by Küchler, Sørensen and Forman (Forman & Sørensen, 2008; Sørensen & Küchler, 2010). Recently, the interest in developing tractable processes resulted in the introduction by Cuchiero, Keller-Ressel and Teichmann (Cuchiero et al., 2008) of the unifying family of polynomial Markovian processes, whose generator maps polynomials into polynomials of (at most) the same degree.

**The Kolmogorov-Pearson diffusions.** We study here some analytical and probabilistic properties of diffusion processes

$$dX_t = \sigma(X_t) dB_t + b(X_t) dt$$  

with quadratic diffusion coefficient

$$a(x) = \frac{\sigma^2(x)}{2} = \epsilon(a_2 x^2 + a_1 x + a_0), \quad \epsilon > 0, \quad a_2 > 0$$  

and linear drift

$$b(x) = -\theta(x - \mu) = -\theta x + p.$$
Remark 1 When \( \theta > 0 \), taking the parameter \( \epsilon = \theta \) in the diffusion coefficient yields the Sørensen parametrization, while letting \( \epsilon \to 0 \) allows studying the diffusion with small noise.

The second notation for the drift is necessary in order to be able to let the two parameters \( \theta \) and \( p \) go to 0 independently. The first notation for the drift reveals the mean \( \mu \) of this family and identifies \( \theta > 0 \) as the “speed of reverting” to \( \mu \) of the dynamical system obtained by letting the noise parameter \( \epsilon \) go to 0.

A classification of Kolmogorov-Pearson diffusions may be achieved by using the degree of the polynomial \( a(x) \) from the diffusion coefficient (the denominator of the Pearson equation (1)), the sign of its leading coefficient \( a_2 \) and the discriminant \( \Delta(a) \) in the quadratic case (see Table 1).

Table 1 The scale and speed density for Kolmogorov-Pearson diffusions

<table>
<thead>
<tr>
<th>Name</th>
<th>( a(x) )</th>
<th>Scale density ( s(x) )</th>
<th>Speed density</th>
</tr>
</thead>
<tbody>
<tr>
<td>Brownian Motion ((\theta = 0))</td>
<td>1/2</td>
<td>( e^{-2px} )</td>
<td>( 2 e^{2px} )</td>
</tr>
<tr>
<td>Ornstein Uhlenbeck</td>
<td>1/2</td>
<td>( e^{-2px+\delta x^2} )</td>
<td>( e^{-\theta(x-\mu)^2} )</td>
</tr>
<tr>
<td>CIR/Squared OU/Gamma</td>
<td>( x )</td>
<td>( x^{-p_0 e^{\delta x}} )</td>
<td>( x_0 e^{-p_0 e^{\delta x}} )</td>
</tr>
<tr>
<td>Jacobi/Beta, ( a_1 &gt; 0 )</td>
<td>( x(a_1-a_2x) )</td>
<td>( \frac{e^{2px}}{a_1-a_2x^2} )</td>
<td>( \frac{x_0 e^{-p_0 e^{\delta x}}}{a_1-a_2x^2} )</td>
</tr>
<tr>
<td>Rec. gamma /IGBM</td>
<td>( a_2x^2 )</td>
<td>( \sigma(x) )</td>
<td>( \frac{\sigma(x)}{p_0 e^{\delta x}} )</td>
</tr>
<tr>
<td>Student/hypergeometric</td>
<td>( a_2x^2 + a_0 )</td>
<td>( \sigma(x) )</td>
<td>( \frac{\sigma(x)}{p_0 e^{\delta x}} )</td>
</tr>
<tr>
<td>Fisher-Snedecor</td>
<td>( x(a_1 + a_2x) )</td>
<td>( \frac{e^{2px}}{a_1 + a_2x^2} )</td>
<td>( \frac{x_0 e^{-p_0 e^{\delta x}}}{a_1 + a_2x^2} )</td>
</tr>
</tbody>
</table>

The Table 2 below recalls the Feller boundary classification and the recurrence conditions for non-regular Kolmogorov-Pearson diffusions, which consist of the non-integrability at \( x = l, x = r \) of the scale density (Aït-Sahalia, 1996, pp 415, 416, assumption A1 (iii))). Recall that a boundary is called regular if both the scale and speed measure of the state space are finite in a neighborhood, natural if neither is finite, and exit-non entrance/entrance-non exit if the criteria of Definition 2 hold.

Table 2 Boundary classification and recurrence conditions for Kolmogorov-Pearson diffusions

<table>
<thead>
<tr>
<th>Name</th>
<th>Left boundary ( l )</th>
<th>Right boundary ( r )</th>
<th>Positive recurrence</th>
</tr>
</thead>
<tbody>
<tr>
<td>Ornstein Uhlenbeck</td>
<td>( -\infty ) - natural</td>
<td>( \infty ) - natural</td>
<td>( \theta \geq 0 )</td>
</tr>
<tr>
<td>CIR/Squared OU/ Gamma</td>
<td>0 - exit for ( p \leq 0 )</td>
<td>regular for ( 0 &lt; p &lt; 1 )</td>
<td>( \theta &gt; 0 ) or ( \theta = 0 ) and ( p \geq 1 )</td>
</tr>
<tr>
<td>Jacobi/Beta</td>
<td>0 - exit for ( \frac{p}{a_1} \leq 0 )</td>
<td>regular for ( 0 &lt; \frac{p}{a_1} \leq 1 )</td>
<td>( \frac{p}{a_1} \leq 1, \frac{a_1}{a_2} - \frac{p}{a_1} \leq 0 )</td>
</tr>
<tr>
<td>Reciprocal Gamma/ IGBM</td>
<td>0 - entrance for ( p &gt; 0 )</td>
<td>regular for ( 0 &lt; \frac{p}{a_1} \leq 1 )</td>
<td>( \frac{p}{a_1} \geq 1, \frac{a_1}{a_2} - \frac{p}{a_1} \geq 1 )</td>
</tr>
<tr>
<td>Student/hypergeometric</td>
<td>( -\infty ) - natural</td>
<td>( \infty ) - natural</td>
<td>( p \geq 0, \frac{a_1}{a_2} \geq -1 )</td>
</tr>
<tr>
<td>Fisher-Snedecor</td>
<td>0 - entrance for ( \frac{p}{a_1} \geq 1 )</td>
<td>regular for ( \frac{p}{a_1} \leq 1 )</td>
<td>( \frac{p}{a_1} \geq 1, \frac{a_1}{a_2} \geq -1 )</td>
</tr>
</tbody>
</table>

The Ornstein-Uhlenbeck, CIR/Gamma and Jacobi/Beta diffusions with all moments and complete orthogonal polynomial bases have by now been extensively studied and widely applied, especially in the ergodic case. However, the first results on the statistical analysis of the heavy-tailed diffusions (3) (the Reciprocal gamma/GARCH/IGBM, Student/hypergeometric and Fisher-Snedecor diffusion) which all have
only a finite number of moments and also non empty continuum spectrum\(^2\), are more recent. We study these processes below, under the assumption of non-regular boundaries, which ensures the uniqueness of the diffusion, without the need to specify boundary conditions.

**The Student parametrization.** Completing the square, we may write (3) as

\[
dX_t = \theta(\mu - X_t) dt + \sqrt{2\alpha_2 \left( (X_t - \mu)^2 + \delta^2 \right)} dB_t, \quad t \geq 0.
\]  

(6)

This parametrization makes sense for the whole Kolmogorov-Pearson family (by allowing \(\delta^2 \leq 0\)), but it is especially convenient when \(\delta \in \mathbb{R}, \alpha_2 > 0\), in which case it produces diffusions living on \((\mathbb{R}, \infty)\).

**Content.** In section 2 we present the three cases of heavy-tailed Kolmogorov-Pearson diffusions (Reciprocal gamma, Fisher-Snedecor and Student diffusions). In subsection 2.4 we give the explicit expressions for the transition densities of these diffusions, obtained by inverting the corresponding Laplace transforms; the case of the Student diffusion seems to have been missing from the literature\(^3\).

We review in section 3.7 general theory concerning the resolvent of diffusions, which is a product of two special monotone solutions of the associated Sturm-Liouville equation.

Finally, in section 3 we provide for convenience some general theory of one-dimensional diffusion processes. Note that diffusion operators have been for more than half a century the focus of three parallel literatures which largely ignore each other: analysis, probability and physics, with different methods and terminologies, which renders reviewing a not so trivial task.

2 **Heavy-tailed Kolmogorov-Pearson diffusions**

We observe the class of diffusion processes

\[
dX_t = \sigma(X_t) dB_t + b(X_t) dt
\]

(7)

with the linear drift \(b(x) = b_1 x + b_0\) and the quadratic squared diffusion coefficient

\[
\sigma^2(x) = 2\alpha(x) = 2(a_2 x^2 + a_1 x + a_0) = \sigma_2^2 x^2 - \sigma_1^2 x + \sigma_0^2,
\]

see expression (3). Heavy-tailed Kolmogorov-Pearson diffusions with the state space \((l, r)\) are a subclass of this class of diffusions and are characterized by properties given in Table 3. Existence of the unique Markovian weak solution \(X = \{X_t, t \geq 0\}\) of the SDE (7) with the pre-specified marginal density from the Pearson family follows from Bibby et al. (Bibby et al., 2005, Theorem 2.1.(i)). Furthermore, the SDE (7) admits a unique strong solution with the time-homogenous transition densities if it satisfies the following sufficient conditions given by Aït-Sahalia (Aït-Sahalia, 1996, page 415, assumption A1 (i) and (ii)):

\(^2\) For precise condition of discreteness of spectrum, see Mu Fa Chen (Chen, 2004, pg. 667)

\(^3\) In principle, once the Green function is known, its residues and values along an eventual branch cut determine in principal the spectral expansion of the density, by Laplace inversion, but completing all the details of this complex analysis exercise is quite tedious – see (Leonenko & Šuvak, 2010a,b; Avram et al., 2010), for the Reciprocal gamma/GARCH/IGBM and Fisher-Snedecor diffusions, in the nonregular boundaries, positive recurrent case (for regular boundaries, several cases need to be analyzed). The essential feature of the heavy tailed case is the presence of a branch cut in the Green function, which produces a continuous spectrum, besides a finite (possibly empty) set of residues, which produce the discrete spectrum, with polynomial eigenfunctions. This is in contrast with with the Ornstein-Uhlenbeck, CIR and Jacobi diffusions, which are comparatively much easier to invert, having only purely discrete simple spectrum, and a complete set of (classical) orthogonal polynomials eigenfunctions (Hermite, Laguerre and Jacobi, respectively).
- the drift coefficient \( b(x) \) and the diffusion coefficient \( \sigma(x) \) are continuously differentiable in \( x \) and \( \sigma^2(x) \) is strictly positive on the whole diffusion state space,
- the integral of the speed density \( m(x) \) of diffusion \( X \) converges at both boundaries of the diffusion state space.

According to Aït-Sahalia (Aït-Sahalia, 1996), these conditions are considerably less restrictive than the global Lipschitz and the linear growth conditions which are usually imposed on drift and diffusion coefficients to obtain existence and uniqueness of a strong solution. Existence of strong solutions of stochastic differential equations defining Fisher-Snedecor, Reciprocal gamma and symmetric Student diffusions is verified in (Avram et al., 2010, Section 3), (Leonenko & Šuvak, 2010a, Section 3) and (Leonenko & Šuvak, 2010b, Section 3), respectively.

Table 3 Classification criteria for heavy-tailed Kolmogorov-Pearson diffusions

<table>
<thead>
<tr>
<th>Discriminant of ( a(x) )</th>
<th>Kolmogorov-Pearson diffusion</th>
<th>State space</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \Delta(a) = 0 )</td>
<td>Reciprocal gamma diffusion</td>
<td>((l, r) = (0, \infty))</td>
</tr>
<tr>
<td>( \Delta(a) &gt; 0 )</td>
<td>Fisher-Snedecor diffusion</td>
<td>((l, r) = (0, \infty))</td>
</tr>
<tr>
<td>( \Delta(a) &lt; 0 )</td>
<td>Student diffusion</td>
<td>((l, r) = (-\infty, \infty))</td>
</tr>
</tbody>
</table>

In Table 3 \( \Delta(a) \) is the discriminant of the \( a(x) \), where \( a_2 > 0 \) and \( a_1 \) and \( a_0 \) are such that the \( \sqrt{a_2 X_t^2 + a_1 X_t + a_0} \) is well defined when \( X_t \) is in the diffusion state space. This parametrization of the SDE (7) could be understood as a general parametrization of the Student/hypergeometric diffusion. It is often useful to know the parametrization of heavy-tailed Kolmogorov-Pearson diffusions used in papers (Leonenko & Šuvak, 2010a,b; Avram et al., 2011), which we call here "the standard parametrization" (see Table 4).

Table 4 Standard parametrization of Kolmogorov-Pearson diffusions

<table>
<thead>
<tr>
<th>Kolmogorov-Pearson diffusion</th>
<th>Standard parametrization</th>
</tr>
</thead>
<tbody>
<tr>
<td>Reciprocal gamma</td>
<td>( b_1 = -\theta, b_0 = \frac{\theta \alpha}{\beta - 1}, a_2 = \frac{\theta}{\beta - 1}, a_1 = 0, a_0 = 0 ) ( \theta &gt; 0, \alpha &gt; 0, \beta &gt; 1 )</td>
</tr>
<tr>
<td>Fisher-Snedecor</td>
<td>( b_1 = -\theta, b_0 = \frac{\theta \beta}{\alpha - 2}, a_2 = \frac{\theta \beta}{\alpha - 2}, a_1 = \frac{\theta \beta}{\alpha - 2 \beta}, a_0 = 0 ) ( \theta &gt; 0, \alpha &gt; 0, \beta &gt; 2 )</td>
</tr>
<tr>
<td>skew-Student</td>
<td>( b_1 = -\theta, b_0 = \theta \mu, a_2 = \theta \nu^{-1}, a_1 = -\frac{\theta \mu}{\nu - 1}, a_0 = \frac{\theta \mu \nu}{\nu - 1} ) ( \theta &gt; 0, \nu &gt; 1, \delta &gt; 0, \mu, \mu' \in \mathbb{R} )</td>
</tr>
</tbody>
</table>

In terms of the standard parametrization the scale and the speed densities for heavy-tailed Kolmogorov-Pearson diffusions are given in Table 5.

2.1 Oscillatory/non-oscillatory classification of natural boundaries

The infinitesimal generator

\[
G f(x) = \frac{1}{m(x)} \left( \frac{f'(x)}{s(x)} \right)' = a(x)s(x)\left( f'(x) \exp \left\{ \frac{x}{\int b(u) \frac{1}{a(u)} \, du} \right\} \right)'
\]
plays the crucial role in classification of boundaries of the diffusion state space.

For O/NO classification of the natural boundaries \( l = -\infty \) and \( r = \infty \) of Student diffusion, the standard procedure requires transformation of the Sturm-Liouville equation

\[
-a(x)f''(x) + b(x)f'(x) + \lambda f(x) = 0, \quad \lambda \geq 0
\]  

(9)

to the Liouville normal form (Fulton et al. (Fulton et al., 2005, pg. 4), see also 3.3 of Appendix). Observe therefore the Sturm-Liouville equation (9) in the self-adjoint form:

\[
-\left((a(x))^{b_1/2a_1} \exp \left\{ -\frac{a_1 b_1 - 2a_2 b_0}{a_2 \sqrt{-\Delta(a)}} \arctan \left( \frac{a'(x)}{\sqrt{-\Delta(a)}} \right) \right\} f'(x) \right) =
\]

\[
\lambda f(x)(a(x))^{(b_1-2a_1)/2a_2} \exp \left\{ -\frac{a_1 b_1 - 2a_2 b_0}{a_2 \sqrt{-\Delta(a)}} \arctan \left( \frac{a'(x)}{\sqrt{-\Delta(a)}} \right) \right\},
\]  

(10)

where \( \Delta(a) = a_1^2 - 4a_0 a_2 \) is the discriminant of the polynomial \( a(x) \) (note that in the case of Student diffusion \( \Delta(a) < 0 \), see Table 3). Generally, this transformation eliminates the first derivative (drift) term from the Sturm-Liouville equation (9) and transforms it to the equation

\[
-g''(u) + Q(u)g(u) = \lambda g(u).
\]  

(11)

Equation (11) is known as the Liouville normal form of the Sturm-Liouville equation (10) or the one-dimensional Schrödinger equation. The function \( Q(u) \) is called the potential function and is defined by the expression

\[
Q(u) = \frac{h''(u)}{h(u)},
\]  

(12)

where

\[
u = u(x) = \int \sqrt{m(x)s(x)} \, dx, \quad h(u) = \sqrt{\frac{m(x(u))}{s(x(u))}}, \quad x(u) = u^{-1}(x)
\]  

(13)

and \( g(u) = h(u)f(x(u)) \), where \( f(x) \) is the solution of the original Sturm-Liouville equation (9). According to (Fulton et al., 2005, pg. 6), the O/NO classification is invariant with respect to the Liouville transformation.

In the Student case we have the following:

\[
\frac{1}{s(x)} = (a(x))^{b_1/2a_1} \exp \left\{ -\frac{a_1 b_1 - 2a_2 b_0}{a_2 \sqrt{-\Delta(a)}} \arctan \left( \frac{a'(x)}{\sqrt{-\Delta(a)}} \right) \right\},
\]  

(14)

\[
m(x) = (a(x))^{(b_1-2a_2)/2a_2} \exp \left\{ -\frac{a_1 b_1 - 2a_2 b_0}{a_2 \sqrt{-\Delta(a)}} \arctan \left( \frac{a'(x)}{\sqrt{-\Delta(a)}} \right) \right\},
\]  

(15)
\[
    u(x) = \sqrt{\frac{1}{a_2}} \log \left( a'(x) + 2\sqrt{a_2} a(x) \right),
\]

(16)

\[
    x(u) = u^{-1}(x) = \frac{e^{-\sqrt{a_2} x}}{4a_2} \left( (a_1 - e^{\sqrt{a_2} x})^2 - 4a_0a_2 \right).
\]

(17)

By taking derivatives of the function \( h(u) \) (see expression (13)) it follows that the potential function is given by

\[
    Q(u) = \frac{e^{4\sqrt{a_2} u}(a_2 - b_1)^2}{4a_2 \left( e^{2\sqrt{a_2} u} - \Delta(a) \right)^2} - \frac{e^{2\sqrt{a_2} u}(b_1 - 2a_2)(a_1b_1 - 2a_2b_0)}{a_2 \left( e^{2\sqrt{a_2} u} - \Delta(a) \right)^2} - \frac{4e^{\sqrt{a_2} u}(b_1 - 2a_2)(a_1b_1 - 2a_2b_0)\Delta(a) - (a_2 - b_1)^2 \Delta^2(a)}{4a_2 \left( e^{2\sqrt{a_2} u} - \Delta(a) \right)^2} +
\]

\[
    \frac{e^{2\sqrt{a_2} u} \left( -8a_1a_2b_0b_1 + a_1^2 \left( 5a_2^2 - 6a_2b_1 + 3b_1^2 \right) - 4a_2 \left( -2a_2b_0^2 + a_0 \left( 5a_2^2 - 6a_2b_1 + b_1^2 \right) \right) \right)}{2a_2 \left( e^{2\sqrt{a_2} u} - \Delta(a) \right)^2}.
\]

The natural boundaries \( l = -\infty \) and \( r = \infty \) of the Sturm-Liouville equation (9) remain unchanged under the Liouville transform, i.e. the corresponding boundaries of the equation (11) are \( l^* = u(l) = -\infty \) and \( r^* = u(r) = \infty \). O/NO classification of the boundaries \( l^* = -\infty \) and \( r^* = \infty \) depends on the behavior of the potential function \( Q(u) \) near these boundaries. Since the last three terms in the expression for \( Q(u) \) vanish as \( u \to -\infty \) and \( u \to \infty \), it follows that

\[
    \lim_{u \to -\infty} Q(u) = \lim_{u \to \infty} Q(u) = \frac{(a_2 - b_1)^2}{4a_2}.
\]

According to (Fulton et al., 2005, Theorem 6) both \( l^* = -\infty \) and \( r^* = \infty \) are O/NO boundaries of the equation (11) with unique positive cutoff

\[
    A = A(a_2, b_1) = \frac{(a_2 - b_1)^2}{4a_2}.
\]

(18)

Furthermore, these boundaries are NO for \( \lambda < A \) and O for \( \lambda > A \). According to Dunford and Schwartz (Dunford & Schwartz, 1963, Corollary 57, pg. 1481) (see also Linetsky (Linetsky, 2004, Theorem 3, pg. 349)), both boundaries are NO for \( \lambda = A \). This classification of boundaries remains invariant under the Liouville transform (see (Fulton et al., 2005, pg. 6)), i.e. both \( l = -\infty \) and \( r = \infty \) are O/NO boundaries of the Sturm-Liouville equation (9) with unique positive cutoff

\[
    A = A(a_2, b_1) = \frac{(a_2 - b_1)^2}{4a_2}.
\]

Furthermore, both boundaries are NO for \( \lambda \leq A \) and O for \( \lambda > A \).

For Reciprocal gamma and Fisher-Snedecor diffusions the left boundary of the diffusion state space is \( l = 0 \) and it is regular, entrance or exit, depending on parameter values (see Table 2). For both diffusion the right boundary \( r = \infty \) is natural. Since the cutoff between the discrete and the absolutely continuous spectrum is determined by asymptotic behavior of the potential function near natural boundaries it means that by taking the parametrization from Table 4 the cutoff formula (18) holds also for these two diffusions living on \((0, \infty)\). Cutoffs between the discrete and the absolutely continuous spectrum for Reciprocal gamma, Fisher-Snedecor and symmetric Student diffusions in terms of the parametrization from Table 4 could be found in (Leonenko & Šuvak, 2010a), (Leonenko & Šuvak, 2010b) and (Avram et al., 2010), respectively.
2.2 Spectrum of the Sturm-Liouville operator

Explicit form of the spectral representation of the transition density of the diffusion process is implied by the structure of the spectrum of the corresponding Sturm-Liouville operator \((-G)\). Furthermore, according to (Linetsky, 2004, pg. 350), if the corresponding potential function \(Q(x)\) has bounded variation on some subinterval \((c, \infty)\) of the positive halfline, in the continuous part of the spectrum of the operator \((-G)\) with natural boundary \(r = \infty\) there are no gaps containing simple eigenvalues.

Spectral category and the structure of the spectrum of the heavy-tailed Kolmogorov-Pearson diffusions is summarized below according to the O/NO classification of boundaries of the diffusion state space \((l, r)\) (see (Fulton et al., 2005, pg. 23-27) and (Linetsky, 2004, pg. 348, Theorem 2)):

- \(l\) NO boundary, \(r\) O/NO natural boundary (Reciprocal gamma and Fisher-Snedecor diffusions)
  These diffusions belong to Linetsky’s spectral category II. In particular, \(l = 0\) is NO boundary, while \(r = \infty\) is O/NO boundary with unique cutoff
  \[
  \Lambda = \Lambda(a_2, b_1) = \frac{(a_2 - b_1)^2}{4a_2}.
  \]
  Since \(r = \infty\) is NO for \(\lambda = \Lambda\), the Sturm-Liouville operator \((-G)\) has a finite set of simple eigenvalues in \([0, \Lambda]\) and an essential spectrum \(\sigma_e(-G) = [\Lambda, \infty)\). Hence, the operator \((-G)\) has a discrete spectrum \(\sigma_d(-G)\) in \([0, \infty)\), i.e. \(\sigma_d(-G) \subset [0, \Lambda]\), and a purely absolutely continuous spectrum \(\sigma_{ac}(-G)\) of multiplicity one in \((\Lambda, \infty)\). For more details on boundary classification for these two diffusions see (Leonenko & Šuvak, 2010a) and (Avram et al., 2012).

- \(l\) and \(r\) are natural O/NO boundaries (Student diffusion)
  These diffusions belong to Linetsky’s spectral category III. In particular, the Sturm-Liouville operator \((-G)\) has a finite set of simple eigenvalues in \([0, \Lambda]\) and an essential spectrum \(\sigma_e(-G) = [\Lambda, \infty)\). Hence, the operator \((-G)\) has a discrete spectrum \(\sigma_d(-G)\) in \([0, \infty)\), i.e. \(\sigma_d(-G) \subset [0, \Lambda]\), and a purely absolutely continuous spectrum \(\sigma_{ac}(-G)\) of multiplicity two in \((\Lambda, \infty)\). For more details on boundary classification for symmetric case of Student diffusion see (Leonenko & Šuvak, 2010b).

2.2.1 Discrete part of the spectrum

In order to calculate eigenvalues (elements of the discrete part of the spectrum), and corresponding eigenfunctions of the Sturm-Liouville operator \((-G)\), we observe the Sturm-Liouville differential equation (9) in the self-adjoint form

\[
-\frac{d}{dx} (a(x)m(x)f_n'(x)) = \lambda_n f_n(x)m(x),
\]

where \(\lambda_n\) is the real spectral parameter representing the simple eigenvalues and \(f_n(x)\) is the corresponding polynomial eigenfunction (see Nikiforov and Uvarov (Nikiforov et al., 1988)). Eigenfunctions \(f_n(x)\) are supposed to be normalized and orthogonal with respect to the speed density of diffusion living on the state space \((l, r)\). Multiplying both sides of the equation (20) by \(f_n(x)\) and taking the integral from \(l\) to \(r\), where \((l, r)\) is the diffusion state space, yields

\[
- \int_l^r f_n(x) \frac{d}{dx} (a(x)m(x)f_n'(x)) \, dx = \lambda_n \int_l^r f_n^2(x)m(x) \, dx.
\]
Now according to orthogonality of the normalized eigenfunctions \( f_n(x) \) with respect to \( m(x) \) it follows that

\[
\lambda_n = -\int f_n(x) \frac{d}{dx} \left( a(x)m(x)f'_n(x) \right) \, dx.
\]

Since \( f_n(x) \) is the polynomial of the \( n \)-th degree, \( f_{2n}^2(x) \) is the polynomial of the degree \( 2n \) and therefore the finiteness of the number of simple eigenvalues follows from the fact that there exists the number \( N \in \mathbb{N} \) such that

\[
\int_i x^{2n} m(x) \, dx < \infty, \quad n = 0, 1, \ldots, N \quad \text{and} \quad \int_i x^{2n} m(x) \, dx = \infty, \quad n = N + 1, N + 2 \ldots
\]

Since the \( n \)-th moment, \( n \in \mathbb{N} \), of the heavy-tailed Pearson distribution exists under the condition \( n < (a_2 - b_1)/2a_2 \) (it follows from the study of particular cases of heavy-tailed Kolmogorov-Pearson diffusions, see (Leonenko & Šuvak, 2010a,b; Avram et al., 2010) and also the speed density 15), it follows that for Kolmogorov-Pearson diffusions \( N = [(a_2 - b_1)/2a_2] \). Therefore, the discrete part of the spectrum of the operator \((-G)\) is of the form \( \{ \lambda_n, n = 0, 1, \ldots, [(a_2 - b_1)/2a_2] \} \). Eigenvalues \( \lambda_n \) are given by the explicit expression

\[
\lambda_n = n ((1 - n)a_2 - b_1), \quad n \in \{0, 1, \ldots, [(a_2 - b_1)/2a_2] \}.
\]

Eigenvalues of the Sturm-Liouville operators of particular heavy-tailed Kolmogorov-Pearson diffusions in standard parametrization are given in Table 6.

<table>
<thead>
<tr>
<th>Kolmogorov-Pearson Diffusion</th>
<th>Eigenvalues</th>
</tr>
</thead>
<tbody>
<tr>
<td>Reciprocal Gamma</td>
<td>( n(\beta - n), \quad n \leq \frac{\beta}{2} )</td>
</tr>
<tr>
<td>Fisher-Snedecor</td>
<td>( n(\beta - 2n), \quad n \leq \frac{\beta}{2} )</td>
</tr>
<tr>
<td>Student</td>
<td>( n(\nu - n), \quad n \leq \frac{\nu}{2} )</td>
</tr>
</tbody>
</table>

Corresponding eigenfunctions are polynomial solutions of the Sturm-Liouville equation (9) given by the Rodrigues formula

\[
\tilde{P}_n(x) = \frac{1}{m(x)} \frac{d^n}{dx^n} \{2^n a^n(x)b(x)\}, \quad n \in \{0, 1, \ldots, [(a_2 - b_1)/2a_2] \}.
\]

Polynomials \( \tilde{P}_n(x) \) form the finite system of polynomials orthogonal with respect to the speed density \( m(x) \), i.e.

\[
\int_{-\infty}^{\infty} \tilde{P}_m(x) \tilde{P}_n(x) m(x) \, dx = 0, \quad m, n \in \{0, 1, \ldots, [(a_2 - b_1)/2a_2] \}, \quad m \neq n.
\]

In particular, the first few non-normalized polynomials are

\[
\begin{align*}
\tilde{P}_0(x) &= 1, \\
\tilde{P}_1(x) &= 2(b_1 x + b_0), \\
\tilde{P}_2(x) &= 4(b_0^2 + a_0)(2a_2 + b_1) + (a_2 + b_1)(2a_1 + (2a_2 + b_1)x) + 2b_0(a_1 + (a_2 + b_1)x), \\
\tilde{P}_3(x) &= 2(2a_2 + b_1)(4a_2 + b_1)(3a_2 + b_1)x^3 + 3(2a_1 + b_0)(2a_2 + b_1)(6a_2 + b_1)x^2 + 20a_0a_2b_0 + 26a_0a_1a_2 + 6(2a_2 + b_1)(2a_1 + b_0)(a_1 + b_0) + a_0(4a_2 + b_1)x + 2b_0^3 + 6a_0b_0b_1 + 6a_1b_0^2 + 8a_0a_1b_1 + 4a_1^2b_0.
\end{align*}
\]
Eigenfunctions (orthogonal polynomials) of the Sturm-Liouville operators of particular heavy-tailed Kolmogorov-Pearson diffusions in standard parametrization are given in Table 7.

Table 7 Bessel, Routh-Romanovski and Fisher-Snedecor polynomials in standard parametrization

<table>
<thead>
<tr>
<th>Kolmogorov-Pearson Diffusion</th>
<th>Orthogonal polynomials</th>
<th>Rodrigues formula</th>
</tr>
</thead>
<tbody>
<tr>
<td>Reciprocal Gamma</td>
<td>Bessel</td>
<td>$x^{\beta+1}e^{\frac{\alpha}{2}\left(2n-(\beta+1)x\right)}$</td>
</tr>
<tr>
<td>Fisher-Snedecor</td>
<td>Fisher-Snedecor</td>
<td>$\frac{x^{\beta+1}e^{\frac{\alpha}{2}\left(2n-(\beta+1)x\right)}}{\Gamma(n+1/2)} \times$</td>
</tr>
<tr>
<td>Student</td>
<td>Routh-Romanovski</td>
<td>$\frac{x^{\beta+1}e^{\frac{\alpha}{2}\left(2n-(\beta+1)x\right)}}{\Gamma(n+1/2)} \times$</td>
</tr>
</tbody>
</table>

For normalization of polynomials with respect to the speed density we must multiply each of them by the normalizing constant given by the general expression

$$K_n = \frac{(-1)^n}{\sqrt{(-1)^n n! d_n I_n}}, \quad n \in \{0, 1, \ldots, \lfloor (a_2 - b_1)/2a_2 \rfloor \},$$

where

$$d_n = 2^n a_2^{n-1} (b_1 + (n-1)a_2) = \frac{\Gamma \left( \frac{b_1}{a_2} + 2n - 1 \right)}{\Gamma \left( \frac{b_1}{a_2} + n \right)}, \quad I_n = 2^n \int_{x}^{r} a^n(x) m(x) \, dx.$$

Normalizing constants for orthogonal polynomials related to heavy-tailed Kolmogorov-Pearson diffusions are given in Table 8.

Table 8 Normalizing constants for orthogonal polynomials in standard parametrization

| Bessel polynomials                        | $(-1)^n \frac{\Gamma \left( \frac{\beta+2n}{2} \right) \Gamma \left( \frac{\beta}{2} \right)}{\Gamma \left( n+1 \right) \Gamma \left( \beta-n+1 \right)} = (-1)^n \frac{\Gamma \left( \frac{\beta}{2} \right)}{\pi^{1/2} \Gamma(\beta)} \left( \prod_{k=0}^{n-1} \left( \frac{\beta}{2} - n - k \right) \right)^{-1}$ |
| Fisher-Snedecor polynomials               | $(-1)^n \frac{\Gamma \left( \frac{\beta}{2} \right)}{\pi \left( n \left( \frac{\beta}{2} + \frac{1}{2} \right) \right)^{n+1/2} \left( \frac{\beta}{2} + n - 2 \right)} \left( \prod_{k=1}^{\infty} \left( \frac{\beta}{2} + k - 2n \right)^{-1} \right)$ |
| Routh-Romanovski polynomials             | $(-1)^n \frac{\Gamma \left( \frac{n}{2} \right) \Gamma \left( \frac{\beta}{2} \right) \Gamma \left( \frac{\beta+1-n}{2} \right)}{\Gamma \left( n+\frac{\beta+1-n}{2} \right) \Gamma \left( \frac{n}{2} + n - k \right) \left( \prod_{k=0}^{\infty} \left( \frac{\beta}{2} + \frac{1}{2} + n - k \right)^{-1} \right) \left( \prod_{k=0}^{\infty} \left( \frac{\beta}{2} + 1 + n - k \right) \right)^{1/2}}$ |

Therefore, the normalized orthogonal polynomials are given by the Rodrigues formula

$$P_n(x) = K_n \hat{P}_n(x), \quad n \in \{0, 1, \ldots, \lfloor (a_2 - b_1)/2a_2 \rfloor \} \text{.}$$

Remark 1 Orthogonality relation for the normalized polynomials $P_n(x)$, i.e. the relation

$$\int_{l}^{r} P_m(x) P_n(x) b(x) \, dx = \delta_{mn}, \quad m, n \in \{0, 1, \ldots, \lfloor (a_2 - b_1)/2a_2 \rfloor \}$$
implies interesting properties of the random variables $P_n(X_t)$, where $X_t$ is from the diffusion with the state space $(l, r)$ and the speed density $m(x)$. In particular, random variables $P_n(X_t)$ are orthonormal, i.e.

$$E[P_m(X_t)P_n(X_t)] = \int_{l}^{r} P_m(x)P_n(x)m(x)\,dx = \delta_{mn}, \quad m, n \in \{0, 1, \ldots, [(a_2 - b_1)/2a_2]\}.$$ 

Since $P_0(x) = 1$, for $m = 0$ and $n \neq 0$ the previous expression takes the form

$$E[P_n(X_t)] = 0, \quad n \in \{1, \ldots, [(a_2 - b_1)/2a_2]\},$$

i.e. $P_n(X_t)$, $n = 1, \ldots, [(a_2 - b_1)/2a_2]$, are orthonormal centered random variables. This property has significant role in statistical analysis of heavy-tailed Kolmogorov-Pearson diffusions (see (Leonenko & Šuvak, 2010a, Sections 5 and 6), (Leonenko & Šuvak, 2010b, Sections 5 and 6) and (Avram et al., 2011, Section 3)).

2.2.2 Essential part of the spectrum

The complex analysis approach for deriving the transition density from the Green’s function (i.e. universe Laplace transform of the Green’s function) in particular cases of heavy-tailed Kolmogorov-Pearson diffusions reveals eigenvalues $\lambda_n$ (22) as simple poles of the Green’s function. In the same procedure the corresponding orthogonal polynomials (23) (see also Table 7) also naturally appear.

From the subsection 2.2 we know that the essential spectrum of the Sturm-Liouville operator $(-\mathcal{G})$ is $\sigma_e(-\mathcal{G}) = [\Lambda, \infty)$. Moreover, the operator $(-\mathcal{G})$ has purely absolutely continuous spectrum of multiplicity one in $[\Lambda, \infty)$, where

$$\Lambda = \frac{(a_2 - b_1)^2}{4a_2},$$

is the unique positive cutoff between the discrete and the absolutely continuous part of the spectrum.

Remark 2 Polynomial eigenfunctions $P_n(x)$, $n \in \{0, 1, \ldots, [(a_2 - b_1)/2a_2]\}$, and (eigen) functions related to absolutely continuous spectrum (i.e. for $\lambda > \Lambda$) belong to orthogonal subspaces of the Hilbert space $\mathcal{H} = L^2((l, r), m(\cdot))$ and therefore are orthogonal with respect to the speed density $m(\cdot)$. This property plays a crucial role in the statistical analysis of this diffusion process (see Reed and Simon (Reed & Simon, 1980)). Namely, polynomials $P_n(x)$ belong to the subspace $\mathcal{H}_{pp}$ of the Hilbert space $L^2((-\infty, \infty), m(\cdot))$ containing functions having only the pure point spectral measure, while (eigen) functions related to absolutely continuous spectrum belong to the subspace $\mathcal{H}_{ac}$ of the same Hilbert space, which contains functions having only the spectral measure which is absolutely continuous with respect to the Lebesgue measure (see Reed and Simon (Reed & Simon, 1980) or Linetsky (Linetsky, 2004, Appendix)). From here it follows that polynomial eigenfunctions and functions related to absolutely continuous spectrum are orthogonal with respect to the density $m(\cdot)$.

2.3 Student diffusion

The Student parametrization of Kolmogorov-Pearson diffusions. Following the formulation of (Shaw, 2008), consider the SDE

$$dX_t = \theta (\mu - X_t) \, dt + \sigma_1 \, dW^{(1)}_t + \sigma_2 \, X_t \, dW^{(2)}_t, \quad t \geq 0,$$

where

$$dX_t = \theta (\mu - X_t) \, dt + \sqrt{\sigma_2^2 \left( \frac{\sigma_1}{\sigma_2} \right)^2 + (1 - \rho^2) \left( \frac{\sigma_1}{\sigma_2} \right)^2} \, dW_t,$$

for $\rho^2 = \frac{\sigma_1^2}{\sigma_2^2}$. This formulation allows for both light and heavy tails, as well as for correlations between the two Brownian motions. For $\rho = 0$, the process becomes a scalar Ornstein-Uhlenbeck process with two Brownian motions, characterized by the coefficients $\sigma_1$ and $\sigma_2$. For $\rho = 1$, the process becomes a linear Ornstein-Uhlenbeck process with two Brownian motions, characterized by the coefficients $\frac{\sigma_1}{\sigma_2}$ and $1 - \frac{\sigma_1^2}{\sigma_2^2}$. For $\rho = 0$ and $\sigma_2 = 1$, the process becomes a linear Ornstein-Uhlenbeck process with one Brownian motion, characterized by the coefficients $\frac{\sigma_1}{\sigma_2}$ and $1 - \frac{\sigma_1^2}{\sigma_2^2}$. For $\rho = 1$ and $\sigma_2 = 1$, the process becomes a scalar Ornstein-Uhlenbeck process with one Brownian motion, characterized by the coefficient $\frac{\sigma_1}{\sigma_2}$.
where $W_t^{(1)}$ and $W_t^{(2)}$, $t \geq 0$, are standard Brownian motions with correlation $\rho$, and $\{W_t, t \geq 0\}$ is a standard Brownian motion resulting from combining the two.

**Remark 2** The diffusion process (26) is Markovian, with infinitesimal generator:

$$\mathcal{G} h(x) = a(x) h''(x) + b(x) h'(x)$$  \hspace{1cm} (27)

where $2a(x) = \sigma^2(x) = \sigma^2_2 \left( \frac{X_t + \rho \frac{a_1}{\sigma_2}}{\sigma_2} \right)^2 + (1 - \rho^2) \left( \frac{a_1}{\sigma_2} \right)^2$.

Simplifying the notation, we will consider the stochastic differential equation (SDE)

$$dX_t = -\theta(X_t - \mu) \, dt + \sqrt{2a_2 \left( (X_t - \mu')^2 + \delta^2 \right)} \, dW_t, \quad t \geq 0,$$

where $\delta > 0$, $2a_2 = \sigma_2^2 = \frac{2\theta}{\nu} > 0$, $\mu, \mu' \in \mathbb{R} > 0$, and $W = \{W_t, t \geq 0\}$ is a standard Brownian motion. Note that by a change of origin, one may always assume that either $\mu$ or $\mu'$ are 0.

Putting $\bar{X}_t := (X_t - \mu')/\delta$, we arrive at

$$d\bar{X}_t = -\theta(\bar{X}_t - \tilde{\mu}) \, dt + \sqrt{2a_2 (1 + \bar{X}_t^2)} \, dW_t, \quad t \geq 0,$$

where we put $\tilde{\mu} := (\mu - \mu')/\delta$.

The infinitesimal operator is

$$\mathcal{G} = a_2 (1 + \bar{x}^2) D_{\bar{x}}^2 - \theta(\bar{x} - \tilde{\mu}) D_{\bar{x}},$$  \hspace{1cm} (30)

and the scale and speed densities are:

$$s(\bar{x}) = (\bar{x}^2 + 1) \frac{1}{2\pi} e^{-\frac{\bar{x}^2}{2} \arctg(\bar{x})}, \quad m(\bar{x}) = \frac{e^{\frac{\bar{x}}{2} \arctg(\bar{x})}}{(\bar{x}^2 + 1)^{\frac{3}{2}}}, \quad x \in \mathbb{R},$$

where $\tilde{a} = a_2/\theta$.

A standardized form is obtained using the Liouville transformation (43) from Section 3.3:

$$y = l(x) = \frac{\arcsinh(x)}{\sigma_2} = \frac{\log(x + \sqrt{x^2 + 1})}{\sigma_2} \Leftrightarrow x = T_y(y) = \sinh(\sigma_2 y).$$  \hspace{1cm} (31)

Noting that $x = \sinh(y) \Rightarrow \sqrt{x^2 + 1} = \cosh(y)$, we find that the diffusion $Y_t = \arcsinh(X_t)$ has the parameters $\sigma_Y(y) = \sigma_2$ and

$$b_Y(y) = \theta \left( \frac{\mu - x}{\sqrt{x^2 + 1}} - \frac{\tilde{\mu} - \tilde{a} x}{\sqrt{\bar{x}^2 + 1}} \right) = \theta \left( \frac{\mu}{\cosh(y)} - \tanh(y)(1 + \tilde{a}) \right).$$

The drift term can then be removed, arriving to the Schrödinger form of the Sturm-Liouville equation:

$$q(y) = c_0 + c_1 \frac{1}{\cosh^2(y)} + c_2 \frac{\sinh(y)}{\cosh^2(y)},$$

where the coefficient

$$c_0 = (1 + \tilde{a}^{-1})^2/8 = \nu^2/8$$  \hspace{1cm} (32)

yields the cutoff between the discrete and the continuous part of the spectrum of the diffusion infinitesimal generator (see (Linetsky, 2007, pg. 283)).
Remark 3 Note that the continuous spectrum cutoff $c_{0}$ in (32) is a function of the degrees of freedom only!

The potential involved is the famous Scarf II potential, (a particular case of which is the Rosen Morse II potential). This famous potential, already studied by Darboux and assigned as exercise 7 in the 1926 textbook (Ince, 1956, p 132), is often used as example in the physics literature (see for example (Fischer et al., 1993)).

A direct approach for solving the corresponding Sturm-Liouville equation was provided by Paulsen (Paulsen & Gjessing, 1997, Theorem A.1, pg. 984), who finds that the monotone solutions are given by the Weyl type fractional integrals

$$\phi_{\lambda}^{+}(x) = \int_{x}^{\infty} (z-x)^{\rho+1} K(z) \, dz,$$

$$\phi_{\lambda}^{-}(x) = \int_{-\infty}^{x} (x-z)^{\rho+1} K(z) \, dz,$$

where $\rho$ is a constant (33) and $K(z)$ is defined in (34).

Example 1 We revisit now Paulsen’ approach (see (Paulsen & Gjessing, 1997, Thm A1) and also (Ince, 1956)) to the Student Sturm-Liouville operator, which is based on representing the solution as a Weyl fractional integral with kernel $K(x,t) = (t-x)^{\rho-1+\rho}$. This approach uses Euler’s transformation (Ince, 1956, 8.31) which decomposes the original operator in a sum $G = \Gamma_{0} - \Gamma_{1} + \ldots (-1)^{p} \Gamma_{p}$ of special type operators, involving an additional parameter $\rho$ chosen to minimize $p$.

For the normalized Student SL equation (Paulsen & Gjessing, 1997, (A2))

$$G = (x^2 + 1) D^2 + (\tilde{\mu} x + \tilde{\mu} \delta) D - \tilde{\lambda},$$

where tilde signifies the corresponding coefficient is divided by $a_{2}$, Euler’s decomposition (Ince, 1956, VIII.31) is $G = \Gamma_{0} - \Gamma_{1} + \Gamma_{2}$ where

$$\Gamma_{0} = G_{0}(x)D^{2} - \rho G_{0}(x)D + \frac{\rho(\rho+1)}{2} G_{0}(x) = (x^2 + 1) D^{2} - 2x\rho D + \rho(\rho+1),$$

$$\Gamma_{1} = G_{1}(x)D - (\rho+1) G_{1}(x) = -(\tilde{\mu} x + \tilde{\mu} \delta + 2x\rho) D + (\rho+1)(\tilde{\mu} + 2\rho),$$

$$\Gamma_{2} = (\rho+1)(\tilde{\mu} + 2\rho) - (\rho+1)(\rho) - \tilde{\lambda} = (\rho+1)(\tilde{\mu} + \rho) - \tilde{\lambda}. $$

The last operator vanishes when $\rho^2 + \rho(1 + \tilde{\mu}) + \tilde{\mu} - \lambda = 0$, for

$$\rho = \frac{1}{2} \left( -1 - \tilde{\mu} \pm \sqrt{(1 + \tilde{\mu})^2 - 4(\tilde{\mu} - \tilde{\lambda})} \right). \tag{33}$$

In order to stress out the dependence of $\rho$ on the spectral parameter $\lambda$ we will denote $\rho$ by $\rho(\lambda)$. With these choices, the order $p$ of $M_{\lambda} = \sum_{p=0}^{p} \Gamma_{i}(z) D^{p-1}$ becomes $p = 1$, i.e. $M_{\lambda} = \Gamma_{0}(z) D_{z} + \Gamma_{1}(z)$, and we can apply directly the results of Pochammer and Jordan (Ince, 1956, XVIII.4), yielding

$$\phi_{\rho(\lambda)}(x) = \int_{\Gamma}(z-x)^{\rho(\lambda)+1} K(z)\, dz,$$

where

$$K(z) = \frac{1}{G_{0}(z)} e^{\int_{0}^{z} \frac{G_{0}(r)}{G_{0}(z)} \, dr} = (z^2 + 1)^{-1} e^{-\int_{0}^{z} \frac{4\rho + 2\mu}{y+1} \, dy} = (z^2 + 1)^{-\beta/2-1} e^{-\delta \arctg(z)}, \tag{34}$$

$$\beta = \tilde{\mu} + 2\rho(\lambda). \tag{35}$$
Finally, the contour must be chosen so that \( \int dz [(z - x)^{\rho(\lambda)}(z^2 + 1)^{-\beta/2}e^{-\delta \arctg(z)}] = 0 \). If \( \rho(\lambda) \) is chosen as the positive root, \( x \) is a zero of the total differential, and it may be checked by limiting arguments that the Weyl type fractional integrals

\[
\phi^{(1)}_{\rho(\lambda)}(x) = \int_{x}^{\infty} (z - x)^{\rho(\lambda)+1} K(z) \, dz, \tag{36}
\]

\[
\phi^{(2)}_{\rho(\lambda)}(x) = \int_{-\infty}^{x} (x - z)^{\rho(\lambda)+1} K(z) \, dz, \tag{37}
\]

are convergent and that the bilinear concomitant \( (z - x)^{\rho(\lambda)}(z^2 + 1)^{-\beta/2}e^{-\delta \arctg(z)} \) is zero at \( \infty \), certifying thus \( \phi^{(1)}_{\rho(\lambda)}(x), \phi^{(2)}_{\rho(\lambda)}(x) \) as two basic solutions of our SL equation. It is easy to check that:

\[
(\phi^{+}_{\lambda})'(y) = -\rho \phi^{+}_{\lambda-1}(y),
\]

\[
(\phi^{-}_{\lambda})'(y) = \rho \phi^{-}_{\lambda-1}(y),
\]

where

\[-1 < \text{Re}(\rho + 1) < 1 + \text{Re}(\beta).\]

These may be furthermore checked to be precisely the increasing and decreasing SL solutions \( \phi^{+}_{\rho(\lambda)}(x) \) and \( \phi^{-}_{\rho(\lambda)}(x) \), respectively, with the Wronskian

\[
W_{\rho(\lambda)} = (\phi^{+}_{\rho(\lambda)}(x))' \phi^{-}_{\rho(\lambda)}(x) - \phi^{+}_{\rho(\lambda)}(x) (\phi^{-}_{\rho(\lambda)}(x))' = -\rho(\lambda) (\phi^{+}_{\rho(\lambda)-1}(x) \phi^{-}_{\rho(\lambda)}(x) + \phi^{+}_{\rho(\lambda)}(x) \phi^{-}_{\rho(\lambda)-1}(x)) =
\]

\[= \rho(\lambda) \left( \int_{x}^{\infty} (z - x)^{\rho(\lambda)} K(z) dz \int_{-\infty}^{x} (x - z)^{\rho(\lambda)+1} K(z) dz + \int_{x}^{\infty} (z - x)^{\rho(\lambda)+1} K(z) dz \right). \]

**Remark 4** For example, two linearly independent monotone solutions and their Wronskians for the resolvents/Green functions \( G_{\lambda}(x,y)/m(y) = w_{\lambda}(x)\phi^{+}_{\lambda}(y), x < y \) for Reciprocal gamma and Fisher-Snedecor diffusions are given in table 9.

where \( M_{\kappa,\mu}(z) = \frac{M_{\kappa,\mu}(z)}{\Gamma(1+2\mu)} \), \( M_{\kappa,\mu}(z) \) and \( W_{\kappa,\mu}(z) \) are respectively the first and the second Whittaker function (see (Luke, 1969, pg. 134)), while \( 2F_{1}(a,b;c;z) \) is the Gauss hypergeometric function (see (Luke, 1969, pg. 39) and preceding discussion).

In principle, once the Green function is known, its residues and values along an eventual branch cut determine in principal the spectral expansion of the density, by Laplace inversion, but completing all the details of this complex analysis exercise is quite tedious, for the Reciprocal gamma/GARCH/IGBM and Fisher-Snedecor diffusions for the non-regular boundaries and positive recurrent case see (Leonenko & Šuvak, 2010a), (Leonenko & Šuvak, 2010b) and (Avram et al., 2011) (for regular boundaries several cases need to be analyzed). The essential feature of the heavy tailed case is the presence of a branch cut in the Green function, which produces a continuous spectrum, besides a finite (possibly empty) set of residues, which produce the discrete spectrum with polynomial eigenfunctions. This is in contrast with the Ornstein-Uhlenbeck, CIR and Jacobi diffusions, which are comparatively much easier to invert, having only purely discrete simple spectrum and a complete set of (classical) orthogonal polynomials eigenfunctions (Hermite, Laguerre and Jacobi, respectively).
### Table 9  
Linearly independent monotone solutions and their Wronskians for Reciprocal gamma and Fisher-Snedecor diffusions

<table>
<thead>
<tr>
<th>Diffusion</th>
<th>Monotone solution $\phi_{\lambda}^+(x)$</th>
<th>Monotone solution $\phi_{\lambda}^-(y)$</th>
<th>Wronskian $w_{\lambda}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Reciprocal gamma diffusion</td>
<td>$\frac{\alpha^2}{\lambda^2} f\left(\frac{\alpha+1}{\mu+1} \phi_{\lambda}(y) \right)$</td>
<td>$\frac{\alpha^2}{\lambda^2} f\left(\frac{\alpha+1}{\mu+1} \phi_{\lambda}(y) \right)$</td>
<td>$\frac{\alpha^2}{\lambda^2} f\left(\frac{\alpha+1}{\mu+1} \phi_{\lambda}(y) \right)$</td>
</tr>
<tr>
<td>Fisher-Snedecor diffusion</td>
<td>$\frac{\alpha^2}{\lambda^2} f\left(\frac{\alpha+1}{\mu+1} \phi_{\lambda}(y) \right)$</td>
<td>$\frac{\alpha^2}{\lambda^2} f\left(\frac{\alpha+1}{\mu+1} \phi_{\lambda}(y) \right)$</td>
<td>$\frac{\alpha^2}{\lambda^2} f\left(\frac{\alpha+1}{\mu+1} \phi_{\lambda}(y) \right)$</td>
</tr>
</tbody>
</table>

### 2.4 Spectral representation of transition density

Eigenvalues $\lambda_n$ given by (21) are precisely simple poles of the Green’s function (they appear in the inverse Laplace transformation of the Green’s function). Similarly to spectral representation of the transition density for the regular Sturm-Liouville problems, the discrete part of the spectral representation of the transition density for heavy-tailed Kolmogorov-Pearson diffusions is hence of the form

$$p_d(x; y, t) = \sum_{n=0}^{[\alpha_2 - \beta_1]/2\alpha_2} e^{-\lambda_n t} P_n(x) P_n(y) m(x),$$

where $m(\cdot)$ is the speed density. The non-normalized continuous part of the spectral representation of the transition density for heavy-tailed Kolmogorov-Pearson diffusions is of one of the following forms, depending on spectral category of diffusion:

- Diffusions belonging to the spectral category II (Reciprocal gamma and Fisher-Snedecor diffusion):

$$p_c(x; y, t) = m(x) \int_{\lambda}^{\infty} e^{-\lambda t} \phi_{\lambda}(x) \phi_{\lambda}(y) d\lambda,$$

where the increasing functions $\phi_{\lambda}(\cdot)$ for Reciprocal gamma and Fisher-Snedecor diffusions are given in Table 9.

- According to Linetsky (Linetsky, 2004), for diffusions belonging to spectral category III (Student diffusion) spectral representation of transition density is of the following form:

$$p_s(x; y, t) = m(x) \int_{\lambda}^{\infty} e^{-\lambda t} \left(\phi_{\lambda}(x) \phi_{\lambda}(y) + \phi_{\lambda}(x) \phi_{\lambda}(y) + \phi_{\lambda}(y) \phi_{\lambda}(x) \phi_{\lambda}(y) \right) d\lambda,$$

where monotone solutions $\phi_{\lambda}(\cdot)$ and $\phi_{\lambda}(\cdot)$ are given by (36) and (37), respectively.

For explicit expressions for spectral representations of transition densities of Reciprocal gamma diffusion, special case of Student diffusion (symmetric Student diffusion) and Fisher-Snedecor diffusion see (Leonenko & Šuvak, 2010a, Theorem 3.1.), (Leonenko & Šuvak, 2010b, Section 4.5.) and (Avram et al., 2010, Theorem 4.1.), respectively.
References


## Appendix

### 3 One dimensional diffusions

One dimensional diffusions are solutions of stochastic differential equations of the form

\[ dX_t = \sigma(X_t)dB_t + b(X_t)dt, \]  

(38)

defined by two functions \( \sigma(x), b(x), x \in \mathcal{D} = (l, r), -\infty \leq l < r \leq \infty \), assumed here to be continuous, and with \( \sigma(x) > 0, \forall x \in (l, r) \). We focus mostly on the case \( l = 0, r = \infty \), out of which the cases of finite \( \mathcal{D} \) and \( \mathcal{D} = \mathbb{R} \) may be easily reconstructed.
Equivalently, a diffusion may be viewed as a Markovian semigroup of operators, with infinitesimal generator

\[ \mathcal{G} = \mathcal{G}_{a,b} := a(x)D^2 + b(x)D, \quad x \in (l,r) \]  

(39)

where \( D = \frac{\partial}{\partial x} \), \( a(x) := \frac{\sigma^2(x)}{2} \). The associated transition probabilities \( p(t,x,y) \) defined by

\[ E_x f(X_t) = \int p(t,x,y)f(y)dy \]

satisfy the backward and forward Kolmogorov equations

\[ p_t = \mathcal{G} p, \quad p_t = \mathcal{L} y p, \quad p(0,x,y) = \delta(x - y) \]

where

\[ \mathcal{G} = \mathcal{G}_1D, \quad \mathcal{L} = D\mathcal{L}_1, \]

with

\[ \mathcal{G}_1 = a(x)D + b(x), \quad \mathcal{L}_1(\cdot) = D(a(x)) - b(x). \]

Also, whenever well-defined, the time evolving expectations \( f(t,x) := E_x f(X_t) \) satisfy the backward Kolmogorov equation

\[ f_t = \mathcal{G} f. \]

The equations (38), (39) describe the diffusion process while it remains in the interior of its state space. To achieve uniqueness of the process/semigroup, "appropriate boundary conditions" must be added for "attainable/accessible" boundary points \( e \) (see Definition 1 below).

The probabilistic representation of a diffusion semigroup is:

\[ (e^{\mathcal{G}} f)(x) = E_x[f(X_t), H_t > t], \]

where

\[ H_y = \inf\{ t : X_t = y \}. \]

**Note:** In applications, the diffusion model (38) may be viewed as a hypothesis on the existence of parameters \( b(x) \) and \( a(x) := \frac{\sigma^2(x)}{2} \) so that the "residual process" \( dB(t) \) is white noise. For this to be practical, it is useful of course to restrict oneself to models with few parameters, like the Kolmogorov-Pearson diffusions.

### 3.1 Diffusions as symmetric operators

An alternative parametrization of diffusion processes is provided by the scale and speed densities, which are defined by the crucial property of transforming the diffusion operator and its adjoint into equivalent forms

\[ \mathcal{G} f(x) = \frac{1}{m(x)} \left( \frac{f'(x)}{\sigma(x)} \right)', \quad \mathcal{G}^* f(x) = \left( \frac{1}{\sigma(x)} \left( \frac{f(x)}{m(x)} \right)' \right)' \]  

(40)

which are self-adjoint with respect to the speed and scale measures \( m(x)dx, \sigma(x)dx \), respectively, and have therefore real spectrum.
For this, the scale and speed densities must satisfy the equations
\[ \frac{s'(x)}{s(x)} = -\frac{b(x)}{a(x)} \Leftrightarrow G_1 s(x) = 0, \quad \frac{m'(x)}{m(x)} = \frac{b(x) - a'(x)}{a(x)} \Leftrightarrow L_1 m(x) = 0, \]
which are satisfied by
\[ s(x) = e^{-B(x)}, B(x) = \int_x^x b(u) a(u) du, \quad m(x) = \frac{s(x)}{a(x)}. \tag{41} \]

Note:
1. The scale/harmonic function \( S(x) \), defined up to a constant by \( S'(x) = s(x) \), satisfies \( G_S(x) = 0 \) (equivalently, by Dynkin’s formula see (Øksendal, 2003, Section 7.4.), the transformed diffusion \( S(X_t) \) has 0 drift).
2. The ”speed” function \( M(x) \) satisfying the adjoint equation \( L M(x) = 0 \).

If \( M([l, r]) := M(r) - M(l) < \infty \), then the speed density \( m(x) = \) will yield, after normalizing, the stationary distribution. The transition density with respect to the speed density \( \tilde{p}_t(x, y) = p_t(x, y) m(y) \) is a symmetric function.

A change of variables \( Z_t = g(X_t) \), with \( g \) a homeomorphism (i.e. strictly monotone and continuous) puts, by Dynkin’s formula, the SDE (38) into the form
\[ dZ_t = dg(X_t) = (\sigma g')(X_t)dB(t) + G(g)(X_t)dt \tag{42} \]
with associated generator
\[ G_z = a(x)g'^2(x)D_{zz} + (Gg)(x)D_z \]
where \( x = x(z) = \frac{\ln g(z)}{g'(z)} \) is the inverse of \( z = g(x) \). The new speed and scale densities are
\[ m_Z(z) = m_X(\frac{\ln g(z)}{g'(z)}), \quad s_Z(z) = s_X(\frac{\ln g(z)}{g'(z)}). \]

1. The particular case \( z = S(x) \), where \( S \) is the scale function, puts the transformed diffusion into a martingale form:
\[ dZ_t = s(X_t)\sigma(X_t)dB(t) \Leftrightarrow Z_T - Z_0 = \int_0^T \sigma_Z(Z_t)dB(t), \sigma_Z(Z_t) = s(X_t)\sigma(X_t). \]

2. The particular case of Liouville transformation: \( z = l(x) = \int_x^x \frac{dz}{\sigma(z)} \) puts the transformed diffusion \( Z_t = l(X_t) \) into a ”perturbed dynamical system/constant variance” form:
\[ dZ_t = b_Z(Z_t)dt + dB(t) \tag{43} \]
where the ”Liouville drift” is:
\[ b_Z(z) = \left( \frac{b}{\sigma} - \sigma' \frac{1}{2} \right) \left( \frac{1}{T(z)} \right) \]
with \( T(z) \) denoting the inverse Liouville transformation.
3.2 Diffusions with interior killing

More generally, diffusions with "interior killing" may be associated to second order elliptic operators

$$G = G_{a,b,c} := a(x)D^2 + b(x)D - c(x), x \in (l,r)$$ (44)

where $c(x) \geq 0$ represents a "killing rate at $x$" with which the process is transferred to a "cemetery" (they arise for example after applying an h-transform to a non-killed diffusion). Let

$$\xi := \inf\{t : X_t \notin D\},$$

denote the killing time, where $D$ is the domain with the attainable boundaries included. The distribution of the last position before killing is:

$$P_x[X_\xi - \in A] = \int_0^t ds \int_A \tilde{p}_s(x,y)k(y)dy$$

where $k(x) := c(x)m(x)$ denotes the "killing density".

In this case, the probabilistic representation of the semigroup is:

$$\left(\exp tG\right)f(x) = E_x[\exp -\int_0^t c(X_s)ds f(X_t), H_t > t] = E_x[f(X_t), \tilde{H} > t],$$

where $\tilde{H} = \min[H_t, \inf\{t \geq 0 : \int_0^t c(X_s)ds \geq \epsilon\}]$, where $\epsilon$ is an exponential random variable with mean 1.

3.3 Transformations between diffusions

We recall here some formulas involved in the transformation of diffusions. The transformation

$$f(x,t) = h(x)v(z,t), \quad z = g(x)$$

brings the backward Kolmogorov equation $\left(\frac{\partial}{\partial t} - G\right)f(x,t) = 0$ to the form

$$\frac{\partial}{\partial t}v(z,t) = \left(\left(\frac{g'}{h}\right)^2a[x]v_{zz}(z,t) + \left[\frac{\partial}{\partial x} + \frac{2h'}{h}g'a[x]\right]v_z(z,t) + \frac{Gh}{h}(x)v(z,t)\right)$$ (45)

(by Dynkin's formula).

This may be viewed as the backward Kolmogorov equation of a new diffusion, whose generator will be denoted by $G^{(h)}$.

1. The particular case $h = 1$ is a simple change of variables $Z_t = g(X_t)$, discussed in section 3.1. The particular case $g(x) = x$, called gauge transformation/Doob's h-transform may be interpreted as a diffusion with killing, with generator

$$G^{(h)}f(x) := h^{-1}G(hf)(x) = (Gf)(x) + 2\frac{h'(x)}{h(x)}a(x)f'(x) + \frac{Gh(x)}{h(x)}f(x)$$

This corresponds probabilistically to considering the weighted semigroup

$$P_t^{(h)}f(x) := E_x\left[\frac{h(X_t)}{h(x)}e^{-\int_0^t \Phi_h(X_s)ds}f(X_t)\right]$$

We will also use the notation

$$G^{(h)} = G_h + h^{-1}Gh, \quad G_h := G + 2\frac{h'(x)}{h(x)}a(x)D_x.$$
By using these transformations, one may reduce the study of a large class of "solvable Markov processes" to the study of processes with hypergeometric Sturm-Liouville equations – see (Albanese et al., 2001; Albanese & Kuznetsov, 2007; Kuznetsov, 2004; Campolieti & Makarov, 2009).

Several particular transformations may make the study of a diffusion more convenient:

1. One transformation which removes the drift is

\[ h(x) = m(x)^{-1/2}, \quad z = g(x) = k \int^x \frac{du}{a(u)}. \]

Indeed, the result of the \( h^- \) transform (multiplication by \( h \)) is:

\[ \mathcal{G}_h := \mathcal{G}^{(h)} - h^{-1} \mathcal{G} h = a(x) D_{xx} + (b(x) + 2a(x) \frac{h'}{h}) D_x + \]

\[ = a(x) D_{xx} + (b(x) - 2a(x) \frac{1}{2} \frac{m'(x)}{m(x)^{3/2} m(x)^{-1/2}}) D_x \]

\[ = a(x) D_{xx} + (b(x) - a(x) \frac{b(x) - a'(x)}{a(x)}) D_x = D_x (a(x) D_x). \]

Therefore, the resulting final drift is

\[ b_Z(z) = \mathcal{G}_h g(x) = (D_x (a(x) D_x)) g(x) = D_x k = 0 \]

is 0.

Finally, using

\[ h^{-1} h' = - \frac{1}{2} \frac{m'}{m}, \quad h^{-1} h'' = - \frac{1}{2} \{ M, x \} := - \frac{1}{2} \left[ \frac{m'}{m} - \frac{1}{2} \left( \frac{m'}{m} \right)^2 \right] \]

\[ = - \frac{1}{2} \left( \frac{b - a'}{a} \right)' - \frac{1}{2} \left( \frac{b - a'}{a} \right)^2, \]

\( \{ M, x \} \) is the so-called Schwarzian derivative), the Fokker-Planck equation is put in "Schrödinger form"

\[ v_t = \frac{k^2}{a(x)} v_{zz}(z) - V(x) v(z) = 0, \]

where the potential is \( V = h^{-1} \mathcal{G} h = h^{-1} \mathcal{G} h = \frac{k^2 - 2ah' + (a')^2 + 2a(b' - a'')}{4a}. \)

2. The choice \( g(x) = x, h(x) = \sqrt{\tilde{g}(x)} \) removes the drift and transforms the diffusion generator to the form

\[ \mathcal{G}^{(h)} f(x) = a(x) f''(x) + \mathcal{G} h(x) f(x) = a(x) (f''(x) - I(x) f(x)) \]

where

\[ I(x) = \left( \frac{h'(x)}{h(x)} \right)^2 - \left( \frac{h'(x)}{h(x)} \right)' = \left( \frac{b(x)}{2a(x)} \right)^2 + \left( \frac{b(x)}{2a(x)} \right)' \]

\[ = \frac{b^2(x) + 2a(x) b'(x) - 2b(a')}{4a^3(x)} \]

is also called the weak/Bose invariant (note that \( \frac{h'(x)}{h(x)} = - \frac{h}{2a} \)).

3. Using first the Liouville transformation \( z = l(x) = \int^x \frac{du}{\sigma(u)} \) to transform to \( \tilde{\mathcal{G}} = \frac{1}{2} D_{zz} + b_Z(z) D_z \) and then the gauge transformation with \( h(x) = \sqrt{\sigma(x) \tilde{g}(x)} \) sets the drift to 0 and normalizes the diffusion coefficient, putting the backward Kolmogorov equation in Liouville normal form

\[ (\frac{\partial}{\partial t} - \frac{\partial^2}{\partial z^2} - J(z)) g(z, t) = 0 \]
is an important invariant, in the sense that two diffusions may be transformed into each other if their invariants differ by a constant and a shift of the independent variables – see Lemma 5.7 of (Albanese & Kuznetsov, 2007) or (Kuznetsov, 2004).

### 3.4 Canonical forms: Krein and Schrödinger

Liouville normal form of the Sturm-Liouville equation is

\[-g''(u) + (Q(u) - \lambda) g(u) = 0,\]  

where \(Q(u)\) is the potential function:

- **Reciprocal Gamma**: \(\theta \frac{\Gamma(\beta - 1)}{\Gamma(\beta - 2)} \left( \alpha^2 e^{-2u\sqrt{\theta - 1}} - 2\alpha(\beta + 1)e^{-u\sqrt{\theta - 1}} + \beta^2 \right)\)
- **Fisher-Snedecor**: \(\theta \frac{\Gamma(\beta - 2)}{\Gamma(\beta - 3)} \left( \beta^2 + (\alpha - 1)(\alpha - 3) \csch^2 \left( u \sqrt{\frac{\theta}{2(\beta - 2)}} \right) - \right)\)
- **Symmetric Student**: \(\theta \frac{\nu}{\nu - 1} \left( \nu - (\nu + 2) \sech^2 u \sqrt{\frac{\nu}{\nu - 1}} \right)\)
- **Skew Student**: \(\theta \frac{\nu}{\nu - 1} \left( \delta^2 (a + 1)^2 + \sech^2 (uv\sqrt{\theta a}) \right)\)

### Reference


### 3.5 Hitting times and Feller's classification of diffusion state space boundaries

Let

\[H_y = \inf \{ t : X_t = y \}\]

The (non)attainability of diffusion boundaries depends on the (in)finiteness of expected hitting times

\[\Sigma(x, y) = \mathbb{E}_x[H_y] = \begin{cases} \int_x^y g(z) M[l, z] \, dz, & x < y \\ \int_y^y g(z) M[z, r] \, dz & x \geq y \end{cases}\]

where

\[M[x, y] = \int_x^y m(z) \, dz,\]

denotes the speed measure (see for example Karlin and Taylor (Karlin & Taylor, 1981, pg. 227) or (4.2) (Mao, 2006), for the case of reflecting diffusions).
Feller’s classification scheme is based on the functions

\[
\Sigma(l_+, \varepsilon) = \int_{l_+}^{\varepsilon} M[z, \varepsilon] g(z) \, dz, \quad \Sigma(\varepsilon, r_-) = \int_{\varepsilon}^{r_-} M[\varepsilon, z] g(z) \, dz, \quad (49)
\]

where \( \varepsilon \in I \) is arbitrary and the notations \( \Sigma(l_+, \cdot) \) and \( \Sigma(\cdot, r_-) \) signify that a limit must be evaluated.

**Definition 1** A boundary \( e (e = l \text{ or } e = r) \) of the diffusion state space is attainable/accessible if \( \Sigma_e < \infty \) and unattainable/unaccessible if \( \Sigma_e = \infty \).

If the boundary \( r = \infty \) is attainable, the diffusion “explodes” with positive probability.

**Definition 2** According to Feller’s boundary classification scheme – see (2.7) (Albanese & Kuznetsov, 2007), the boundary \( l \) is said to be:

- non-singular/regular if it is accessible, and \( S(x) \text{m}(x) \in L_1[l, l + \varepsilon] \),
- exit-non entrance, if it is accessible, and \( S(x) \text{m}(x) \notin L_1[l, l + \varepsilon] \)
- entrance-non exit, if it is unaccessible, and \( S(x) \text{m}(x) \in L_1[l, l + \varepsilon] \)
- natural, if it is unaccessible, and \( S(x) \text{m}(x) \notin L_1[l, l + \varepsilon] \).

The same classification applies for \( r \), with the obvious necessary modifications.

**Remark 5** Probabilistic interpretations of Feller’s classification scheme with respect to the behavior of the diffusion process at the boundary \( e \in \{l, r\} \):

- regular boundary - diffusion can both enter and leave from the regular boundary,
- entrance boundary - it is often natural to start the diffusion from the entrance boundary, but such a boundary could never be reached from the interior of the state space - starting from the entrance boundary, the process quickly moves to the interior of its state space but never returns to the entrance boundary;
- exit boundary - for a diffusion process the exit boundary must always be an absorbing point (a trap state), since it is impossible to preserve the continuity of sample paths of the Markov process after attaining the exit boundary,
- natural (Feller) boundary - diffusion can neither start at natural boundary nor reach it in a finite expected time so natural boundaries are always omitted from the state space \( I \).

**Example 2** Cf. (Borodin & Salminen, 2002, pg. 73), for all values of \( \nu \), the Bessel process with generator

\[
\frac{1}{2} D^2 + \frac{\nu + \frac{1}{2}}{x} D
\]

has a natural boundary at infinity (not exit-nor entrance).

The boundary 0 is

\[
\begin{cases}
\text{entrance-not-exit} & \nu \geq 0 \\
\text{regular (both exit and entrance)} & -1 < \nu < 0 \\
\text{exit-not-entrance} & \nu \leq -1,
\end{cases}
\]

Let now \( \bar{D} \) denote the diffusion state space, obtained by joining the attainable boundary points to \( D \). A diffusion is called regular if

\[
P_x[H_y < \infty] > 0, \forall x, y \in \bar{D}.
\]
3.6 Feller diffusions

For regular boundaries, several possible behaviors after the boundary has been reached are consistent with the Markov property and with continuity (absorbing, reflecting, elastic, sticky).

The complete picture providing the one to one correspondence between "probabilistically valid" boundary conditions and semigroups, and probabilistic constructions in all cases emerged in the paper (Itô & McKean, 1963). A pedagogical review was provided recently by Kostrykin, Potthoff, and Schrader (Kostrykin et al., 2010), who consider the "unified boundary condition":

\[ p_0 f(e) - p_1 f^+(e) + p_2 G f(e) = 0, p_i \in [0, 1], \sum_i p_i = 1. \]

The cases \( p_0 = 1, p_1 = 1 \) and \( p_2 = 1 \) correspond respectively to Dirichlet, Neumann and Wentzell boundary conditions.

For elastic boundary defined by \( p_1 + p_1 = 1 \), this becomes

\[ f'(e) = \alpha f(e), \alpha = \frac{p_1}{p_0} \in [0, \infty] \]

Here \( \alpha = \infty \) corresponds to instantaneous killing at \( e \), \( \alpha = 0 \) corresponds to reflection with no killing, and intermediate parameters correspond to delayed killing at the time \( T_0 \) when the local time at \( e \) has exceeded an exponentially distributed random variable with mean \( \alpha^{-1} \).

**Remark 6** Even though the parallelism between the analytic theory of Sturm-Liouville operators with boundary conditions on one hand, and that of probabilistic diffusion semigroups on the other hand, has already been largely established by Feller (Feller, 1952), these theories have not yet fully merged, as one may infer from the divergence of boundary classifications used in Sturm-Liouville theory and probability (for example, regularity of a boundary \( e \) requires finiteness in the first, but not so in the second), and from the recent appearance of new boundary classifications like Van Kampen’s – see (Gabrielli & Cecconi, 2008), Appendix A.

Furthermore, a complete reference seems still hard to find today, due to the fact that many authors like (Itô & McKean, 1974), (Pitman & Yor, 2003) left killing and transient cases as an exercise to the reader – see also (Steinsaltz & Evans, 2007), pg. 1289.

For Reciprocal gamma and Fisher-Snedecor diffusions the resolvents/Green functions for \( x < y \) are given in table 10, where corresponding \( \phi^\pm(x) \) are given in table 11.

<table>
<thead>
<tr>
<th>Table 10</th>
<th>Green’s function ( G(x, y)/m(y) )</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Diffusion</strong></td>
<td><strong>Green’s function</strong></td>
</tr>
<tr>
<td>Reciprocal Gamma</td>
<td>( \frac{1}{\beta} F \left( \frac{-\beta}{2} + \sqrt{\frac{\beta^2}{4} + \frac{\alpha(\beta - 1)}{\gamma}} \right) \phi^-(x) \phi^+(y) )</td>
</tr>
<tr>
<td>Fisher-Snedecor</td>
<td>( \frac{\alpha^{\frac{\alpha}{2}} \beta^{-\frac{\beta}{2}} \Gamma \left( \frac{1 + \sqrt{\frac{\beta^2}{4} + \frac{\alpha(\beta - 2)}{\gamma}}}{2} \right) \phi^-(x) \phi^+(y) )</td>
</tr>
</tbody>
</table>
Table 11 $\phi^\pm(x)$ for Reciprocal gamma and Fisher-Snedecor diffusions

<table>
<thead>
<tr>
<th>Diffusion</th>
<th>Green’s function $G(x, y)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Reciprocal Gamma</td>
<td>$\frac{\beta - 1}{\alpha} \left( \frac{x}{y} \right)^{\frac{\alpha + 1}{2}} \frac{1}{\Gamma \left( \frac{x}{y} \right)} \frac{\mu(s)}{\mu(s) + \frac{1}{4}} \left( -\frac{\beta}{2} + \mu(s) \right) M_{\frac{\beta + 1}{2}, \mu(s)} \left( \frac{\beta}{2} \right) W_{\frac{\beta + 1}{2}, \mu(s)} \left( \frac{\beta}{2} \right)$</td>
</tr>
<tr>
<td>Fisher-Snedecor</td>
<td>$\frac{\alpha (\beta - 2)}{2 \theta} x \frac{1}{\beta - 1} (ax + \beta)^{-\frac{3}{2}} \left( \frac{\lambda t}{\beta} + \sqrt{\frac{\alpha x}{\beta} + \frac{\alpha (\beta - 2)}{2 \theta}} \right) G_{\lambda t} \left( \frac{\beta}{2} \right) \left( 1 + 2 \frac{\alpha x}{\beta} + \frac{\alpha (\beta - 2)}{2 \theta} \right) \times f_1(x, s) f_2(x, s)$</td>
</tr>
</tbody>
</table>

3.7 The resolvent, Sturm-Liouville equations and first passage

The fundamental characteristic of diffusions, the transition density $p_t(x, y)$, is typically unknown explicitly. However, a "quasi-explicit" formula (57) is available for the Green function/resolvent density/density at a random exponential time

$$G_\lambda(x, y) = \int_0^\infty e^{-\lambda t} p_t(x, y) dt. \quad (50)$$

The cornerstone of diffusion analysis is an operator version of (50), concerning the "resolvent", i.e. the Laplace transform in time of the transition operators semigroup:

$$R_\lambda f(x) := \int_0^\infty e^{-\lambda t} P_t f(x) dt = \int_0^\infty e^{-\left(\lambda I - G\right) t} u(x) dt = (\lambda I - G)^{-1} f(x).$$

The resolvents are integral operators, whose kernel $G_\lambda(x, y)$ may be determined by looking for an image $u(x) = u_f(x) = (\lambda I - G)^{-1} f(x)$, for arbitrary $f$, i.e. for a solution of the second order non-homogeneous ODE

$$(\mathcal{G} - \lambda) u(x) + f(x) = 0, \quad (51)$$

with the "respective boundary conditions". Lagrange’s variation of parameters formula yields one particular solution of (51):

$$u_f(x) = \int_0^\infty H_\lambda(x, y) f(y) dy, \quad (52)$$

where $H_\lambda$ may be written as a product:

$$H_\lambda(x, y) = \phi_1(x) \frac{\phi_2(y)}{W(\phi_1, \phi_2)(y)a(y)} I[y \leq x] + \phi_2(x) \frac{\phi_1(y)}{W(\phi_1, \phi_2)(y)a(y)} I[y > x]$$

$$= \frac{m(y)}{w_\lambda} \begin{cases} \phi_1(x) \phi_2(y), & y \leq x \\ \phi_2(x) \phi_1(y), & y > x \end{cases}, \quad (53)$$

where $\phi_1, \phi_2$ are any two independent solutions of the **homogeneous Sturm Liouville equation**

$$\mathcal{G} u(x) - \lambda u(x) = 0, \quad (54)$$

with non zero Wronskian

$$w_\lambda = w_{\phi_1, \phi_2} = \frac{W(\phi_1, \phi_2)(y)}{a(y)} \quad (55)$$
(cf. the Abel-Liouville formula I,5(3), (Poole, 1936)). Indeed, (53) is just the formula for one particular solution of the inhomogeneous equation (51).

We stress that φ₁,φ₂ will not necessarily satisfy the diffusion domain boundary conditions, unless a special choice is made. In (56), we introduce two special "fundamental" particular solutions of (54), φₓ⁺, which do satisfy the domain boundary conditions, and for which the kernel defined in (53) is the Green function/resolvent density, for all the possible "Feller boundary conditions" (unifying thus "miraculously" different probabilistic behaviors reflection, absorption, killing) – see (Borodin & Salminen, 2002), pg. 19 and section 3.6.

**Remark 7** Note that the Laplace transforms of hitting times are forced, by continuity of paths and the Markov property, to be of the form:

\[ H_\lambda(y|x) := E_x e^{-\lambda H_y} = \begin{cases} \phi_1^{-1}(x), & x \leq y \\ \phi_2^{-1}(x), & x \geq y \end{cases} \]

(56)

where φₓ⁺(x) are defined up to a constant. To fix them, one traditional choice, called Weyl functions (Kostenko et al., 2010) is to take wₓ⁻¹φₓ⁺(x), ensuring thus that the Wronskian with respect to the scale function is identically 1.

As easily checked, φₓ⁺(x) are increasing/decreasing positive solutions of (54) (see (Borodin & Salminen, 2002, pg. 18) and (Itô & McKean, 1974, pg. 128)). After specification of boundary conditions at regular points, such functions are unique, up to a constant, as may be inferred by noting that the processes \( e^{-\lambda t} \phi_x^\pm(X_t) \) are martingales, and applying the optional stopping theorem.

**Example 3** For Brownian motion on \((-\infty, \infty)\), φₓ(x) = e^±√2λx. For Brownian motions on \((0, \infty)\) killed and reflected at 0, φₓ⁻ stays the same, while φₓ⁺ becomes

\[ \phi_x^+ = 2 \sinh(\sqrt{2\lambda}x) = e^{\sqrt{2\lambda}x} - \left( \frac{e^{\sqrt{2\lambda}x}}{e^{-\sqrt{2\lambda}x}} \right) \bigg|_{x=0} e^{-\sqrt{2\lambda}x} \]

and

\[ \phi_x^- = 2 \cosh(\sqrt{2\lambda}x) = e^{\sqrt{2\lambda}x} - \left( \frac{(e^{\sqrt{2\lambda}x})'}{(e^{-\sqrt{2\lambda}x})'} \right) \bigg|_{x=0} e^{-\sqrt{2\lambda}x}, \]

respectively. For general transformation formulas when adding killing or reflection at an interior point, see (Hulley & Platen, 2007), Lemmas 2.3, 2.5.

For doubly reflected Brownian motion on \([0, 1]\), φₓ±(x) = sinh(√λx), sinh(√λ(1 - x)).

The functions φₓ±(x) are called "fundamental solutions" in Borodin and Salminen (Borodin & Salminen, 2002), but we prefer calling them monotone solutions, since the name fundamental solutions is already used for the transition kernel \( p_t(x,y) \) (Peskir, 2006; Goard, 2006), as well as in other contexts, like the related Krein’s string representation of diffusions (Comtet & Tourigny, 2011).

**Remark 8** In conclusion, for all Feller diffusions, the formula:

\[ \frac{G_\lambda(x,y)}{m(y)} = w_\lambda^{-1} \phi_1^+(x \wedge y) \phi_-^-(x \vee y) = w_\lambda^{-1} \begin{cases} \phi_1^+(x)\phi_1^-(y), & x \leq y \\ \phi_1^-(y)\phi_1^+(x), & x \geq y \end{cases} \]

(57)

will yield the resolvent density and the transition density \( p_t(x,y) \) by Laplace inversion in an unified manner, disregarding of the boundary conditions! One could recover this way the well-known densities of the processes in Example 3.
For example, Green’s function for the doubly reflected Brownian motion on $[0, 1]$ is

$$G_\lambda(x, y) = \frac{\sinh(\sqrt{\lambda} \min(x, y)) \sinh(\sqrt{\lambda} (1 - \max(x, y)))}{\sqrt{\lambda} \sinh(\sqrt{\lambda})}.$$ 

**Remark 9** Note also a second formula for the killed first passage probabilities:

$$H_\lambda(y|x) = E_x e^{-\lambda H_y} = \begin{cases} \phi_\lambda^+(x) & x \leq y \\ \phi_\lambda^-(y) & x \geq y \end{cases} = \frac{G_\lambda(x, y)}{G_\lambda(y, y)},$$

with an obvious probabilistic explanation.

**Remark 10** The monotone Sturm-Liouville solutions change after a homeomorphism $Z_t = g(X_t)$ to

$$\phi_\lambda^\pm(g(x), \lambda) = \phi_\lambda^\pm(x, \lambda),$$

and their Wronskian with respect to the scale $w_\lambda$ remains unchanged (see for example (Zirbel, 1997, exp. (2.19))).

The monotone solutions intervene also in other Feynman-Kac problems, arising either analytically by separation of variables, or probabilistically, by excursion theory methods, as illustrated in (Pitman & Yor, 2003).

For example, putting $H = \min[H_a, H_b]$, the two-sided exit probabilities are given by:

$$E_x[e^{-\lambda H}; I_{H_b < H_a}] = \frac{S_\lambda[a, x]}{S_\lambda[a, b]}$$

where $S_\lambda[x, y] = \phi_\lambda^+(x)\phi_\lambda^-(y) - \phi_\lambda^-(x)\phi_\lambda^+(y)$.

Note that in the case $\lambda = 0$, one of the fundamental harmonic functions becomes constant. Then, putting $S[x, y] := \int_x^y s(x) dx$ and assuming $S^-(x) := S[x, r] := \int_x^r s(x) dx < \infty$, we find – see for example (Borodin & Salminen, 2002), pg. 14 or (3.5) (Keilson, 1965) – that

$$P_x[\tau_a < \tau_r] = \frac{S[x, r]}{S[a, r]}$$

(and similarly for the other boundary $t$).

The monotone solutions intervene also in the ”key formula” for the Laplace exponent of the inverse of the local time at $y$:

$$\psi^y(\lambda) = \frac{1}{2} \left( \frac{\phi_\lambda^+(y)'}{\phi_\lambda^-(y)} - \frac{\phi_\lambda^-(y)'}{\phi_\lambda^+(y)} \right) = \frac{1}{2 \lambda G_\lambda(x, y)} = \frac{W_\lambda}{2 \phi_\lambda^+(y)\phi_\lambda^-(y)},$$

see Theorem 1 and (40) of (Pitman & Yor, 2003). Note that both the terms in the first parenthesis are Laplace exponents of Levy subordinators, and that they satisfy the ”Riccati version” of the Sturm-Liouville equation:

$$\psi' + \psi^2 + \frac{b}{a} \psi = \lambda a,$$

obtained by putting $\psi = \frac{\phi_\lambda'}{\phi_\lambda}$.

**Remark 11** The solutions of the equation (61) may be obtained by a continued fractions numerical algorithm, due to Euler (Comtet & Tourigny, 2011, (4.2)).
3.8 Weyl-Titchmarsh theory and Schrödinger operators

Diffusion operators are the focus of three parallel literatures: analysis, physics and probability, with
different methods and terminologies. A brief attempt of "dictionary" is provided below.
The cornerstone of the analysis literature are the three Weyl-solutions, named after a procedure due to Weyl which ends up constructing the monotone solutions of probabilistic diffusion theory.
The cornerstone of the probabilistic literature is excursion theory, which among other things separates
a) the conceptual roles of the operator and of the boundary conditions, and b) provides a probabilistic interpretation of the Wronskian of the monotone solutions, as the Laplace exponent of the Levy process of "excursions". This Wronskian, called "Weyl m-function", may be seen as the central object of both theories, since it characterizes uniquely the operator, and allows a determination of its spectral function (Kostenko et al., 2010, Lemma 3.3) and a classification of its spectrum (Kostenko et al., 2010, Cor 3.5).
The big methodological divorce may be seen at this level, since the probabilistic literature ignores spectral aspects typically, and Weyl’s methodology totally, while analysts prefer analyzing boundary conditions on a case by case basis, ignoring the Feller-McKean-Ito unification.
For a nice review of possible asymptotic behaviors of the eigenfunctions, corresponding to the various types of spectrum, see (Gilbert, 2005).
On the practical level of computing exact solutions, the richest literature is that of physicists, for whom the search for "solvable diffusions" with exact solutions is a quite venerable line of research. While traditionally this was often achieved by ”pilgrimage to handbooks of special functions” (Mielnik & Rosas-Ortiz, 2004), there exist also systematic transformation approaches, whose roots may be traced back to works of Euler, Darboux, and Pochammer – see (Ince, 1956, VIII,XVIII), (Natanzon, 1979), and to the factorization ”tricks” of Dirac and Schrödinger in the physics literature (Dereziński & Wrochna, 2011).
For a paper combining this tradition with the probabilistic Cameron-Martin change of drift formula, see (Fischer et al., 1993). Finally, a beginning of unification of the Feller probabilistic and Weyl analysis treatment of diffusions was recently provided in (Kolb, 2009), who discusses the relation between the Feller regular/singular and the Weyl limit circle/limit point cases.

3.9 Solving Sturm-Liouville equations for Kolmogorov Pearson diffusions

Consider the differential operator generating the Kolmogorov-Pearson diffusion semigroups:
\[(GV)(x) := \sum_{i=0}^{2} a_{i} x^{i} D^{2} + (rx + p) D \]\nand the associated Sturm-Liouville equation
\[GV - \lambda V = 0.\]

In this section we investigate the connection between the monotone SL solutions \(\phi^{\pm}_{\lambda}(x)\) and the scale function \(S(x) = \phi^{\pm}_{0}(x)\).
Note that with respect to the "Riccati version" (61) of the Sturm-Liouville equation, this amounts to relating a nonhomogeneous Riccati equation to its homogeneous part.
We employ a transform method originating with Poincaré in the context of solving differential equations with polynomial coefficients, and described for example in (Ince, 1956, VIII, XVIII) and (Jacobsen & Jensen, 2007, Prop.2). This transform generalizes the classical Laplace or Fourier transforms, by making
the choice of integration contour specific to the problem to be solved, and by employing "intertwining kernels" more general than the exponential.

The method consists in looking for solutions of the form

$$V(x) = \int_{\Gamma} K(x,z) \hat{V}(z) dz$$  \hspace{1cm} (64)

where the function $\hat{V}(z)$ and the integration contour $\Gamma$ are yet to be determined.

The operator to be studied $G_x$ and the kernel must satisfy a relation of the form

$$G_x K(x,z) = M_x K(x,z)$$

where $M_x$ is an operator depending on $z$ and $D_x$ only (see (Ince, 1956, 8.2 (B)) and (Ince, 1956, 8.31, pg.192) for two examples).

For example, one may use an exponential transform

$$V(x) = \int_{\Gamma} e^{xz} \hat{V}(z) dz = \int_{\Gamma} e^{xz} z^{-1} V^\ast(z) dz.$$  \hspace{1cm} (65)

Plugging this into the SL equation (63) and applying integration by parts, one finds that two conditions (Ince, 1956, VIII.2,C,D) must be satisfied:

1. The kernel $\hat{V}(z)$ must satisfy the adjoint operator

$$\hat{M}_z \hat{V}(z) = 0.$$  \hspace{1cm} (66)

In the Laplace case for example, $\hat{M}_z$ is the transform of the initial operator, obtained by extending linearly the transformation $x \mapsto -D_z, D_x \mapsto z$, i.e.

$$x^i (D_x)^k \mapsto (-D_z)^i z^k.$$

2. To remove boundary effects, the integration contour $\Gamma$ must be either

(a) an arc connecting zeros of the bilinear concomitant $P[K(x,z), \hat{V}(z)]$ (Ince, 1956, V.3,(B)), so that boundary contributions cancel, or
(b) a closed contour.

As a particular example of the latter, when $\hat{V}(z)$ is the usual Laplace transform, $\Gamma$ is a Bromwich contour, a limiting case.

**Example 4** For the student SL operator, the operator and its Laplace adjoint is:

$$G = (a_2 x^2 + a_0) D^2 + (-\theta x + p) D - \lambda \mapsto \hat{M} = a_2 D_z^2 z^2 + a_0 z^2 + \theta D_z z + pz - \lambda$$

and the kernel must satisfy

$$[a_2 D_z^2 z^2 + a_0 z^2 + \theta D_z z + pz - \lambda] \hat{V}(z) = 0$$

$$a_2 (z^2 \hat{V}(z))'' + \theta (z \hat{V}(z))' + (a_0 z^2 + pz - \lambda) \hat{V}(z) = 0 \Leftrightarrow$$

$$a_2 (z V_\lambda''(z))'' + \theta (V_\lambda'(z))' + \left(\psi(z) - \lambda \right) V_\lambda'(z) = 0 \Leftrightarrow$$

$$a_2 z V_\lambda''(z)'' + (\theta + 2a_2) (V_\lambda')' + \left(\psi(z) - \lambda \right) V_\lambda(0) = 0$$

where $V_\lambda(z) := z \hat{V}_\lambda(z), \psi(z) = a_0 z^2 + pz$. This may be viewed as a expected discount problem for a CIR process, but does not seem easier than the original SL equation.
3.10 Geodesics and large deviations asymptotics

It is a remarkable fact that the transition density of Brownian motion is provided exactly by its semi-classical limit approximation

\[ p_T(x,y) = \sqrt{\det \left( -(2\pi)^{-1} \frac{\partial^2}{\partial x \partial y} V_T(x,y) \right)} \exp\left( -V_T(x,y) \right), \]

where the first term is the so called Van Vleck prefactor, and the second, \( V_T(x,y) \) is the action functional/large deviations functional defined by the variational problem:

\[ V_T(x,y) = \inf \int_0^T \frac{u_t^2}{2} dt, \quad x_0 = x, \quad x_T = y, \quad x'_t = b(x_t) + \sigma(x_t)u_t \Leftrightarrow \]

\[ V_T(x,y) = \frac{1}{2} \inf \int_0^T L(x'_t, x_t) dt, \quad x_0 = x, \quad x_T = y, \quad L(x'_t, x_t) = \left( \frac{x'_t - b(x_t)}{\sigma(x_t)} \right)^2 \]  \( (67) \)

(for jump-diffusions, the asymptotic first-passage problem is obtained formally from the Brownian one by replacing the quadratic "Brownian Lagrangian" \( L(u) = \frac{u^2}{2} \) by an arbitrary convex Lagrangian, keeping thus the "linear-convex" structure of our Lagrangian problem).

The minimizing paths of the variational problem (67) satisfy the Euler-Lagrange equation

\[ \frac{dL_{x'}}{dt} = L_x \Rightarrow x'' = b'b - \frac{a'}{2a} \left( x' \right)^2 + b^2. \]  \( (68) \)

For Brownian motion for example, they are straight lines \( x_t = x + \frac{b}{\sqrt{\pi T}} (y - x) \), yielding \( V_T(x,y) = \frac{(y-x)^2}{2T} \).