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Application of growth functions in the predictions of live weight of domestic animals

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Abstract. We consider several most frequently used growth functions with the aim of predicting live weight of domestic animals. Special attention is paid to the possibility of estimating well the saturation level of animal weight and defining life cycle phases based on animal weight. Parameters of the growth function are most often estimated on the basis of measurement data by applying the Least Squares (LS) principle. These nonlinear optimization problems very often refer to a numerically very demanding and unstable process. In practice, it is also possible that among the data there might appear several measurement errors or poor measurement samples. Such data might lead not only to unreliable, but very often to wrong conclusions. The Least Absolute Deviations (LAD) principle can be successfully applied for the purpose of detecting and minorizing the effect of such data. On the other hand, by using known properties of LAD-approximation it is possible to significantly simplify the minimizing functional, by which parameters of the growth function are estimated. Implementation of two such possibilities is shown in terms of methodology.

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Key words: growth function, animal weight growth, life cycle phase, saturation level, least absolute deviations, LAD, Least Squares, LS

1 Introduction

The growth of animals is a result of many biological processes where genotype determines their maximum expression, while environment determines the stage to which this genetic potential can be utilized. It is also clear that the growth is a significant physiological activity for all domestic animals, but it is of special interest when meat animals such as pigs, poultry, beef and others are concerned since the growth is nowadays considered as the material base of animal production. In practice, selection criteria usually applied for improving the growth characteristics in domestic animals are daily live weight gain, food conversion ratio, ultrasound backfat thickness and different slaughter traits as well as some meat quality traits. For example, pork producers have economic interest to produce lean pork as efficiently as possible because of consumer demand and trading based on carcass lean percentage. Knowing the parameters in a certain model enables prediction of future live weights of animals or their tissues, organs etc. along the time scale. In this light, modelling of animal growth can be of help in decision making processes regarding the optimal slaughter age/weight, selection of feeding regime, diet composition, etc.

We consider several most frequently used growth functions simulating animal weight growth. From that standpoint, special attention is paid to the possibility of estimating well the saturation level of animal weight and defining life cycle phases based on animal weight.

It is well known that for some growth function parameter estimation on the basis of the given measurement data is a very demanding numerical procedure, which cannot always be successfully carried out by ready-made software (*Mathematica, Matlab, SAS, Statistica*). In this paper, estimation of the parameter vector $\mathbf{a} \in \mathbb{R}^n$ for some model growth function $t \mapsto f(t;\mathbf{a})$ is estimated on the basis of measurement data (t_i, y_i), $i = 1,\dots,m$, $m \geq n$, where $0 < t_1 < \ldots < t_m$, and $y_i > 0$, primarily by applying the Least Absolute Deviations (LAD) principle[2] (Bazaraa et al., 2006; Cadzow, 2002). In that way it is possible that among the data there appear severe measurement errors or poor measurement samples known in the literature as "outliers" or "wild points" (Watson, 1980). Such data might lead not only to unreliable, but very often to wrong conclusions. The LAD-principle can be successfully applied for the purpose of detecting such data. On the other hand, by using known properties of LAD-approximation it is possible to simplify the minimizing function significantly, by which parameters of the growth function are

[2]The principle is attributed to Josip Rudjer Bošković (1711-1787), Croatian scientist (mathematician, physicist, astronomer and philosopher) born in Dubrovnik. Powerful computers have recently caused great interest in and popularity of that principle, which can be seen in numerous papers published in journals as well as international conferences dealing with this issue. A sequence of such conferences is dedicated to J. R. Bošković (Dodge, 1987).

estimated. If the measurement data are contaminated only by normal distributed random errors, LAD-approximation may be used as an excellent initial approximation for searching for the best Least Squares (LS) approximation. Implementation of two such possibilities is shown in terms of methodology.

For the purpose of estimating the saturation level and determining life cycle phases in animal weight growth we consider a few most frequently used growth functions:

(a) Logistic function with parameters A, b, c (Jukić and Scitovski, 2003; Kralik et al., 2007; Pfeifer et al., 1984; Ratkowsky, 1990)

$$f(t; A, b, c) = \frac{A}{1 + be^{-ct}}, \qquad A, b, c > 0, \tag{1}$$

which is a solution of the differential equation

$$\frac{dy}{dt} = cy(A - y), \qquad A, c > 0. \tag{2}$$

(b) Generalized logistic function with parameters A, b, c, γ (Jukić and Scitovski, 1998; Kralik et al., 2007; Ratkowsky, 1990; Scitovski et al., 2006)

$$f(t; A, b, c, \gamma) = \frac{A}{(1 + be^{-c\gamma t})^{1/\gamma}}, \qquad A, b, c, \gamma > 0, \tag{3}$$

which is a solution of the differential equation

$$\frac{dy}{dt} = cy\left(1 - \left(\frac{y}{A}\right)^{\gamma}\right), \qquad A, c, \gamma > 0. \tag{4}$$

(c) Gompertz function with parameters A, b, c (Franses, 1994; Jukić et al., 2004; Kralik et al., 2007; Ratkowsky, 1990; Vouri et al., 2006)

$$f(t; A, b, c) = e^{A - be^{-ct}}, \qquad A, b, c > 0, \tag{5}$$

which is a solution of the differential equation

$$\frac{dy}{dt} = cy(A - \ln y), \qquad A, c > 0. \tag{6}$$

(d) Von Bertalanffy growth function with parameters A, b, c (Koehn et al., 2007; Kralik et al., 2007; Ratkowsky, 1990)

$$f(t; A, b, c) = A\left(1 - be^{-ct}\right)^3, \qquad A, b, c > 0, \tag{7}$$

which is a solution of the differential equation

$$\frac{dy}{dt} = \alpha y^{\frac{2}{3}} - \beta y, \qquad \alpha, \beta > 0, \tag{8}$$

whereby the connection between parameters (α, β) and (A, b, c) is given by $A = \frac{\alpha}{\beta}$, $b = \frac{K}{\alpha}$, $c = \frac{\beta}{3}$, where K is the integration constant of differential equation (8).

General growth functions which combine several other growth functions can also be found in the literature. E.g. a well-known Richardson growth function (Ratkowsky, 1990)

$$f(t; A, b, c) = A\left(1 - be^{-ct}\right)^{\delta}, \qquad A, b, c > 0,$$

was developed as a generalization of the von Bertalanffy growth function. Specially, for $\delta = -1$, this growth fundtion becomes a logistic function, and for $\delta \to \infty$, it becomes a Gompertz function.

The paper is organized as follows. In *Section 2* the problem of estimating best LAD-optimal parameters in nonlinear growth functions is considered. In *Subsection 2.1* and *Subsection 2.2* two approaches are proposed on the basis of which it is sometimes possible to considerably simplify the search for best LAD-parameters. Thereby *Subsection 2.1* considers the case in which the growth function contains at least one linear parameter, whereas *Subsection 2.2* gives a modification of a well-known log-linearization method, which for most frequently considered growth functions gives best LAD-optimal parameters in only a few steps. *Section 3* shows the application of the LAD-principle to estimation of optimal parameters of the mentioned growth functions, which is then used for determining characteristic points of the mentioned growth functions and estimation of life cycle phases. In *Section 4* LS and LAD-optimal parameters obtained on the basis of our own measurement data are compared. Finally, in *Section 5* main conclusions are given.

2 Least absolute deviation method

In applied research parameter vector $\mathbf{a} \in \mathbb{R}^n$ of the growth function $t \mapsto f(t; \mathbf{a})$ is most frequently estimated on the basis of measurement data (t_i, y_i), $i = 1, \dots, m$, $m \geq n$, by applying the well-known Least Squares (LS) principle[3], that leads to a nonlinear minimization problem

$$F_2(\mathbf{a}) = \sum_{i=1}^{m} (f(t_i; \mathbf{a}) - y_i)^2 \to \min_{\mathbf{a} \in \mathbb{R}^n}. \tag{9}$$

Thereby in applied research different linearizations are used without prior justification, which may lead to unreliable, and very often wrong conclusions.

For example, what happens frequently is that instead of minimizing functional (9) functional

$$\Phi(\mathbf{a}) = \sum_{i=1}^{m} (\ln f(t_i; \mathbf{a}) - \ln y_i)^2 \to \min_{\mathbf{a} \in \mathbb{R}^n} \tag{10}$$

is minimized. Functional Φ can be significantly simpler than functional F_2, but parameter vector $\hat{\mathbf{a}}$ obtained by minimizing functional Φ can significantly differ from the real value obtained by

[3]It is associated with the German mathematician C. F. Gauss (1777-1855), who applied this principle for the first time in his paper from 1805 in which he studied motion of celestial bodies.

minimizing functional F_2 (McCallum, 2000; Franses, 1994; Hadeler et al., 2007; Jukić et al., 2004; Scitovski and Kosanović, 1985).

In this paper we will primarily show the application of the LAD-principle for estimation of the optimal parameter vector $\mathbf{a}^* \in \mathbb{R}^n$ of the aforementioned growth functions. Generally, the problem of searching for the LAD-optimal parameter vector $\mathbf{a}^* \in \mathbb{R}^n$ of the growth function $t \mapsto f(t_i; \mathbf{a})$ is reduced to searching for the stationary point of the functional

$$F(\mathbf{a}) = \sum_{i=1}^{m} |f(t_i; \mathbf{a}) - y_i| = \|\mathbf{r}(\mathbf{a})\|_1 \to \min_{\mathbf{a}\in\mathbb{R}^n}, \tag{11}$$

where $\mathbf{r}(\mathbf{a}) = (r_1(\mathbf{a}), \ldots, r_m(\mathbf{a}))^T$, $r_i(\mathbf{a}) = f(t_i; \mathbf{a}) - y_i$, $i \in I = \{1, \ldots, m\}$. According to Bazaraa et al. (2006); Demjanov and Vasiljev (1981); Gonin and Money (1989); Ruszczynski (2006), vector $\mathbf{a}^* = (a_1^*, \ldots, a_n^*)^T \in \mathbb{R}^n$ is a stationary point of functional (11), if for every $i \in I_0(\mathbf{a}^*) = \{i \in I : r_i(\mathbf{a}^*) = 0\}$ there exists $\lambda_i \in [-1, 1]$ such that

$$\mathbf{0} = \sum_{i\in I_0(\mathbf{a}^*)} \lambda_i \operatorname{grad} r_i(\mathbf{a}^*) + \sum_{i\in I\setminus I_0(\mathbf{a}^*)} \sigma_i(\mathbf{a}^*) \operatorname{grad} r_i(\mathbf{a}^*), \tag{12}$$

where $\operatorname{grad} r_i(\mathbf{a}^*) = \left(\frac{\partial r_i(\mathbf{a}^*)}{\partial a_1}, \ldots, \frac{\partial r_i(\mathbf{a}^*)}{\partial a_n}\right)^T$, and $\sigma_i(\mathbf{a}^*) = \operatorname{sign}(r_i(\mathbf{a}^*))$, $i \in I \setminus I_0(\mathbf{a}^*)$.

In the case when the growth function is linear in parameters, very efficient methods for solving this problem can be found in the literature (Bartels et al., 1978; Cadzow, 2002; Cupec et al., 2009; Sabo and Scitovski, 2008; Yan, 2003). If the growth function is nonlinear in parameters, the problem of minimizing functional (11) is a numerically very demanding nondifferentiable nonlinear minimization problem. For solving this problem there exist general methods (Kelley, 1999) and corresponding ready-made software (*Mathematica, Matlab, SAS, Statistica*). It often happens that by using given software either minimization of functional (11) cannot be done or it gives wrong solutions for most growth functions given in *Section 1*.

2.1 Partial linear growth functions

With the majority of growth functions one or more parameters occurs linearly. This fact may be used for the purpose of simplifying the procedure of searching for LAD-parameters.

Without loss of generality, suppose that growth function $t \mapsto f(t; a_1, \ldots, a_n)$, is linear in the parameter a_1 and that it can be written e.g. in the following form

$$f(t; a_1, \ldots, a_n) = a_1 g(t; a_2, \ldots, a_n),\ (a_1, a_2, \ldots, a_n)^T \in \mathbb{R}^n. \tag{13}$$

Moreover, let us suppose that $(a_1^*, a_2^*, \ldots, a_n^*)^T \in \mathbb{R}^n$ are optimal LAD-parameters of growth function (13) obtained on the basis of given data points (t_i, y_i), $i \in I = \{1, \ldots, m\}$, i.e.

$$\min_{(a_1,\ldots,a_n)^T\in\mathbb{R}^n} F(a_1, a_2, \ldots, a_n) = F(a_1^*, a_2^*, \ldots, a_n^*),$$

where

$$F(a_1, a_2, \ldots, a_n) = \sum_{i=1}^{m} |a_1 g(t_i; a_2, \ldots, a_n) - y_i| \,.$$

In accordance with the well-known property of weighted median (Sabo and Scitovski, 2008), there exists index $i_0 \in I$ such that

$$\begin{aligned} F(a_1^*, a_2^*, \ldots, a_n^*) &= \sum_{i=1}^{m} |g(t_i; a_2^*, \ldots, a_n^*)| \left| a_1^* - \frac{y_i}{g(t_i; a_2^*, \ldots, a_n^*)} \right| \\ &\geq \sum_{i=1}^{m} |g(t_i; a_2^*, \ldots, a_n^*)| \left| \alpha_0^* - \frac{y_i}{g(t_i; a_2^*, \ldots, a_n^*)} \right|, \end{aligned}$$

where $\alpha_0^* := \frac{y_{i_0}}{g(t_{i_0}; a_2^*, \ldots, a_n^*)}$. Note that $F(a_1^*, a_2^*, \ldots, a_n^*) = F(\alpha_0^*, a_2^*, \ldots, a_n^*)$, i.e. there exists a best LAD-solution, such that the graph of the corresponding growth function passes through at least one data point (t_{i_0}, y_{i_0}). Based upon that fact, we eliminate parameter a_1, by means of which we obtain a growth function with the remaining $n-1$ parameters $a_2, \ldots, a_n$

$$\phi_0(t; a_2, \ldots, a_n) = y_{i_0} \frac{g(t; a_2, \ldots, a_n)}{g(t_{i_0}; a_2, \ldots, a_n)}, \qquad (a_2, \ldots, a_n)^T \in \mathbb{R}^{n-1},$$

whereby the corresponding LAD-problem becomes simpler and gives values of parameters $\hat{a}_2, \ldots, \hat{a}_n$. After that, for the LAD-optimal value of parametar a_1 we obtain

$$\hat{a}_1 = \frac{y_{i_0}}{g(t_{i_0}; \hat{a}_2, \ldots, \hat{a}_n)},$$

and the corresponding LAD-sum is $\hat{F}$. Among all acceptable possibilities for choosing the data point (t_{i_0}, y_{i_0}), $i_0 \in I$, we choose the one that gives the smallest LAD-sum.

On the basis of general results (Bazaraa et al., 2006; Schöbel, 1999, 2003; Watson, 1980; Yan, 2003), it is also possible to conduct the described procedure of searching for LAD-optimal parameters in a more general case in which a greater number of parameters occurs linearly in the growth function. An illustration of this method is given by the example in *Section 3.1.*

2.2 Log-linearization

For the purpose of solving optimization problem (11) for some special growth functions, a well-known log-linearization method will be modified and an iterative procedure will be proposed by which the problem of determining the stationary point of functional (11) is reduced to a sequence of simpler weighted LAD-problems.

Let us first notice that by applying the Lagrange mean value theorem directly to logarithmic function $u \mapsto \ln u$ there directly follows

Lemma 1. *For every $\alpha, \beta > 0$, $\alpha < \beta$ there exists $\xi(\alpha, \beta) \in \langle \alpha, \beta \rangle$ such that*

$$\xi(\alpha, \beta)(\ln \beta - \ln \alpha) = \beta - \alpha.$$

According to *Lemma 1*, there exists $\xi(\mathbf{a}) = (\xi_1(\mathbf{a}), \ldots, \xi_m(\mathbf{a}))^T \in \mathbb{R}^m$, where $\xi_i(\mathbf{a})$ is a number between $f(t_i;\mathbf{a})$ and y_i. Therefore, functional (11) can be written as

$$F(\mathbf{a}) = \sum_{i=1}^{m} |\xi_i(\mathbf{a})|\,|\ln f(t_i;\mathbf{a}) - \ln y_i|.$$

Motivated by the previous formula, let us define the following iterative procedure.

Let $\mathbf{a}^{(0)} \in \mathbb{R}^n$ be an arbitrary vector. For $k = 0, 1, \ldots$, we define

$$\mathbf{a}^{(k+1)} = \operatorname*{argmin}_{\mathbf{a}\in\mathbb{R}^n} G_k(\mathbf{a}), \qquad \text{where} \tag{14}$$

$$G_k(\mathbf{a}) = \sum_{i=1}^{m} |f(t_i;\mathbf{a}^{(k)})|\,|\ln f(t_i;\mathbf{a}) - \ln y_i|. \tag{15}$$

The following theorem shows that if sequence $\mathbf{a}^{(k)}$, $k = 0, 1, \ldots$, defined by the iterative procedure (14–15) starting with some term $\mathbf{a}^{(\mu)}$, $\mu \geq 1$ becomes stationary, then in that way a stationary point of functional F is found.

Theorem 1. *If there exists an index μ such that in the iterative process (14–15) there holds $\mathbf{a}^{\mu+1} = \mathbf{a}^{\mu}$, then the vector $\mathbf{a}^* = \mathbf{a}^{\mu}$ is a stationary point of functional F given by (11).*

Proof. According to the assumption, let $\mathbf{a}^* = \mathbf{a}^{\mu}$ be a minimum point of the functional G_μ. Therefore, this is at the same time a stationary point of the functional G_μ, i.e. if for every $i \in I_0'(\mathbf{a}^*) = \{i \in I : \ln f(t_i, \mathbf{a}^*) - \ln y_i = 0\}$ there exists $\lambda_i \in [-1, 1]$ such that

$$\mathbf{0} = \sum_{i\in I_0'(\mathbf{a}^*)} \lambda_i \operatorname{grad} r_i(\mathbf{a}^*) + \sum_{i\in I\setminus I_0'(\mathbf{a}^*)} \sigma_i'(\mathbf{a}^*) \operatorname{grad} r_i(\mathbf{a}^*), \tag{16}$$

where $\sigma_i'(\mathbf{a}^*) = \operatorname{sign}(\ln f(t_i;\mathbf{a}^*) - \ln y_i)$, $i \in I \setminus I_0'(\mathbf{a}^*)$. Note that $I_0'(\mathbf{a}^*) = I_0(\mathbf{a})$ and $\sigma_i'(\mathbf{a}^*) = \sigma_i(\mathbf{a}^*)$. Hence from (16) there follows

$$\mathbf{0} = \sum_{i\in I_0(\mathbf{a}^*)} \lambda_i \operatorname{grad} r_i(\mathbf{a}^*) + \sum_{i\in I\setminus I_0(\mathbf{a}^*)} \sigma_i(\mathbf{a}^*) \operatorname{grad} r_i(\mathbf{a}^*),$$

i.e. $\mathbf{a}^*$ is a stationary point of the functional F. □

Hence, if the sequence $\mathbf{a}^{(k)}$, $k = 0, 1, \ldots$, defined by the iterative procedure (14–15) starting with some term $\mathbf{a}^{(\mu)}$, $\mu \geq 1$ becomes stationary, then in that way we found a good candidate for a global minimizer of the functional F. The advantage of the given method lies in the fact that in every step of the iterative procedure (14-15) we solve a simpler LAD-problem in which at least one parameter becomes linear. In this case we can apply a property of the LAD-principle described in *Subsection 2.1*, by which there exists a best LAD-solution, such that the graph of the corresponding growth function passes through at least one data point. In addition to that, as illustrated by the example given in *Subsection 3.2*, the iterative procedure (14–15) mostly ends in only a few steps.

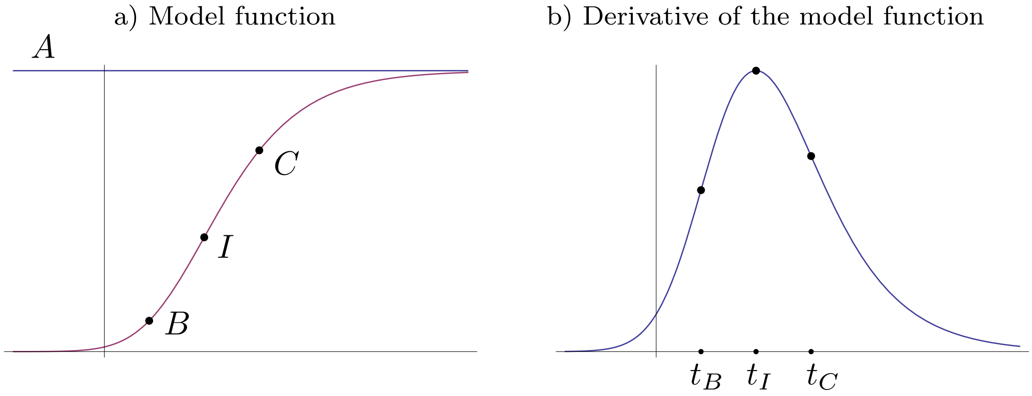

Figure 1: Growth function and its derivative

3 Applications of the LAD-principle in the prediction of live weight of domestic animals

All growth functions mentioned in *Section* 1 are of the sigmoidal form (see Fig1.a) with one inflection point $I = (t_I, f(t_I))$, their derivatives are bell-shaped (see Fig1.b) with two inflection points $B = (t_B, f(t_B))$ and $C = (t_C, f(t_C))$.

Upon estimation of parameters of the growth function on the basis of the given measurement data, it is possible to estimate the saturation level and determine life cycle phases (Kralik et al., 1999), that are defined by means of the aforementioned points I, B, C:

- Preparation phase – forming $t \in [0, t_B]$;
- Phase of intensive growth $t \in [t_B, t_C]$;
- Phase of growth retardation $t > t_C$.

For growth functions mentioned in *Introduction*, in Table 1 we list formulas for characteristic points I, B, C.[4]

3.1 Estimation of LAD-parameters of the generalized logistic function

Since by a generalized logistic function (3) parameter A occurs linearly, we can apply the principle described in *Section 2.1.* Optimal LAD-parameters of the generalized logistic function (3) are determined such that for every data point (t_{i_0}, y_{i_0}), $i_0 \in I$, we solve a simpler LAD problem for the growth function

$$\phi(t; b, c, \gamma) = y_{i_0} \frac{1 + be^{-c\gamma t_{i_0}}}{(1 + be^{-c\gamma t})^{1/\gamma}}, \qquad b, c, \gamma > 0,$$

[4]All evaluations and illustrations were done by *Mathematica 6* on a PC (CPU: 2.00 GHz Intel Core 2 Duo processor, Memory: 1.99 GB DDR2) on the basis of our own software.

Growth function	Saturation level	Inflection point	t_B	t_C
Logistic	A	$\left(\frac{\ln b}{c}, \frac{A}{2}\right)$	$\frac{1}{c}\ln\frac{b}{2+\sqrt{3}}$	$\frac{1}{c}\ln\frac{b}{2-\sqrt{3}}$
Gen. log.	A	$\left(\frac{1}{c\gamma}\ln\frac{b}{\gamma}, \frac{A}{(1+\gamma)^{1/\gamma}}\right)$	$\frac{1}{c\gamma}\ln\frac{2b}{\gamma(\gamma+3)+\gamma\sqrt{(\gamma+1)(\gamma+5)}}$	$\frac{1}{c\gamma}\ln\frac{2b}{\gamma(\gamma+3)-\gamma\sqrt{(\gamma+1)(\gamma+5)}}$
Gompertz	e^A	$\left(\frac{\ln b}{c}, e^{A-1}\right)$	$\frac{1}{c}\ln\frac{2b}{3+\sqrt{5}}$	$\frac{1}{c}\ln\frac{2b}{3-\sqrt{5}}$
Von Bertalanffy	A	$\left(\frac{\ln 3b}{c}, \frac{8A}{27}\right)$	$\frac{1}{c}\ln b(4-\sqrt{7})$	$\frac{1}{c}\ln b(4+\sqrt{7})$

Table 1: Characteristic points of growth functions

and obtain corresponding values of parameters $\hat{b}, \hat{c}, \hat{\gamma}$. After that, for the value of parametar A we obtain

$$\hat{A} = y_{i_0}\left(1 + \hat{b}e^{-\hat{c}\hat{\gamma}t_{i_0}}\right)^{1/\hat{\gamma}},$$

and the corresponding LAD-sum is $\hat{F}$. Among all acceptable possibilities for choosing the data (t_{i_0}, y_{i_0}) we choose the one that gives the smallest LAD-sum.

In the following example we illustrate the mentioned method on the basis of our own measurement data.

Example 1. *Given are broiler weight measurement data* (t_i, y_i), $i = 1, \dots, 8$, *where* t_i *and* y_i *are time intervals in weeks and weights of broilers in* kg, *respectively.*

t_i	1	2	3	4	5	6	7	8
y_i	.165	.443	.861	1.401	2.022	2.676	3.312	3.891

Table 2 shows values of parameters $\hat{b}, \hat{c}, \hat{\gamma}, \hat{A}$ obtained in the previously described way if we fix some of the first five data points. Point $(1, .165)$ is not acceptable as a fix point because in this case the parameter γ attains a negative value. Similarly, other points that are not mentioned are also unacceptable. Since the smallest LAD-sum is attained if we fix the third point $(3, .861)$, $A^* = 5.78457$, $b^* = 1.58802$, $c^* = 1.50596$, $\gamma^* = 0.232189$ are obtained as LAD-optimal parameters of the generalized logistic function.

Fix point	$\hat{b}$	$\hat{c}$	$\hat{\gamma}$	$\hat{A}$	$\hat{F}$
(1,.165)	7.11434	0.494918	-17.1631	0.09244	4.20831
(2,.443)	1.58584	1.09853	0.207094	12.7766	1.22962
(3,.861)	1.58802	1.50596	0.232189	5.78457	0.12453
(4,1.401)	2.96276	1.0482	0.361854	5.58901	0.20645
(5,2.022)	3.30886	1.06215	0.374271	5.49061	0.17392

Table 2: Choice of the fix point

3.2 Estimation of LAD-parameters of the Gompertz function

The problem of searching for the optimal parameters of Gompertz function (5) according to the LAD-principle leads to minimization of functional (11), which is in this case of the form

$$F(a,b,c) = \sum_{i=1}^{m} \left| e^{a-be^{-ct_i}} - y_i \right| \to \min_{a,b,c>0} . \tag{17}$$

General methods of nondifferentiable minimization of this functional rarely give a solution because of the overflow which often appears in the numerical procedure. In this case the corresponding functional (15) in the iterative procedure (14-15) is

$$G_k(a,b,c) = \sum_{i=1}^{m} e^{a_k - b_k e^{-c_k t_i}} \left| a - be^{-ct_i} - \ln y_i \right| , \tag{18}$$

whereby two parameters occur linearly. The problem of minimizing functional (18) can be considered as a onedimensional minimization problem:

$$\min_{c>0} \psi^{(k)}(c), \tag{19}$$

whereby the value of the function $\psi^{(k)}$ in some point $\hat{c}$ is

$$\psi^{(k)}(\hat{c}) = \min_{a,b>0} \sum_{i=1}^{m} w_i^{(k)} |a - be^{-ct_i} - \ln y_i|, \qquad w_i^{(k)} = e^{a_k - b_k e^{-c_k t_i}}. \tag{20}$$

Minimization problem (20) can be solved by applying the Two Points Method (Sabo and Scitovski, 2008) and onedimensional minimization problem (19) can be solved by the Brent method (Brent, 1973) or some of methods mentioned in Kelley (1999). In the following example we illustrate the mentioned method on the basis of our own measurement data.

Example 2. *Given are pig weight measurement data* (t_i, y_i), $i = 1, \dots, 26$ *where* t_i *and* y_i *are time intervals in days and weights of pigs in* kg, *respectively.*[5]

t_i	49	53	58	61	68	75	82	89	96	103	110	117	124
y_i	23.0	23.7	29.0	30.8	33.6	39.0	44.0	48.8	49.0	59.7	67.4	72.0	77.7

t_i	131	138	145	152	159	166	173	180	187	194	201	208	215
y_i	84.1	87.9	94.7	101.7	110.0	113.4	123.6	120.0	123.6	130.6	139.3	136.1	144.8

Optimal parameters of the Gompertz function will be searched for by the iterative procedure (14-15) combined by the Two Points Method (Cupec et al., 2009) and the Brent method (Brent, 1973), where the corresponding functional G_k is given by (18). The flow of the iterative procedure is shown in *Table 3*. Note that in several steps only the given method attains optimal values very efficiently.

[5] The data originate from the "Agrofarmer" farm owned by the Ivančić family. For illustration, it suffices to use the data referring only to one pig. More detailed data were collected for the purpose of D.Vincek's PhD dissertation.

k	a_k	b_k	c_k	$F(a_k, b_k, c_k)$
0	0	0	0	2 081.5000
1	5.36523	3.74714	0.0105252	42.9155
2	5.34997	3.77618	0.0107468	42.4849
3	5.35321	3.76976	0.0106988	42.4567
4	5.35321	3.76964	0.0106987	42.4567

Table 3: The iterative procedure

4 Comparison of LAD and LS optimal parameters

For broiler growth measurement data from *Example 1* and pig growth measurement data from *Example 2* we estimate LAD and LS optimal parameters for all growth functions mentioned in *Section 1*. After that we determine characteristic points from *Table 1*, on the basis of which we can determine the saturation level and estimate life cycle phases of the animal weight growth in question.

Let us first consider measurement data from *Example 1*. In *Table 4* optimal LS-parameters are given in brackets below optimal LAD-parameters for every growth function. We also calculate corresponding characteristic points I, B, C.

For every optimal LS-parameter of all growth functions in *Table 5* asymptotic confidence intervals are given that are obtained by means of *Mathematica* module `NonlinearRegress`. *Figure 2* shows the corresponding area of all curves whose parameters are placed in corresponding asymptotic confidence intervals.

Growth function	Parameters A	b	c	γ	Inflection point	$(t_B, f(t_B))$	$(t_C, f(t_C))$
Logistic	4.92744	26.5123	0.575061	–			
	(4.61277)	(29.9927)	(0.626874)	–	$(5.7, 2.5)$	$(3.4, 1.04)$	$(8.0, 3.89)$
Gen. log.	5.78457	1.58802	1.50596	0.232189			
	(6.22662)	(0.147201)	(9.80803)	(0.029831)	$(5.5, 2.35)$	$(2.5, 0.64)$	$(8.5, 4.14)$
Gompertz	1.86191	4.64243	0.27774	–			
	(1.84738)	(4.68099)	(0.282193)	–	$(5.5, 2.37)$	$(2.1, 0.47)$	$(9.0, 4.39)$
Von Bertalanffy	8.01065	0.898499	0.179395	–			
	(8.81407)	(0.876403)	(0.163196)	–	$(5.5, 2.37)$	$(1.1, 0.14)$	$(10.0, 4.9)$

Table 4: LS and LAD- optimal parameters and characterictic points of the growth function based on data from Example 1

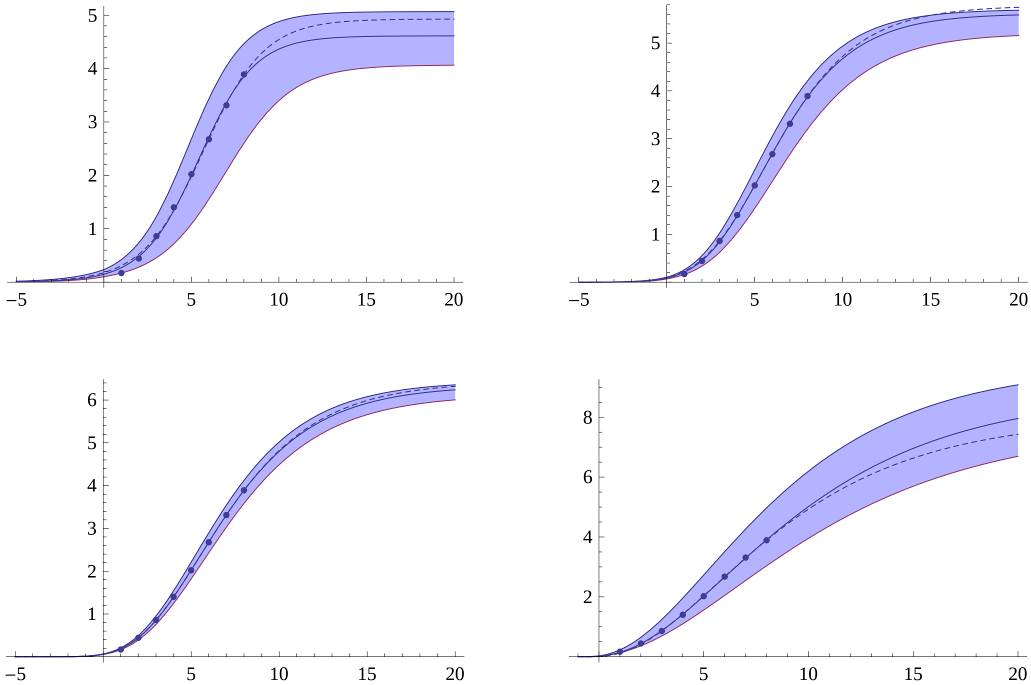

Figure 2: Area of all curves with parameters from confidence intervals based on data referring to broilers

Growth function	Confidence interval A	b	c
Logistic	$\langle 4.067, 5.159\rangle$	$\langle 20.593, 39.393\rangle$	$\langle 0.528, 0.725\rangle$
Gen. log.	$\langle 5.206, 6.027\rangle$	$\langle 1.537, 1.759\rangle$	$\langle 1.441, 1.691\rangle$
Gompertz	$\langle 1.813, 1.882\rangle$	$\langle 4.589, 4.772\rangle$	$\langle 0.272, 0.292\rangle$
Von Bertalanffy	$\langle 7.773, 9.895\rangle$	$\langle 0.854, 0.899\rangle$	$\langle 0.145, 0.181\rangle$

Table 5: Confidence intervals for optimal LS-parameters based on data from Example 1

Similar holds for measurement data from *Example 2*. In *Table 6* optimal LS-parameters are given in brackets below optimal LAD-parameters for every growth function. We also calculate corresponding characteristic points I, B, C.

For every optimal LS-parameter of all growth functions in *Table 7* asymptotic confidence intervals are given that are obtained by means of *Mathematica* module `NonlinearRegress`, and *Figure 3* shows the corresponding area of all curves whose parameters are placed in corresponding asymptotic confidence intervals.

Growth function	Parameters A	b	c	γ	Inflection point	$(t_B, f(t_B))$	$(t_C, f(t_C))$
Logistic	160.517	16.2013	0.021953	–			
	(163.143)	(17.1395)	(0.022077)	–	$(126.9, 80.3)$	$(66.9, 33.9)$	$(186.9, 126.6)$
Gen. log.	206.361	0.266801	0.175882	0.06463			
	(170.236)	(7.59008)	(0.0269173)	(0.70018)	$(124.7, 78.3)$	$(37.6, 17.2)$	$(211.9, 143.0)$
Gompertz	5.3532	3.76964	0.010699	–			
	(5.3139)	(3.8704)	(0.011277)	–	$(124.0, 77.7)$	$(34.1, 15.4)$	$(214.0, 144.2)$
Von Bertalanffy	239.972	0.805731	0.007629	–			
	(241.083)	(0.809021)	(0.007662)	–	$(115.7, 71.4)$	$(11.9, 4.3)$	$(219.5, 147.8)$

Table 6: LS and LAD- optimal parameters and characterictic points of the growth function based on data from Example 2

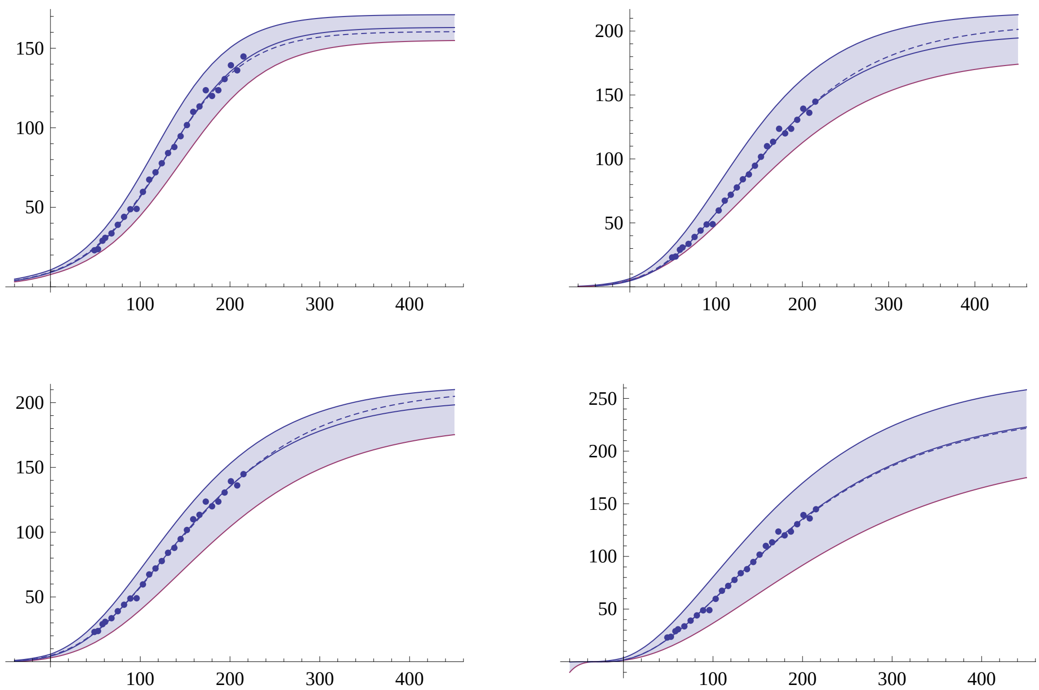

Figure 3: Area of all curves with parameters from confidence intervals based on data referring to pigs

Growth function	Confidence interval A	b	c
Logistic	$\langle 155.137, 171.150\rangle$	$\langle 15.093, 19.186\rangle$	$\langle 0.020, 0.024\rangle$
Gen. log.	$\langle 180.041, 216.692\rangle$	$\langle 0.255, 0.294\rangle$	$\langle 0.164, 0.207\rangle$
Gompertz	$\langle 5.215, 5.413\rangle$	$\langle 3.603, 4.138\rangle$	$\langle 0.010, 0.013\rangle$
Von Bertalanffy	$\langle 203.285, 278.882\rangle$	$\langle 0.763, 0.855\rangle$	$\langle 0.006, 0.009\rangle$

Table 7: Confidence intervals for optimal LS-parameters based on data from Example 2

5 Summary and conclusions

Animal growth is a result of many biological processes in which genotype determines their maximum expression, whereas environment determines the stage to which this genetic potential can be utilized. In practice, daily live weight gain, food conversion ratio, ultrasound backfat thickness and different slaughter traits as well as some meat quality traits make selection criteria that are usually applied for improving the growth characteristics in domestic animals. Various nonlinear growth functions described in the paper are used for the purpose of mathematical modeling of the animal weight growth. Most parameters in these models occur nonlinearly, and their estimation on the basis of measurement data is often a numerically very demanding and unstable process. In addition to that, among the data so-called "outliers" are present very often. Since LS-optimal parameters are heavily dependent on such data, in order to minorize a bad influence of outliers, the application of the LAD-approach is preferred. If the measurement data are contaminated only by normal distributed random errors, LAD-approximation may be used as an excellent initial approximation for searching for the best Least Squares (LS) approximation.

In most growth functions found in the literature, at least one parameter occurs linearly. In that case, we can apply a property of the LAD-principle by which there exists a best LAD-solution such that the graph of the corresponding growth function passes through at least one data point (see *Section 2.1*). In that way, the number of parameters of the minimizing functional might be reduced. This method is in *Section 3.1* illustrated by an example of the generalized logistic function.

The paper also considers one modification of the well-known log-linearization method, which can be successfully applied in many situations. Beside Theorem 1, a new, very successful method was developed, by means of which a stationary point of the minimizing functional can be determined by solving a sequence of simpler LAD-problems. Thereby, in practical applications it is shown that for the purpose of obtaining a satisfactory approximation of the parameters it suffices to conduct only a few iterations. This method is in *Section 3.2* illustrated by an example of the Gompertz function.

Some other mathematical model functions that occur more frequently in other scientific areas (e.g. economy, marketing, medicine, engineering, etc.) can be considered in a similar way.

The aforementioned two methods for estimation of LAD-optimal parameters in growth functions represent a theoretical contribution of this paper. The application of these methods to some most frequently used growth functions, as well as estimation of the saturation level of animal weight and definition of life cycle phases based on animal weight represent a possible contribution of this paper to the practice.

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