Levi subgroups of p-adic Spin(2n+1)

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Abstract

We explicitly describe Levi subgroups of odd spin groups over algebraic closure of a p-adic field.

1 Introduction

Let F be an algebraic closure of a p-adic field. For $n \in \mathbb{N}$, let $Spin(2n+1, F)$ be the split simply-connected algebraic group of type B_n. $Spin(2n+1, F)$ is a double covering, as algebraic groups, of the odd special orthogonal group $SO(2n+1, F)$. In the representation theory, it is very important to know what the Levi subgroups in considered group look like. In some other classical groups, such as already mentioned $SO(n, F)$, the Levi subgroups are isomorphic to a product of some general linear groups and another $SO(m, F)$, where $m \leq n$, i.e. product of some general linear groups and classical group of a smaller rank and of a same type. But, this is not the case for spin groups, which implies that some different techniques for investigating these groups have to be used. Examples of Levi subgroups of $Spin(5, F)$ can be found in [2], so we assume $n > 2$.

Here is an outline of the paper. Section 2 presents some preliminaries, mainly from [3] and [6]. In the third section, we have case-by-case consideration of Levi subgroups. The same method was used by Asgari in [1] to determine the Levi subgroups of a simply-connected group of type F_4.

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2 Preliminaries

Fix a maximal torus T of $Spin(2n+1, F)$ and a Borel subgroup B containing T. The based root system associated to $(Spin(2n+1, F), B, T)$,

$(X, \Sigma, X^\vee, \Sigma^\vee)$, is given by

$$X = \mathbb{Z}e_1 \oplus \mathbb{Z}e_2 \oplus \cdots \oplus \mathbb{Z}e_{n-1} \oplus \mathbb{Z}\frac{e_1 + \cdots + e_n}{2}$$
$$X^\vee = \mathbb{Z}(e_1^\vee - e_2^\vee) \oplus \mathbb{Z}(e_2^\vee - e_3^\vee) \oplus \cdots \oplus \mathbb{Z}(e_{n-1}^\vee - e_n^\vee) \oplus \mathbb{Z}2e_n^\vee$$

Let $\Sigma = \{\alpha_1, \alpha_2, \ldots, \alpha_n\}$ be a system of simple roots, where $\alpha_1 = e_1 - e_2$, $\alpha_2 = e_2 - e_3$, $\ldots$, $\alpha_{n-1} = e_{n-1} - e_n$, $\alpha_n = e_n$. We denote the associated coroots by $\Sigma^\vee = \{\alpha_1^\vee, \alpha_2^\vee, \ldots, \alpha_n^\vee\}$, where $\alpha_1^\vee = e_1^\vee - e_2^\vee$, $\alpha_2^\vee = e_2^\vee - e_3^\vee$, $\ldots$, $\alpha_{n-1}^\vee = e_{n-1}^\vee - e_n^\vee$, $\alpha_n^\vee = 2e_n^\vee$ (observe that $e_1, \ldots, e_n$ are chosen in the standard way, such that $\langle e_i, e_j^\vee \rangle = \delta_{i,j}$).

Every standard Levi subgroup corresponds to some subset θ of Σ. Subgroup corresponding to θ will be denoted by M_θ. Each M_θ is an almost direct product of a connected component of its center and its derived group. Connected component of the center of M_θ will be denoted by A_θ, while derived group of M_θ will be denoted by M'_θ. In other words,

$$M_\theta \simeq \frac{A_\theta \times M'_\theta}{A_\theta \cap M'_\theta}$$

Since $Spin(2n+1, F)$ is a simply-connected group, the derived group of each M_θ is also simply-connected, so it can be obtained directly from θ, i.e. from its root system. It is well - known that

$$A_\theta = (\bigcap_{\beta \in \theta} ker\beta)^0$$

so A_θ can also be obtained from the set of simple roots θ. After obtaining A_θ and M'_θ (which will be considered case-by-case, depending on the type of θ), we can construct their almost direct product to finally obtain M_θ.

The maximal torus of $Spin(2n+1, F)$ will be denoted by T. We have the next proposition ([1], Proposition 3.1.2 or [4], page 108), which holds for simply-connected groups:

Proposition 2.1 *Each $t \in T$ can be written uniquely as*

$$t = \prod_{i=1}^{n} \alpha_i^\vee(t_i), t_i \in F^*$$

Kernels of simple roots in Σ can now be described as follows:

Proposition 2.2 *Let $t \in ker\alpha_i$. Then*

$$\alpha_i(t) = \alpha_i(\prod_{j=1}^{n} \alpha_j^{\vee}(t_j)) = \prod_{j=1}^{n} t_j^{\langle \alpha_i, \alpha_j^{\vee} \rangle} = 1$$

This implies:

- *if $i = 1$, then $t_1^2 = t_2$*
- *if $2 \leq i \leq n-2$, then $t_i^2 = t_{i-1}t_{i+1}$*
- *if $i = n-1$, then $t_i^2 = t_{i-1}t_{i+1}^2$*
- *if $i = n$, then $t_i^2 = t_{i-1}$*

Let $z = \alpha_n^{\vee}(-1)$. From [1], Corollary 3.1.3, follows that the center of $Spin(2n+1, F)$ equals $\{1, z\} \simeq \mathbb{Z}_2$. From now on, z stands for the non-trivial element of the center of $Spin(2n+1, F)$, for some $n \geq 1$. We introduce the notion of the general spin groups, following Asgari [1]. These groups are defined in the following way:

$$GSpin(2n+1, F) = \frac{GL(1, F) \times Spin(2n+1, F)}{\{(1,1), (-1, z)\}}, n \geq 1,$$

$$GSpin(1, F) = GL(1, F).$$

The derived group of a general spin group is a spin group.

Advantage of general spin groups is that their Levi subgroups are isomorphic to a product of general linear groups and a general spin group of a smaller rank. This was proved in [1], using root datum of general spin groups. Another proof can be found in this manuscript.

3 LEVI SUBGROUPS

Let us fix some notation. Let $\theta \subset \Sigma$, $\theta \neq \emptyset$. Here and subsequently, we will write θ as a union of connected components of its Dyinkin diagram,

$$\theta = \theta_1 \cup \theta_2 \cup \cdots \cup \theta_k$$

where $\theta_i \cap \theta_j = \emptyset$ for $i \neq j$. We choose $\theta_1, \ldots, \theta_k$ in such a way that for $\alpha_{i_1} \in \theta_{j_1}$ and $\alpha_{i_2} \in \theta_{j_2}$, where $j_1 < j_2$, then $i_1 < i_2$. For $1 \leq i \leq k$, let

$n_i = |\theta_i|$. For shorten notation, we write l_i instead of $\sum_{1\leq j\leq i} n_j$. Now it follows that, if min_i is the minimal index such that $\alpha_{min_i} \in \theta_i$, then $\theta_i = \{\alpha_{min_i}, \alpha_{min_i+1}, \ldots, \alpha_{min_i+n_i-1}\}$. Also, if $\alpha_{i_1} \in \theta_{j_1}$ and $\alpha_{i_2} \in \theta_{j_2}$, where $j_1 < j_2$, then $i_2 - i_1 > 1$.

We write ζ_k for the $k-$th primitive root of identity in F^* and I_n for $n \times n$ identity matrix.

Now we begin case-by-case consideration:

(1) Suppose $\alpha_1 \in \theta$, $\alpha_{n-1}, \alpha_n \notin \theta$. Obviously, $\alpha_1 \in \theta_1$, $min_1 = 1$ and $min_k + n_k - 1 < n - 1$.

We obtain M'_θ using [4], Chapter 5., Theorem 1.33, Lemma 1.35 and Example 1.36 (pages 109-111), where derived group of M_θ is described. In this case, M'_θ is isomorphic to $SL(n_1+1, F) \times SL(n_2+1, F) \times \cdots \times SL(n_k+1, F)$.

Let $\lambda_1 = t_1$. From Proposition 2.2. we get $t_2 = \lambda_1^2$, $t_3 = \lambda_1^3$, $\ldots$, $t_{n_1} = \lambda_1^{n_1}$, $t_{n_1+1} = \lambda_1^{n_1+1}$. Next, put $\lambda_2 = t_{n_1+2}$, $\lambda_3 = t_{n_1+3}$, $\ldots$, $\lambda_{min_2-n_1} = t_{min_2}$. If $min_2 = n_1+2$, then let $\mu_1 = \lambda_1^{n_1+1}$; let $\mu_1 = \lambda_{min_2-n_1-1}$ otherwise.

From Proposition 2.2. again, we obtain

$t_{min_2+1} = t^2_{min_2} t^{-1}_{min_2-1} = \lambda^2_{min_2-n_1}\mu_1^{-1}$,

$t_{min_2+2} = t^2_{min_2+1} t^{-1}_{min_2} = \lambda^4_{min_2-n_1}\mu_1^{-2}\lambda^{-1}_{min_2-n_1} = \lambda^3_{min_2-n_1}\mu_1^{-2}$,

$t_{min_2+3} = t^2_{min_2+2} t^{-1}_{min_2+1} = \lambda^4_{min_2-n_1}\mu_1^{-3}$,

$\vdots$

$t_{min_2+n_2-1} = \lambda^{n_2}_{min_2-n_1}\mu_1^{-n_2+1}$,

$t_{min_2+n_2} = \lambda^{n_2+1}_{min_2-n_1}\mu_1^{-n_2}$.

This equations cover kernels of all the roots in θ_2, so for each root between θ_2 and θ_3 we put $\lambda_{min_2-n_1+1} = t_{min_2+n_2+1}$, $\lambda_{min_2-n_1+2} = t_{min_2+n_2+2}$, $\ldots$, $\lambda_{min_3-l_2} = t_{min_3}$. If $min_3 = min_2+n_2+1$, then let $\mu_2 = \lambda^{n_2+1}_{min_2-n_1}\mu_1^{-n_2}$; let $\mu_2 = \lambda_{min_3-l_2-1}$ otherwise. Repeating the procedure similar to that in the previous paragraph, we get

$t_{min_3+1} = t^2_{min_3} t^{-1}_{min_3-1} = \lambda^2_{min_3-l_2}\mu_2^{-1}$,

$\vdots$

$t_{min_3+n_3-1} = \lambda^{n_3}_{min_3-l_2}\mu_2^{-n_3+1}$,

$t_{min_3+n_3} = \lambda^{n_3+1}_{min_3-l_2}\mu_2^{-n_3}$.

We continue by repeating this process for all the remaining subsets

$\theta_4,\ldots,\theta_k$ of θ. At the end we get $t_{min_k+n_k-1} = \lambda^{n_k}_{min_k-l_{k-1}}\mu^{-n_k+1}_{k-1}$ and $t_{min_k+n_k} = \lambda^{n_k+1}_{min_k-l_{k-1}}\mu^{-n_k}_{k-1}$.

Since in this case $min_k + n_k < n$, we also have to put $\lambda_{min_k-l_{k-1}+1} = t_{min_k+n_k+1}$, $\ldots$, $\lambda_{n-l_k} = t_n$.

Finally, we have:

$$\begin{aligned} A_\theta &= \{\alpha^\vee_1(\lambda_1)\alpha^\vee_2(\lambda^2_1)\cdots\alpha^\vee_{n_1+1}(\lambda^{n_1+1}_1)\alpha^\vee_{n_1+2}(\lambda_2)\cdots\alpha^\vee_{min_2}(\lambda_{min_2-n_1})\cdot \\ &\quad \alpha^\vee_{min_2+1}(\lambda^2_{min_2-n_1}\mu^{-1}_1)\alpha^\vee_{min_2+2}(\lambda^3_{min_2-n_1}\mu^{-2}_1)\cdots \\ &\quad \alpha^\vee_{min_2+n_2}(\lambda^{n_2+1}_{min_2-n_1}\mu^{-n_2}_1)\alpha^\vee_{min_2+n_2+1}(\lambda_{min_2-n_1+1})\cdots\alpha^\vee_{min_3}(\lambda_{min_3-l_2})\cdot \\ &\quad \alpha^\vee_{min_3+1}(\lambda^2_{min_3-l_2}\mu^{-1}_2)\cdots\alpha^\vee_{min_3+n_3}(\lambda^{n_3+1}_{min_3-l_2}\mu^{-n_3}_2)\cdots \\ &\quad \alpha^\vee_{min_k+n_k}(\lambda^{n_k+1}_{min_k-l_{k-1}}\mu^{-n_k}_{k-1})\alpha^\vee_{min_k+n_k+1}(\lambda_{min_k-l_{k-1}+1})\cdots\alpha^\vee_n(\lambda_{n-l_k}) : \\ &\quad \lambda_1,\cdots,\lambda_{n-l_k}\in F^*\} \simeq (F^*)^{n-l_k} \end{aligned}$$

After identifying A_θ with $GL(1,F)^{n-l_k} \simeq (F^*)^{n-l_k}$, we fix (as in [4], Example 1.36) an identification of M'_θ with $SL(n_1+1,F)\times SL(n_2+1,F)\times\cdots\times SL(n_k+1,F)$ under which the element $\alpha^\vee_1(\lambda_1)\alpha^\vee_2(\lambda^2_1)\cdots\alpha^\vee_{n_1}(\lambda^{n_1}_1)$ goes to the diagonal element $diag(\lambda_1,\lambda_1,\ldots,\lambda_1,\lambda^{-n_1}_1)$ of $SL(n_1+1,F)$, $\alpha^\vee_{min_2}(\lambda_{min_2-n_1})\alpha^\vee_{min_2+1}(\lambda^2_{min_2-n_1}\mu^{-1}_1)\cdots\alpha^\vee_{min_2+n_2-1}(\lambda^{n_2}_{min_2-n_1}\mu^{-n_2+1}_1)$ to $diag(\lambda_{min_2-n_1},\ldots,\lambda_{min_2-n_1},\lambda^{-n_2}_{min_2-n_1})$ of $SL(n_2+1,F)$ and proceed in the same way for all connected components $\theta_3,\ldots,\theta_k$ (similar identifications are used in all cases). Using this identifications, we conclude that in $A_\theta\bigcap M'_\theta$ we have:
$\lambda^{n_1+1}_1 = 1, \lambda_2 = \lambda_3 = \cdots = \mu_1 = 1,$
$\lambda^{n_2+1}_{min_2-n_1} = 1, \lambda_{min_2-n_1+1} = \lambda_{min_2-n_1+2} = \cdots = \mu_2 = 1,$
$\lambda^{n_3+1}_{min_3-l_2} = 1,\ldots,\mu_{k-1} = 1, \lambda^{n_k+1}_{min_k-l_{k-1}} = 1,$
$\lambda_{min_k-l_{k-1}+1} = \cdots = \lambda_{n-l_k} = 1,$
therefore

$$\begin{aligned} A_\theta\cap M'_\theta &= \{\alpha^\vee_1(\lambda_1)\alpha^\vee_2(\lambda^2_1)\cdots\alpha^\vee_{n_1}(\lambda^{n_1}_1)\alpha^\vee_{min_2}(\lambda_{min_2-n_1})\cdots \\ &\quad \alpha^\vee_{min_2+n_2-1}(\lambda^{n_2}_{min_2-n_1})\cdots\alpha^\vee_{min_k+n_k}(\lambda^{n_k}_{min_k-l_{k-1}}) : \\ &\quad \lambda^{n_1+1}_1 = 1, \lambda^{n_2+1}_{min_2-n_1} = 1,\ldots,\lambda^{n_k+1}_{min_k-l_{k-1}} = 1\} \\ &\simeq \langle\zeta_{n_1+1}\rangle\times\langle\zeta_{n_2+1}\rangle\times\cdots\times\langle\zeta_{n_k+1}\rangle \end{aligned}$$

It follows immediately that

$$\begin{aligned} M_\theta &\simeq \frac{(F^*)^{n-l_k} \times SL(n_1+1,F) \times \cdots \times SL(n_k+1,F)}{\langle \zeta_{n_1+1}\rangle \times \cdots \times \langle \zeta_{n_k+1}\rangle} \\ &\simeq \frac{F^* \times SL(n_1+1,F)}{\langle \zeta_{n_1+1}\rangle} \times \cdots \times \frac{F^* \times SL(n_k+1,F)}{\langle \zeta_{n_k+1}\rangle} \times (F^*)^{n-l_k-k} \\ &\simeq GL(n_1+1,F) \times \cdots \times GL(n_k+1,F) \times GL(1,F)^{n-l_k-k} \end{aligned}$$

because the mapping $F^* \times SL(n,F) \to GL(n,F)$, $(x,S) \mapsto xI_n \cdot S$, is a surjective homomorphism whose kernel is isomorphic to $\langle \zeta_n \rangle$.

(2) Suppose $\alpha_1, \alpha_{n-1}, \alpha_n \notin \theta$. Of course, $min_k + n_k - 1 < n-1$. M'_θ is again isomorphic to $SL(n_1+1,F) \times SL(n_2+1,F) \times \cdots \times SL(n_k+1,F)$. We start with $\lambda_1 = t_1,\ \lambda_2 = t_2,\ \ldots, \lambda_{min_1} = t_{min_1}$. It follows $t_{min_1+1} = \lambda^2_{min_1}\lambda^{-1}_{min_1-1}, \ldots, t_{min_1+n_1-1} = \lambda^{n_1}_{min_1}\lambda^{-n_1+1}_{min_1-1}$ and $t_{min_1+n_1} = \lambda^{n_1+1}_{min_1}\lambda^{-n_1}_{min_1-1}$. We can now proceed analogously to the case **(1)**:

$$\begin{aligned} A_\theta &= \{\alpha_1^\vee(\lambda_1)\cdots\alpha^\vee_{min_1}(\lambda_{min_1})\alpha^\vee_{min_1+1}(\lambda^2_{min_1}\lambda^{-1}_{min_1-1})\cdots \\ &\quad \alpha^\vee_{min_1+n_1}(\lambda^{n_1+1}_{min_1}\lambda^{-n_1}_{min_1-1})\cdots\alpha^\vee_{min_k}(\lambda_{min_k-l_{k-1}})\cdots \\ &\quad \alpha^\vee_{min_k+n_k}(\lambda^{n_k+1}_{min_k-l_{k-1}}\mu^{-n_k}_{k-1})\alpha^\vee_{min_k+n_k+1}(\lambda_{min_k-l_{k-1}+1})\cdots \\ &\quad \alpha^\vee_n(\lambda_{n-l_k}) : \lambda_1, \cdots, \lambda_{n-l_k} \in F^*\} \\ &\simeq (F^*)^{n-l_k} \end{aligned}$$

In $A_\theta \cap M'_\theta$ we have:
$\lambda_1 = \cdots = \lambda_{min_1-1} = 1, \lambda^{n_1+1}_{min_1} = 1,$
$\lambda_{min_1+1} = \cdots = \lambda_{min_2-n_1-1} = \mu_1 = 1, \lambda^{n_2+1}_{min_2-n_1} = 1,$
$\vdots$
$\lambda_{min_{k-1}-l_{k-2}} = \cdots = \lambda_{min_k-l_{k-1}-1} = \mu_{k-1} = 1,$
$\lambda^{n_k+1}_{min_k-l_{k-1}} = 1, \lambda_{min_k-l_{k-1}+1} = \cdots = \lambda_{n-l_k} = 1.$

Therefore, $A_\theta \cap M'_\theta \simeq \langle\zeta_{n_1+1}\rangle \times \langle\zeta_{n_2+1}\rangle \times \cdots \times \langle\zeta_{n_k+1}\rangle$ and, again,

$$M_\theta \simeq GL(n_1+1,F) \times \cdots \times GL(n_k+1,F) \times GL(1,F)^{n-l_k-k}$$

(3) Suppose $\alpha_1, \alpha_{n-1}, \alpha_n \in \theta$. Obviously, $min_1 = 1$ and $min_k + n_k = n+1$.

M'_θ is isomorphic to $SL(n_1+1,F) \times SL(n_2+1,F) \times \cdots \times SL(n_{k-1}+1,F) \times Spin(2n_k+1,F)$.

On the set $\theta \setminus \theta_k = \theta_1 \cup \theta_2 \cup \cdots \cup \theta_{k-1}$ we apply the same analysis as in the case **(1)** and get
$\lambda_1 = t_1, \ldots, \lambda_1^{n_1+1} = t_{n_1+1}, \lambda_2 = t_{n_1+2},$
$\vdots$
$\lambda_{min_{k-1}-l_{k-2}} = t_{min_{k-1}},$
$\vdots$
$t_{min_{k-1}+n_{k-1}-1} = \lambda_{min_{k-1}-l_{k-2}}^{n_{k-1}} \mu_{k-2}^{-n_{k-1}+1},$
$t_{min_{k-1}+n_{k-1}} = \lambda_{min_{k-1}-l_{k-2}}^{n_{k-1}+1} \mu_{k-2}^{-n_{k-1}}.$

Next, put $\lambda_{min_{k-1}-l_{k-2}+1} = t_n$. From Proposition 2.2 applied to the set θ_k we obtain: $t_{n-1} = t_{n-2} = \cdots = t_{n-n_k} = \lambda^2_{min_{k-1}-l_{k-2}+1}$. We have two possibilities which are considered separately:

- $min_{k-1} + n_{k-1} = n - n_k$
  It follows directly that $min_{k-1} - l_{k-2} = n - l_k$ and $\lambda_{n-l_k}^{n_{k-1}+1} \mu_{k-2}^{-n_{k-1}} = \lambda^2_{n-l_k+1}$.
  So, $A_\theta \simeq (F^*)^{n-l_k}$.
  In $A_\theta \cap M'_\theta$ we have:
  $\lambda_1^{n_1+1} = 1, \lambda_2 = \lambda_3 = \cdots = \mu_1 = 1,$
  $\lambda_{min_2-n_1}^{n_2+1} = 1, \lambda_{min_2-n_1+1} = \lambda_{min_2-n_1+2} = \cdots = \mu_2 = 1,$
  $\vdots$
  $\lambda_{n-l_k}^{n_{k-1}+1} = 1 = \lambda^2_{n-l_k+1}.$
  that implies $A_\theta \cap M'_\theta \simeq \langle \zeta_{n_1+1} \rangle \times \langle \zeta_{n_2+1} \rangle \times \cdots \times \langle \zeta_{n_{k-2}+1} \rangle \times \langle \zeta_{2(n_{k-1}+1)} \rangle$ (this $2(n_{k-1}+1)$-th root of identity comes from the last equation). This gives,

  $$M_\theta \simeq GL(n_1+1, F) \times \cdots \times GL(n_{k-2}+1, F) \times GL(1, F)^{n-l_k-k} \times \frac{GL(1, F) \times SL(n_{k-1}+1, F) \times Spin(2n_k+1, F)}{B},$$

  where $B = \{(\zeta, \zeta^2 \cdot I_{n_{k-1}+1}, \zeta^{n_{k-1}+1}) : \zeta^{2(n_{k-1}+1)} = 1\}$. Observe that the set $\{\zeta^{n_{k-1}+1} : \zeta^{2(n_{k-1}+1)} = 1\}$ can be identified with $\{1, z\}$, the center of $Spin(2n_k+1, F)$.

- $min_{k-1} + n_{k-1} < n - n_k$
  We put $\lambda_{min_{k-1}-l_{k-2}+2} = t_{min_{k-1}+n_{k-1}+1}$, $\lambda_{min_{k-1}-l_{k-2}+3} = t_{min_{k-1}+n_{k-1}+2}$, $\ldots, \lambda_{n-l_k} = t_{n-n_k-1}$.

Again, $A_\theta \simeq (F^*)^{n-l_k}$, while in $A_\theta \cap M'_\theta$ we have
$\lambda_1^{n_1+1} = 1, \lambda_2 = \lambda_3 = \cdots = \mu_1 = 1,$
$\vdots$
$\lambda_{min_{k-1}-l_{k-2}}^{n_{k-1}+1} = 1,\ \mu_{k-2} = 1,$
$\lambda^2_{min_{k-1}-l_{k-2}+1} = 1,\ \lambda_{min_{k-1}-l_{k-2}+2} = \cdots = \lambda_{n-l_k} = 1$, that implies
$A_\theta \cap M'_\theta \simeq \langle\zeta_{n_1+1}\rangle \times \langle\zeta_{n_2+1}\rangle \times \cdots \times \langle\zeta_{n_{k-1}+1}\rangle \times \langle\zeta_2\rangle.$
Observe that $\langle\zeta_2\rangle \simeq \{(1,1),(-1,z)\}$. We thus get,

$$\begin{aligned} M_\theta &\simeq GL(n_1+1,F) \times \cdots \times GL(n_{k-1}+1,F) \times GL(1,F)^{n-l_k-k} \times \\ &\quad \frac{GL(1,F) \times Spin(2n_k+1,F)}{\langle\zeta_2\rangle} \\ &\simeq GL(n_1+1,F) \times \cdots \times GL(n_{k-1}+1,F) \times GL(1,F)^{n-l_k-k} \times \\ &\quad GSpin(2n_k+1,F) \end{aligned}$$

(4) Suppose $\alpha_1, \alpha_n \in \theta, \alpha_{n-1} \notin \theta$. Clearly, $min_1 = 1$, $\theta_k = \{\alpha_n\}$ and $n_k = 1$. M'_θ is isomorphic to $SL(n_1+1,F) \times SL(n_2+1,F) \times \cdots \times SL(n_{k-1}+1,F) \times Spin(3,F)$. This case can be handled in much the same way as the case **(3)**, so we only state final results.

- if $min_{k-1} + n_{k-1} = n-1$, then

$$\begin{aligned} M_\theta &\simeq GL(n_1+1,F) \times \cdots \times GL(n_{k-2}+1,F) \times GL(1,F)^{n-l_k-k} \times \\ &\quad \frac{GL(1,F) \times SL(n_{k-1}+1,F) \times Spin(3,F)}{B} \end{aligned}$$

  where $B = \{(\zeta, \zeta^2 \cdot I_{n_{k-1}+1}, \zeta^{n_{k-1}+1}) : \zeta^{2(n_{k-1}+1)} = 1\}$

- if $min_{k-1} + n_{k-1} < n-1$, then

$$\begin{aligned} M_\theta &\simeq GL(n_1+1,F) \times \cdots \times GL(n_{k-2}+1,F) \times GL(1,F)^{n-l_k-k} \times \\ &\quad GSpin(3,F) \end{aligned}$$

(5) Suppose $\alpha_1 \notin \theta, \alpha_{n-1}, \alpha_n \in \theta$. Obviously, $min_1 > 1$ and $min_k + n_k = n+1$. M'_θ is isomorphic to $SL(n_1+1,F) \times SL(n_2+1,F) \times \cdots \times SL(n_{k-1}+1,F) \times Spin(2n_k+1,F)$.

Let $\lambda_1 = t_n$. From Proposition 2.2 we conclude that $t_{n-1} = \cdots = t_{min_k} = t_{min_k-1} = \lambda_1^2$. Next, let $\lambda_2 = t_{min_k-2}, \ldots, \lambda_{min_k-min_{k-1}-n_{k-1}+1} = t_{min_{k-1}+n_{k-1}-1}$.

If $min_{k-1}+n_{k-1}=min_k-1$ then put $\mu_1=\lambda_1^2$ otherwise put $\mu_1=\lambda_{min_k-min_{k-1}-n_{k-1}}$. Using standard calculations, easily follows:
$t_{min_{k-1}+n_{k-1}-2}=\lambda^2_{min_k-min_{k-1}-n_{k-1}+1}\mu_1^{-1}$,
$t_{min_{k-1}+n_{k-1}-3}=\lambda^3_{min_k-min_{k-1}-n_{k-1}+1}\mu_1^{-2}$,
$\vdots$
$t_{min_{k-1}-1}=\lambda^{n_{k-1}+1}_{min_k-min_{k-1}-n_{k-1}+1}\mu_1^{-n_k}$.

In the next step, let $\lambda_{min_k-min_{k-1}-n_{k-1}+2}=t_{min_{k-1}-2}$, $\lambda_{min_k-min_{k-1}-n_{k-1}+3}=t_{min_{k-1}-3}$,..., $\lambda_{min_k-min_{k-2}-n_{k-1}-n_{k-2}+1}=t_{min_{k-2}+n_{k-2}-1}$.

If $min_{k-2}+n_{k-2}=min_{k-1}-1$ then put $\mu_2=\lambda^{n_{k-1}+1}_{min_k-min_{k-1}-n_{k-1}+1}\mu_1^{-n_k}$ otherwise put $\mu_2=\lambda_{min_k-min_{k-2}-n_{k-1}-n_{k-2}}$. The rest of this construction runs as before:
$t_{min_{k-2}+n_{k-2}-2}=\lambda^2_{min_k-min_{k-2}-n_{k-1}-n_{k-2}+1}\mu_2^{-1}$,
$\vdots$
$t_{min_{k-2}-1}=\lambda^{n_{k-2}+1}_{min_k-min_{k-2}-n_{k-1}-n_{k-2}+1}\mu_2^{-n_{k-1}}$,
$\vdots$
$t_{min_1-1}=\lambda^{n_1+1}_{min_k-min_1-l_{k-1}+1}\mu_{k-1}^{-n_1}$.
Also, we have to add $\lambda_{min_k-min_1-l_{k-1}+2}=t_{min_1-2}$, ..., $\lambda_{min_k-l_{k-1}-1}=t_1$. From $min_k+n_k=n+1$ we easily get that $min_k-l_{k-1}-1=n-l_k$.

$$\begin{aligned} A_\theta &= \{\alpha_1^\vee(\lambda_{n-l_k})\alpha_2^\vee(\lambda_{n-l_k-1})\cdots\alpha^\vee_{min_1-2}(\lambda_{min_k-min_1-l_{k-1}+2})\cdot \\ &\quad \alpha^\vee_{min_1-1}(\lambda^{n_1+1}_{min_k-min_1-l_k+n_k+1}\mu^{-n_1}_{k-1})\cdots\alpha^\vee_{min_k-1}(\lambda_1^2)\cdots\alpha_n^\vee(\lambda_1): \\ &\quad \lambda_1,\dots,\lambda_{n-l_k}\in F^*\}\simeq(F^*)^{n-l_k} \end{aligned}$$

In $A_\theta\cap M'_\theta$ we have:
$\lambda_1^2=1,\lambda_2=\cdots=\lambda_{min_k-min_{k-1}-n_{k-1}}=\mu_1=1,\lambda^{n_{k-1}+1}_{min_k-min_{k-1}-n_{k-1}+1}=1$,
$\vdots$
$\mu_{k-1}=1,\lambda^{n_1+1}_{min_k-min_1-l_{k-1}+1}=1,\lambda_{min_k-min_1-l_{k-1}+2}=\cdots=\lambda_{n-l_k}=1$, that implies
$A_\theta\cap M'_\theta\simeq\langle\zeta_{n_1+1}\rangle\times\langle\zeta_{n_2+1}\rangle\times\cdots\times\langle\zeta_{n_{k-2}+1}\rangle\times\langle\zeta_2\rangle$.

Finally,

$$\begin{aligned} M_\theta &\simeq GL(n_1+1,F)\times\cdots\times GL(n_{k-2}+1,F)\times GL(1,F)^{n-l_k-k}\times \\ &\frac{GL(1,F)\times Spin(2n_k+1,F)}{\langle\zeta_2\rangle} \\ &\simeq GL(n_1+1,F)\times\cdots\times GL(n_{k-2}+1,F)\times GL(1,F)^{n-l_k-k}\times \\ &GSpin(2n_k+1,F)\end{aligned}$$

Observe that, for $\theta=\Sigma\setminus\{\alpha_1\}$ we have $\theta=\theta_1$, $k=1$, $n_1=n-1$ and

$$M_{\Sigma\setminus\{\alpha_1\}}\simeq M_\theta = GSpin(2(n-1)+1,F)$$

which implies that $GSpin(2n-1,F)$ is the maximal Levi subgroup of $Spin(2n+1,F)$.

(6) Suppose $\alpha_1,\alpha_{n-1}\notin\theta,\alpha_n\in\theta$. Of course, $min_1>1$ and $n_k=1$. M'_θ is isomorphic to $SL(n_1+1,F)\times SL(n_2+1,F)\times\cdots\times SL(n_{k-1}+1,F)\times Spin(3,F)$. Analysis similar to that in the case **(5)** shows that:

$$\begin{aligned} M_\theta &\simeq GL(n_1+1,F)\times\cdots\times GL(n_{k-2}+1,F)\times GL(1,F)^{n-l_k-k}\times \\ &\frac{GL(1,F)\times Spin(3,F)}{\{1,z\}} \\ &\simeq GL(n_1+1,F)\times\cdots\times GL(n_{k-2}+1,F)\times GL(1,F)^{n-l_k-k}\times \\ &GSpin(3,F)\end{aligned}$$

(7) Suppose $\alpha_1,\alpha_{n-1}\in\theta,\alpha_n\notin\theta$. Clearly, $min_1=1$ and $min_k+n_k=n$. M'_θ is isomorphic to $SL(n_1+1,F)\times SL(n_2+1,F)\times\cdots\times SL(n_k+1,F)$.

Proceeding analogously to the case **(1)** we obtain:

$\lambda_1=t_1,t_2=\lambda_1^2$, $t_3=\lambda_1^3$, $\ldots$, $t_{n_1}=\lambda_1^{n_1}$, $t_{n_1+1}=\lambda_1^{n_1+1}$,
$\lambda_2=t_{n_1+2}$, $\lambda_3=t_{n_1+3}$, $\ldots$, $\lambda_{min_2-n_1}=t_{min_2}$,
$t_{min_2+1}=\lambda^2_{min_2-n_1}\mu_1^{-1}$, $\ldots$, $t_{min_2+n_2}=\lambda^{n_2+1}_{min_2-n_1}\mu_1^{-n_2}$,
$\vdots$
$t_{min_k+n_k-1}=\lambda^{n_k}_{min_k-l_{k-1}}\mu_{k-1}^{-n_k+1}$, $t_n^2=t^2_{min_k+n_k}=\lambda^{n_k+1}_{min_k-l_{k-1}}\mu_{k-1}^{-n_k}$.

Suppose $\theta=\Sigma\setminus\{\alpha_n\}$. Then $k=1$, $n_1=n-1$, $M'_\theta=SL(n,F)$ and $t_n^2=\lambda_1^n=t_1^n$.

If n is even, say $n=2m$, then

$A_\theta = \{\alpha_1^\vee(\lambda_1)\alpha_2^\vee(\lambda_1^2)\cdots\alpha_{n-1}^\vee(\lambda_1^{n-1})\alpha_n^\vee(\lambda_1^m) : \lambda_1 \in F^*\} \simeq F^*$.
Observe that t_k could not be equal $-\lambda_1^m$ in A_θ, because A_θ is a connected component of the center. In $A_\theta \cap M'_\theta$ we have $\lambda_1^m = 1$, so $A_\theta \cap M'_\theta \simeq \langle\zeta^m\rangle$, therefore

$$M_\theta \simeq \frac{GL(1,F)\times SL(n,F)}{\langle\zeta^m\rangle}$$

If n is odd, then $M_\theta \simeq GL(n,F)$, as Shahidi asserts in [5], Remark 2.2.

If θ has more then one component, then $t_n^2 = \lambda_{min_k - l_{k-1}}^{n_k+1}\mu_{k-1}^{-n_k}$.
Since n_k+1 and $-n_k$ are of different parities, if n_k is even or μ_{k-1} isn't equal to λ^m for some $\lambda \in F^*$ and m even, we can proceed in the same way as above and get

$$M_\theta \simeq GL(n_1+1,F)\times\cdots\times GL(n_k+1,F)\times GL(1,F)^{n-l_k-k}$$

Now we have to consider the situation when n_k is odd and $\mu_{k-1} = \lambda^m$, for $\lambda \in F^*$ and m even. If this is the case, then $\mu_{k-1} = \lambda_{min_{k-1}-l_{k-2}}^{n_{k-1}+1}\mu_{k-2}^{-n_{k-1}}$. Again, this implies that n_{k-1} is odd and $\mu_{k-2} = \lambda_{min_{k-2}-l_{k-3}}^{n_{k-2}+1}\mu_{k-3}^{-n_{k-2}}$. We continue in this fashion to obtain $\mu_2 = \lambda_{min_2-n_1}^{n_2+1}\mu_1^{-n_2}$, n_2 is odd, $\mu_1 = \lambda_1^{n_1+1}$ and n_1 is odd. We conclude that n_k is odd and $\mu_{k-1} = \lambda^m$, for $\lambda \in F^*$ and m even, only if n_i is odd for each $1 \le i \le k$ and $min_i + n_i = min_{i+1} - 1$ for each $1 \le i \le k-1$. Observe that this implies $min_k - l_{k-1} = k = n - l_k$. If this is the case, then

$$\begin{aligned} A_\theta \;=\; & \{\alpha_1^\vee(\lambda_1)\alpha_2^\vee(\lambda_1^2)\cdots\alpha_{n_1+1}^\vee(\lambda_1^{n_1+1})\alpha_{min_2}^\vee(\lambda_2)\cdot \\ & \alpha_{min_2+1}^\vee(\lambda_2^2\mu_1^{-1})\alpha_{min_2+2}^\vee(\lambda_2^3\mu_1^{-2})\cdots \\ & \alpha_{min_k}^\vee(\lambda_{n-l_k})\cdots\alpha_{n-1}^\vee(\lambda_{n-l_k}^{n_k}\mu_{k-1}^{-n_k+1})\alpha_n^\vee(\lambda_{n-l_k}^{\frac{n_k+1}{2}}\mu) : \\ & \lambda_1,\cdots,\lambda_{n-l_k} \in F^*, \mu^2 = \mu_{k-1}^{-n_k}\} \simeq (F^*)^{n-l_k} \end{aligned}$$

In $A_\theta \cap M'_\theta$ we have:
$\lambda_1^{n_1+1} = \lambda_2^{n_2+1} = \cdots = \lambda_{k-1}^{n_{k-1}+1} = \lambda_{n-l_k}^{\frac{n_k+1}{2}} = \mu_1 = \mu_2 = \cdots = \mu_{k-1} = 1$, we easily get that $\lambda_{n-l_k}^{n_k+1} = 1$, so $A_\theta \cap M'_\theta \simeq \langle\zeta_{n_1+1}\rangle \times \langle\zeta_{n_2+1}\rangle \times\cdots\times \langle\zeta_{n_k+1}\rangle$ and

$$M_\theta \simeq GL(n_1+1,F)\times\cdots\times GL(n_k+1,F)$$

(8) Suppose $\alpha_1, \alpha_n \notin \theta, \alpha_{n-1} \in \theta$. Clearly, $min_1 > 1$, $\theta \neq \Sigma \setminus \{\alpha_n\}$ and $min_k + n_k = n$. M'_θ is isomorphic to $SL(n_1+1,F)\times SL(n_2+1,F)\times$

$\cdots \times SL(n_k+1,F)$. By the same method as in the case **(7)**, we obtain

$$M_\theta \simeq GL(n_1+1,F) \times \cdots \times GL(n_k+1,F) \times GL(1,F)^{n-l_k-k}.$$

From given cases we deduce the following corollary:

Corollary 3.1 *The Levi subgroups of the general spin group $GSpin(2n+1,F)$ are isomorphic to $GL(n_1,F)\times GL(n_2,F)\times\cdots\times GL(n_k,F)\times GSpin(2m+1,F)$, $m \leq n$.*

Remark: Observe that $\frac{F^* \times SL(n,F)}{\langle \zeta_n \rangle}$ is not isomorphic to $GL(n,F)$ over p-adic field F which is not algebraically closed.

Let F_1 be a p-adic field. We denote algebraic closure of F_1 by $\overline{F}_1$.We have the next exact sequence:

$1 \to \{\pm 1\} \hookrightarrow Spin(2n+1, \overline{F}_1) \xrightarrow{f} SO(2n+1, \overline{F}_1) \to 1,$

where f is a central isogeny. F_1-rational points of $Spin(2n+1)$ may be obtained by using the following exact sequence:

$1 \to \{\pm 1\} \hookrightarrow Spin(2n+1, F_1) \xrightarrow{f} SO(2n+1, F_1) \xrightarrow{\delta} F_1^*/(F_1^*)^2$

(homomorphism δ is called the spinor norm)

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