

Irreducibility of the unitary principal series of p -adic $\widetilde{Sp}(n)$

Marcela Hanzer, Ivan Matic

July 18, 2009

Abstract

Let F be a p -adic field. We prove irreducibility of the unitary principal series of the group $\widetilde{Sp}(n)$ over F using theta correspondence.

1 Introduction

One of the basic problems when studying the induced representations of p -adic reductive groups is a problem of describing the constituents of the principal series representations. Although there were some earlier papers studying this problem, we note that the unitary principal series of classical groups (more generally, the representations parabolically induced from the irreducible discrete series representations) were studied in an uniform way by Goldberg in [4]. His results were based on the study of R -groups. We do not extend the theory of R -groups from the reductive group setting to the metaplectic case, but rather rely on the theta correspondence to carry over known results from the classical groups case to the metaplectic case.

In the second section, we recall the notions of the metaplectic groups $\widetilde{Sp}(n)$, as double-coverings of symplectic groups. We recall of the full lift of an irreducible representation of one group in a dual pair (with respect to the Weil representation of the ambient metaplectic group). We recall the notation for standard parabolic subgroups, parabolic induction and Jacquet modules. In the third section we prove the irreducibility of the unitary principal series representations of $\widetilde{Sp}(n)$ using induction over n .

The authors wish to thank Goran Muić for suggestion to study this problem.

2 Preliminaries

Let F be a non-archimedean local field of odd characteristic. Let $\widetilde{Sp}(n)$ be the unique non-trivial two-fold central extension of symplectic group $Sp(n, F)$, i.e., the following holds:

$$1 \rightarrow \mu_2 \rightarrow \widetilde{Sp}(n) \rightarrow Sp(n, F) \rightarrow 1,$$

where $\mu_2 = \{\pm 1\}$. The multiplication in $\widetilde{Sp}(2)$ (which is as a set given by $Sp(2, F) \times \mu_2$) is given by the Rao's cocycle ([12]).

Let V_0 be an anisotropic quadratic space over F of odd dimension. For more details about the invariants of this space, we refer the reader to [9], p. 75. In each step we add a hyperbolic plane and obtain an enlarged quadratic space, a tower of quadratic spaces and a tower of corresponding orthogonal groups. In the case when r hyperbolic planes are added to the anisotropic space, enlarged quadratic space will be denoted by V_r , while a corresponding orthogonal group will be denoted by $O(V_r)$. For us, it is enough to study the case $\dim V_0 = 1$.

We fix a non-trivial additive character ψ of F and let $\omega_{n,r}$ be the pullback of the Weil representation $\omega_{n(2r+1),\psi}$ of the group $Sp(n(2r+1))$, restricted to the dual pair $\widetilde{Sp}(n) \times O(V_r)$.

Let $GL(n, F)$ be a double cover of $GL(n, F)$, where the multiplication is given by $(g_1, \epsilon_1)(g_2, \epsilon_2) = (g_1g_2, \epsilon_1\epsilon_2(\det g_1, \det g_2)_F)$. Here $\epsilon_i \in \mu_2$, $i = 1, 2$ and $(\cdot, \cdot)_F$ denotes the Hilbert symbol of the field F , and this cocycle on $GL(n, F)$ is actually a restriction of the Rao's cocycle on $Sp(n, F)$ to $GL(n, F)$, if we view this group as the Siegel Levi subgroup of $Sp(n, F)$ ([8], p. 235).

From now on, we fix a character $\chi_{V,\psi}$ of $GL(n, F)$, which is given by $\chi_{V,\psi}(g, \epsilon) = \chi_V(\det g)\epsilon\gamma(\det g, \frac{1}{2}\psi)^{-1}$. Here γ denotes the Weil invariant, while χ_V is a character related to the quadratic form on $O(V_r)$ ([9], p. 37). We denote by $\alpha = \chi_{V,\psi}^2$. Observe that α is a quadratic character on $GL(n)$.

For an irreducible smooth representation π of the group $\widetilde{Sp}(n)$, $\Theta(\pi, r)$ denotes it's full lift on the r -th level of the orthogonal tower, i.e., the isotypic

component in the representation $\omega_{n,r}$, so that there is a surjection $\omega_{n,r} \rightarrow \pi \otimes \Theta(\pi, r)$. More generally, for an irreducible representation π_1 of the l -group G_1 and a smooth representation ζ of the group $G_1 \times G_2$ (where G_2 is again an l -group), $\Theta(\pi_1, \zeta)$ denotes the isotypic component of π_1 in ζ , so that there is a $G_1 \times G_2$ epimorphism $\zeta \rightarrow \pi_1 \otimes \Theta(\pi_1, \zeta)$ (observe that $\Theta(\pi_1, \zeta)$ is a smooth representation of G_2 -[10], Chapter 2, III.3 and III.4).

We study liftings of the genuine (i.e, which do not factor through μ_2) representations of the metaplectic group in the split orthogonal tower, $\dim V_0 = 1$, with a quadratic form $q_0(x) = x^2$, while the character $\chi_{V,\psi}$ is adjusted to form q_0 .

By ν we denote a character of $GL(k, F)$ defined by $|\det|_F$. Further, for an ordered partition $s = (n_1, n_2, \dots, n_j)$ of some $m \leq n$, we denote by P_s a standard parabolic subgroup of $Sp(n, F)$ (consisting of block upper-triangular matrices), whose Levi factor equals $GL(n_1) \times GL(n_2) \times \dots \times GL(n_j) \times Sp(n - |s|, F)$, where $|s| = m = \sum_{i=1}^j n_i$. Then the standard parabolic subgroup \widetilde{P}_s of $\widetilde{Sp}(n)$ is the preimage of P_s in $\widetilde{Sp}(n)$. We have the analogous notation for the Levi subgroups of the metaplectic groups, which are described in more detail in Section 2.2 of [6]. The standard parabolic subgroups (containing the upper triangular Borel subgroup) of $O(r)$ have the analogous description as the standard parabolic subgroups of $Sp(n, F)$. If \widetilde{P}_s is a standard parabolic subgroup of $\widetilde{Sp}(n)$ described above, or P_s a similar standard parabolic subgroup of $O(r)$, the normalized Jacquet module of a smooth representation π of $\widetilde{Sp}(n)$ (respectively, $O(r)$) with respect to \widetilde{P}_s (respectively, P_s) is denoted by $R_{\widetilde{P}_s}(\pi)$ (respectively, $R_{P_s}(\pi)$). We also use $R_{P_{min}}(\pi)$ to denote the Jacquet module of π with respect to the minimal standard parabolic subgroup. We denote by ω_0 a nontrivial character of the group μ_2 , viewed as a representation of $\widetilde{Sp}(0)$.

The following fact, which follows directly from [6], we use frequently while determining composition series of induced representations: for an irreducible genuine representation π of $GL(k, F)$ and an irreducible genuine representation σ of $\widetilde{Sp}(n)$ we have (in the appropriate Grothendieck group)

$$\pi \rtimes \sigma = \widetilde{\pi} \alpha \rtimes \sigma,$$

where $\pi \rtimes \sigma$ denotes the representation of the group $\widetilde{Sp}(n+k)$ parabolically induced from the representation $\pi \otimes \sigma$ of the maximal Levi subgroup $\widetilde{M}_{(k)}$. We follow here the usual notation for parabolic induction for classical groups,

adopted to the metaplectic case ([14],[6]). We also freely use Zelevinsky's notation for the parabolic induction for general linear groups ([15]). We note that every irreducible genuine representation of $\widetilde{Sp}(n)$ can be embedded in the representation parabolically induced from the genuine cuspidal representations of some Levi subgroup (Proposition 4.4 of [6]).

3 Irreducibility of the unitary principal series

Let $\zeta_1, \zeta_2, \dots, \zeta_n$ denote the unitary characters of F^\times . We prove irreducibility of the representation $\Pi := \chi_{V,\psi}\zeta_1 \times \chi_{V,\psi}\zeta_2 \times \dots \times \chi_{V,\psi}\zeta_n \rtimes \omega_0$, using the theta correspondence. Our theorem does not involve any assumptions about the unitary characters ζ_1, \dots, ζ_n or about the residual characteristic of the field F .

Theorem 3.1. *The representation Π is irreducible. There is an epimorphism*

$$\Theta(\Pi, n) \rightarrow \zeta_1 \times \dots \times \zeta_n \rtimes 1.$$

The rest of this section is devoted to the proof of the Theorem 3.1. The proof is by induction over n . The theorem for $n = 1$ is known (cf. [9], p. 89, [7]), while the case $n = 2$ is Proposition 3.5 of [5].

Suppose that theorem holds for all $k \leq n - 1$. We prove that theorem holds for $k = n$. We have divided the proof into a sequence of propositions and lemmas.

Proposition 3.2. *Let π_1 be an irreducible subrepresentation of Π . Then $\Theta(\pi_1, n)$ is different from zero.*

Proof. (of Proposition 3.2) According to the stable range condition (cf. [9], p. 48), $\Theta(\pi_1, 2n) \neq 0$. By [9], Theorem 7.1, every non-zero irreducible quotient τ of $\Theta(\pi_1, 2n)$ has a cuspidal support, which we denote by $[\tau]$, equal to $[\nu^{\frac{m}{2}-n-\frac{1}{2}}, \nu^{\frac{m}{2}-n-2}, \dots, \nu^{\frac{m_0}{2}-n}, \zeta_1, \dots, \zeta_n; 1]$, where $m = \dim V_{2n}$, which implies $[\tau] = [\nu^{n-\frac{1}{2}}, \dots, \nu^{\frac{1}{2}}, \zeta_1, \dots, \zeta_n; 1]$. If we denote by $R_{P_{min}}$ Jacquet module with respect to the minimal parabolic subgroup, then either $R_{P_{min}}(\tau) \geq \nu^{s_{j_1}} \otimes \zeta_{i_0}^{\epsilon_{i_0}} \otimes \dots \otimes 1$ or $R_{P_{min}} \geq \zeta_{i_0}^{\epsilon_{i_0}} \otimes \dots \otimes 1$ holds, where $s_{j_1} \in \{\pm\frac{1}{2}, \dots, \pm(n - \frac{1}{2})\}$ and $\epsilon_{i_0} \in \{\pm 1\}$. If we suppose that the first factor is unitary, Lemma 2.6 from [2] gives

$$0 \neq \text{Hom}(R_{P_{min}}(\tau), \zeta_{i_0}^{\epsilon_{i_0}} \otimes \dots \otimes 1) \cong \text{Hom}(\tau, \zeta_{i_0}^{\epsilon_{i_0}} \times \dots \rtimes 1)$$

so τ is a subrepresentation of $\zeta_{i_0}^{\epsilon_0} \times \cdots \rtimes 1$. Let $\nu^{s_{j_2}}$ denote the first non-unitary factor in $\zeta_{i_0}^{\epsilon_{i_0}} \times \cdots \times 1$. Since $\nu^{s_{j_2}} \times \zeta_j^{\epsilon_j}$ is irreducible for every j and ϵ_j , it follows that τ is also a subrepresentation of $\nu^{s_{j_2}} \times \zeta_{i_0}^{\epsilon_{i_0}} \times \cdots \rtimes 1$. This gives an irreducible subquotient τ' of $\zeta_{i_0}^{\epsilon_{i_0}} \times \cdots \rtimes 1$ such that τ is a subrepresentation of $\nu^{s_{j_2}} \rtimes \tau'$, which implies $R_{P_1}(\tau)(\nu^{s_{j_2}}) \neq 0$ and gives an epimorphism $R_{P_1}(\tau) \rightarrow \nu^{s_{j_2}} \otimes \tau'$, where $R_{P_1}(\tau)(\nu^{s_{j_2}})$ is the isotypic component of $R_{P_1}(\tau)$ along the generalized character $\nu^{s_{j_2}}$. This gives an epimorphism $\omega_{n,2n} \rightarrow \pi_1 \otimes \Theta(\pi_1, 2n) \rightarrow \pi_1 \otimes \tau$, which directly yield to the epimorphisms $R_{P_1}(\omega_{n,2n}) \rightarrow \pi \otimes R_{P_1}(\Theta(\pi_1, 2n)) \rightarrow \pi_1 \otimes \nu^{s_{j_2}} \otimes \tau'$.

According to [9] (we use notation from Proposition 3.3 of [7]), the Jacquet module $R_{P_1}(\omega_{n,2n})$ has the following filtration of the length 2:

$$I_{10} = \nu^{-(n-\frac{1}{2})} \otimes \omega_{n,2n-1} \text{ (a quotient),}$$

$$I_{11} = \text{Ind}_{GL(1,F) \times \widetilde{P_1} \times O(V_{2n-1})}^{M_1 \times Sp(n)} (\chi_{V,\psi} \Sigma'_1 \otimes \omega_{n-1,2n-1}) \text{ (a subrepresentation).}$$

Here Σ'_1 is a twist of the usual $GL(1, F) \times GL(1, F)$ representation on the Schwartz space $C_c^\infty(GL(1, F))$, as explained in ([9],[7]). Let us denote by T an epimorphism $R_{P_1}(\omega_{n,2n}) \rightarrow \pi_1 \otimes \nu^{s_{j_2}} \otimes \tau'$. Suppose $T|_{I_{11}} \neq 0$. Using the second Frobenius isomorphism, we obtain non-zero $GL(1, F)$ -homomorphism $\chi_{V,\psi} \Sigma'_1 \rightarrow \nu^{s_{j_2}}$. Since the isotypic component of $\Theta(\nu^{s_{j_2}}, \chi_{V,\psi} \Sigma'_1)$ equals $\chi_{V,\psi} \nu^{-s_{j_2}}$, we have a non-zero homomorphism $\nu^{s_{j_2}} \otimes \chi_{V,\psi} \nu^{-s_{j_2}} \otimes \omega_{n-1,2n-1} \rightarrow \nu^{s_{j_2}} \otimes \widetilde{R_{\widetilde{P_1}}(\widetilde{\pi}_1)} \otimes \tau'$. Immediately follows that the isotypic component

$\widetilde{R_{\widetilde{P_1}}(\widetilde{\pi}_1)}(\chi_{V,\psi} \nu^{-s_{j_2}})$ is different from zero, which contradicts our assumption that π_1 is the subrepresentation of Π . Obviously, this means $T|_{I_{11}} = 0$ so we can take $T : I_{10} \rightarrow \pi_1 \otimes \nu^{s_{j_2}} \otimes \tau'$ i.e., there is a homomorphism of $GL(1, F) \times O(V_{2n-1}) \times Sp(n)$ -modules $\nu^{-(n-\frac{1}{2})} \otimes \omega_{n,2n-1} \rightarrow \pi_1 \otimes \nu^{s_{j_2}} \otimes \tau'$. Directly follows that there is a non-zero homomorphism $\omega_{n,2n-1} \rightarrow \pi_1 \otimes \tau'$ which implies $\Theta(\pi_1, 2n-1) \neq 0$.

Now we proceed by induction, in an analogous manner.

Suppose that we have proved that $\Theta(\pi_1, 2n-i) \neq 0$ for $i = 0, 1, \dots, k, k \leq n-1$. Now we prove $\Theta(\pi_1, 2n-(k+1)) \neq 0$. Let τ_1 be an irreducible quotient of $\Theta(\pi_1, 2n-k)$. In a similar way as before, we can conclude that there are the epimorphisms $\omega_{n,2n-k} \rightarrow \pi_1 \otimes \Theta(\pi_1, 2n-k) \rightarrow \pi_1 \otimes \tau_1$, the cuspidal support of τ_1 equals $[\nu^{n-k-\frac{1}{2}}, \dots, \nu^{\frac{1}{2}}, \zeta_1, \dots, \zeta_n; 1]$ and there is $s_2 \in \{\pm\frac{1}{2}, \dots, \pm(n-k-\frac{1}{2})\}$ such that $\nu^{s_2} \otimes \cdots \otimes 1$ is a subquotient of $R_{P_{min}}(\tau_1)$. So, there is an irreducible representation τ'' of $O(V_{2n-k-1})$ such that τ_1 is a subrepresentation of $\nu^{s_2} \rtimes \tau''$, i.e., such that there is an epimorphism $R_{P_1}(\tau_1) \rightarrow \nu^{s_2} \otimes \tau''$. This also gives a non-zero epimorphism $T : R_{P_1}(\omega_{n,2n-k}) \rightarrow \pi_1 \otimes \nu^{s_2} \otimes \tau''$. From

the following filtration of the Jacquet module $R_{P_1}(\omega_{n,2n-k})$:

$$I_{10} = \nu^{-(n-k-\frac{1}{2})} \otimes \omega_{n,2n-k-1} \text{ (a quotient),}$$

$$I_{11} = \text{Ind}_{GL(1,F) \times \widetilde{P_1} \times O(V_{2n-k-1})}^{M_1 \times \widetilde{Sp(n)}} (\chi_{V,\psi} \Sigma'_1 \otimes \omega_{n-1,2n-k-1}) \text{ (a subrepresentation),}$$

in the same fashion as before we obtain $T|_{I_{11}} = 0$. So, we may consider $T : \nu^{-(n-k-\frac{1}{2})} \otimes \omega_{n,2n-k-1} \rightarrow \pi_1 \otimes \nu^{s_2} \otimes \tau''$ that implies $\Theta(\pi_1, 2n-(k+1)) \neq 0$. This gives $\Theta(\pi_1, n) \neq 0$, and this is precisely the assertion of the proposition. \square

Previous proposition implies that the cuspidal support of each irreducible quotient of $\Theta(\pi_1, n)$ equals $[\zeta_1, \dots, \zeta_n; 1]$. Observe that the representation $\zeta_1 \times \dots \times \zeta_n \rtimes 1$ of the group $O(V_n)$ is irreducible (because its restriction to the group $SO(V_n)$ is irreducible), so each irreducible subquotient of $\Theta(\pi_1, n)$ equals $\zeta_1 \times \dots \times \zeta_n \rtimes 1$.

Remark. We may conclude that all subrepresentations π_1 of Π appear as quotients of the representation $\Theta(\zeta_1 \times \dots \times \zeta_n \rtimes 1, n)$. If we suppose that the residual characteristic of F is different from 2, the Howe duality conjecture implies that all the subrepresentations of Π are mutually isomorphic, so $\Pi \cong \pi_1 \oplus \dots \oplus \pi_1$. Now the uniqueness of the Whittaker model for the principal series of $Sp(n)$ ([1] and [13]) gives $\Pi \cong \pi_1$ and Π is obviously irreducible.

Now we continue with the proof of Theorem 3.1 without assumptions on residual characteristic.

Proposition 3.3. *Let J_{11} denote the subrepresentation of $R_{\widetilde{P_1}}(\omega_{n,n})$ in Kudla's filtration. Then the following holds:*

$$\begin{aligned} \Theta(\chi_{V,\psi} \zeta_1 \otimes \chi_{V,\psi} \zeta_2 \times \dots \times \chi_{V,\psi} \zeta_n \rtimes \omega_0, R_{\widetilde{P_1}}(\omega_{n,n})) = \\ \Theta(\chi_{V,\psi} \zeta_1 \otimes \chi_{V,\psi} \zeta_2 \times \dots \times \chi_{V,\psi} \zeta_n \rtimes \omega_0, J_{11}). \end{aligned}$$

Proof. $R_{\widetilde{P_1}}(\omega_{n,n})$ has the following filtration:

$$J_{10} = \chi_{V,\psi} \nu^{\frac{1}{2}} \otimes \omega_{n-1,n} \text{ (a quotient),}$$

$$J_{11} = \text{Ind}_{GL(1,F) \times P_1 \times \widetilde{Sp(n-1)}}^{\widetilde{M_1} \times O(V_n)} (\chi_{V,\psi} \Sigma'_1 \otimes \omega_{n-1,n-1}) \text{ (a subrepresentation).}$$

Let T denote the mapping $T : \text{Hom}_{\widetilde{M_1}}(R_{\widetilde{P_1}}(\omega_{n,n}), \chi_{V,\psi} \zeta_1 \otimes \chi_{V,\psi} \zeta_2 \times \dots \times \chi_{V,\psi} \zeta_n \rtimes \omega_0) \rightarrow \text{Hom}_{\widetilde{M_1}}(J_{11}, \chi_{V,\psi} \zeta_1 \otimes \chi_{V,\psi} \zeta_2 \times \dots \times \chi_{V,\psi} \zeta_n \rtimes \omega_0)$ given by restriction. We claim that T is an isomorphism of the vector spaces.

To prove that T is an injection, we choose $f \in \text{Hom}_{\widetilde{M_1}}(R_{\widetilde{P_1}}(\omega_{n,n}), \chi_{V,\psi} \zeta_1 \otimes \chi_{V,\psi} \zeta_2 \times \dots \times \chi_{V,\psi} \zeta_n \rtimes \omega_0)$ such that $T(f) = f|_{J_{11}} = 0$. So, we may consider

$f : J_{10} \rightarrow \chi_{V,\psi}\zeta_1 \otimes \chi_{V,\psi}\zeta_2 \times \cdots \times \chi_{V,\psi}\zeta_n \rtimes \omega_0$. Since $J_{10} = \chi_{V,\psi}\nu^{\frac{1}{2}} \otimes \omega_{n-1,n}$ and, if we assume that $f \neq 0$, we have $\text{Im}f \cong \chi_{V,\psi}\zeta_1 \otimes W$, where $W \hookrightarrow \chi_{V,\psi}\zeta_2 \times \cdots \times \chi_{V,\psi}\zeta_n \rtimes \omega_0$ is a non-zero smooth $GL(n-1) \times O(V_n)$ -module ([10], Chapter 2, III.3). We obtain a contradiction, so we get $f = 0$.

Now we prove that T is a surjection. Let $f \in \text{Hom}(J_{11}, \chi_{V,\psi}\zeta_1 \otimes \chi_{V,\psi}\zeta_2 \times \cdots \times \chi_{V,\psi}\zeta_n \rtimes \omega_0)$. We have the following sequence of \widetilde{M}_1 -invariant spaces:

$$0 \subseteq U \subseteq J_{11} \subseteq R_{\widetilde{P}_1}(\omega_{n,n}),$$

where U denotes the subspace such that J_{11}/U is isomorphic $\chi_{V,\psi}\zeta_1 \otimes \chi_{V,\psi}\zeta_2 \times \cdots \rtimes \omega_0 \otimes \Theta(\chi_{V,\psi}\zeta_1 \otimes \chi_{V,\psi}\zeta_2 \times \cdots \rtimes \omega_0)$ as $\widetilde{M}_1 \times O(V_n)$ -modules.

Clearly, $(R_{\widetilde{P}_1}(\omega_{n,n})/U)/(J_{11}/U) \cong R_{\widetilde{P}_1}(\omega_{n,n})/J_{11} \cong J_{10}$. Observe that J_{11}/U and J_{10} have different central characters (as \widetilde{M}_1 -modules). So, we obtain the following short exact sequence:

$$0 \longrightarrow J_{11}/U \longrightarrow R_{\widetilde{P}_1}(\omega_{n,n})/U \longrightarrow J_{10} \longrightarrow 0,$$

and, using the standard arguments, we get that $R_{\widetilde{P}_1}(\omega_{n,n})/U$ is $\widetilde{GL}(1)$ -finite.

Now it can be easily seen that $R_{\widetilde{P}_1}(\omega_{n,n})/U$ is isomorphic to the direct sum of two submodules with the central characters equal to $\chi_{V,\psi}\zeta_1$ and $\chi_{V,\psi}\nu^{\frac{1}{2}}$. Since these submodules are also \widetilde{M}_1 -invariant, we obtain $R_{\widetilde{P}_1}(\omega_{n,n})/U \cong J_{10} \oplus J_{11}/U$. Since f factors through U , we can consider f as an operator on J_{11}/U . Now it is obvious that T is a surjection.

Using the relations between isotypic components and spaces of homomorphisms ([11], Lemma 1.1) we get

$$\Theta(\chi_{V,\psi}\zeta_1 \otimes \chi_{V,\psi}\zeta_2 \times \cdots \rtimes \omega_0, R_{\widetilde{P}_1}(\omega_{n,n})) \cong \Theta(\chi_{V,\psi}\zeta_1 \otimes \chi_{V,\psi}\zeta_2 \times \cdots \rtimes \omega_0, J_{11}),$$

if we proved that the representation $\Theta(\chi_{V,\psi}\zeta_1 \otimes \chi_{V,\psi}\zeta_2 \times \cdots \rtimes \omega_0, J_{11})$ is admissible. This claim follows from the following proposition. \square

Proposition 3.4. *The following holds:*

$$\Theta(\chi_{V,\psi}\zeta_1 \otimes \chi_{V,\psi}\zeta_2 \times \cdots \rtimes \omega_0, J_{11}) = \zeta_1 \times \zeta_2 \times \cdots \times \zeta_n \rtimes 1.$$

Proof. Observe that there exist epimorphisms $\chi_{V,\psi}\Sigma'_1 \rightarrow \chi_{V,\psi}\zeta_1 \otimes \zeta_1^{-1}$ and, by the inductive assumption, $\omega_{n-1,n-1} \rightarrow \chi_{V,\psi}\zeta_2 \times \cdots \rtimes \omega_0 \otimes \zeta_2 \times \cdots \times \zeta_n \rtimes 1$. Further, the inductive assumption of Theorem 3.1 implies irreducibility

of the representation $\chi_{V,\psi}\zeta_2 \times \cdots \times \chi_{V,\psi}\zeta_n \rtimes \omega_0$ and gives an epimorphism $\Theta(\chi_{V,\psi}\zeta_2 \times \cdots \times \chi_{V,\psi}\zeta_n \rtimes \omega_0, n-1) \rightarrow \zeta_2 \times \cdots \times \zeta_n \rtimes 1$. By inducing, we obtain an epimorphism $J_{11} \rightarrow \chi_{V,\psi}\zeta_1 \otimes \chi_{V,\psi}\zeta_2 \times \cdots \rtimes \omega_0 \otimes \zeta_1 \times \cdots \times \zeta_n \rtimes 1$, which gives the epimorphism $\Theta(\chi_{V,\psi}\zeta_1 \otimes \chi_{V,\psi}\zeta_2 \times \cdots \rtimes \omega_0, J_{11}) \rightarrow \zeta_1 \times \cdots \times \zeta_n \rtimes 1$.

We prove that there is also an epimorphism $\zeta_1 \times \cdots \times \zeta_n \rtimes 1 \rightarrow \Theta(\chi_{V,\psi}\zeta_1 \otimes \chi_{V,\psi}\zeta_2 \times \cdots \rtimes \omega_0)$. We use the second Frobenius isomorphism:

$$\begin{aligned} & \text{Hom}_{\widetilde{M}_1 \times O(V_{n-1})}(J_{11}, \chi_{V,\psi}\zeta_1 \otimes \chi_{V,\psi}\zeta_2 \times \cdots \rtimes \omega_0 \otimes \Theta(\chi_{V,\psi}\zeta_1 \otimes \cdots \rtimes \omega_0, J_{11})) \cong \\ & \text{Hom}_{\widetilde{M}_1 \times M_1}(\chi_{V,\psi}\Sigma'_1 \otimes \omega_{n-1,n-1}, \chi_{V,\psi}\zeta_1 \otimes \chi_{V,\psi}\zeta_2 \times \cdots \rtimes \omega_0 \otimes R_{\overline{P}_1}(\Theta(\chi_{V,\psi}\zeta_1 \otimes \cdots \rtimes \omega_0, J_{11}))). \end{aligned}$$

Generally, for some reductive group G' (and for metaplectic groups), it's parabolic subgroup P' with the Levi subgroup M' and opposite parabolic subgroup \overline{P}' , the second Frobenius isomorphism gives

$$\text{Hom}_{G'}(\text{Ind}_{M'}^{G'}(\pi), \Pi) \cong \text{Hom}_{M'}(\pi, R_{\overline{P}'}(\Pi)), \quad (1)$$

for some smooth representation π (resp., Π) of the group M' (resp., G'). We denote the space of the representation π by V_π .

Above isomorphism can be explicitly described in the following way: Let Ψ denote the embedding

$$\Psi : V_\pi \hookrightarrow R_{\overline{P}'}(\text{Ind}_{M'}(V_\pi)),$$

which corresponds to the open cell $P'\overline{P}'$ in G' ([3]). Now, for some $T \in \text{Hom}_{G'}(\text{Ind}_{M'}^{G'}(\pi), \Pi)$, compose Ψ with the corresponding mapping $T_{\overline{P}'} : R_{\overline{P}'}(\text{Ind}_{M'}^{G'}(\pi)) \rightarrow R_{\overline{P}'}(\Pi)$.

From now on, for every $f \in \text{Hom}_{\widetilde{M}_1 \times O(V_{n-1})}(J_{11}, \chi_{V,\psi}\zeta_1 \otimes \chi_{V,\psi}\zeta_2 \times \cdots \rtimes \omega_0 \otimes \Theta(\chi_{V,\psi}\zeta_1 \otimes \cdots \rtimes \omega_0, J_{11}))$, an element of $\text{Hom}_{\widetilde{M}_1 \times M_1}(\chi_{V,\psi}\Sigma'_1 \otimes \omega_{n-1,n-1}, \chi_{V,\psi}\zeta_1 \otimes \chi_{V,\psi}\zeta_2 \times \cdots \rtimes \omega_0 \otimes R_{\overline{P}_1}(\Theta(\chi_{V,\psi}\zeta_1 \otimes \cdots \rtimes \omega_0, J_{11})))$ obtained in the way described above is denoted by f_0 .

Let $\varphi \in \text{Hom}_{\widetilde{M}_1 \times O(V_{n-1})}(J_{11}, \chi_{V,\psi}\zeta_1 \otimes \chi_{V,\psi}\zeta_2 \times \cdots \rtimes \omega_0 \otimes \Theta(\chi_{V,\psi}\zeta_1 \otimes \cdots \rtimes \omega_0, J_{11}))$ denote the natural epimorphism. Clearly, $\text{Im}\varphi_0 \subseteq \chi_{V,\psi}\zeta_1 \otimes \chi_{V,\psi}\zeta_2 \times \cdots \rtimes \omega_0 \otimes R_{\overline{P}_1}(\Theta(\chi_{V,\psi}\zeta_1 \otimes \cdots \rtimes \omega_0, J_{11}))$. But, since $\text{Im}\varphi_0$ is an $\widetilde{M}_1 \times M_1$ -invariant subspace of the above representation given as a tensor product, where the representation of \widetilde{M}_1 is irreducible, follows that $\text{Im}\varphi_0 = \chi_{V,\psi}\zeta_1 \otimes \chi_{V,\psi}\zeta_2 \times \cdots \rtimes \omega_0 \otimes V'$, where V' is M_1 -invariant subspace of $\Theta(\chi_{V,\psi}\zeta_1 \otimes \chi_{V,\psi}\zeta_2 \times \cdots \rtimes \omega_0, J_{11})$. We can determine V' , since there must be a non-zero mapping $\chi_{V,\psi}\zeta_1 \otimes \Theta(\chi_{V,\psi}\zeta_1, \chi_{V,\psi}\Sigma'_1) \otimes \chi_{V,\psi}\zeta_2 \times \cdots \rtimes \omega_0 \otimes \Theta(\chi_{V,\psi}\zeta_2 \times \cdots \rtimes \omega_0, n-1) \rightarrow \chi_{V,\psi}\zeta_1 \otimes \chi_{V,\psi}\zeta_2 \times \cdots \rtimes \omega_0 \otimes V'$. However, since both of these

isotypic components are irreducible, we obtain $\text{Im}\varphi_0 = \chi_{V,\psi}\zeta_1 \otimes \chi_{V,\psi}\zeta_2 \times \cdots \times \omega_0 \otimes \zeta_1^{-1} \otimes \zeta_2 \times \cdots \times \zeta_n \rtimes 1$.

The mapping φ_0 can be written as $\varphi_0 = \varphi'' \circ \varphi'$, where φ' denotes the canonical epimorphism $\chi_{V,\psi}\Sigma'_1 \otimes \omega_{n-1,n-1} \rightarrow \chi_{V,\psi}\zeta_1 \otimes \zeta_1^{-1} \otimes \chi_{V,\psi}\zeta_2 \times \cdots \times \omega_0 \otimes \zeta_2 \times \cdots \times \zeta_n \rtimes 1$, while φ'' is the inclusion of the preceding representation in $\chi_{V,\psi}\zeta_1 \otimes \chi_{V,\psi}\zeta_2 \times \cdots \times \omega_0 \otimes R_{\bar{P}_1}(\Theta(\chi_{V,\psi}\zeta_1 \otimes \chi_{V,\psi}\zeta_2 \times \cdots \times \omega_0, J_{11}))$. Now we are able to construct the mapping $\text{Ind}(\varphi') : J_{11} \rightarrow \chi_{V,\psi}\zeta_1 \otimes \chi_{V,\psi}\zeta_2 \times \cdots \times \omega_0 \otimes \zeta_1^{-1} \times \zeta_2 \times \cdots \times \zeta_n \rtimes 1$. Applying the second Frobenius isomorphism to φ'' , we obtain $\varphi_1 \in \text{Hom}_{\widetilde{M}_1 \times P_1}^{\widetilde{M}_1 \times O(V_n)}(\chi_{V,\psi}\zeta_1 \otimes \chi_{V,\psi}\zeta_2 \times \cdots \times \omega_0 \otimes \zeta_1^{-1} \times \zeta_2 \times \cdots \times \zeta_n \rtimes 1, \chi_{V,\psi}\zeta_1 \otimes \chi_{V,\psi}\zeta_2 \times \cdots \times \omega_0 \otimes \Theta(\chi_{V,\psi}\zeta_1 \otimes \chi_{V,\psi}\zeta_2 \times \cdots \times \omega_0, J_{11}))$ such that $(\varphi_1)_0 = \varphi''$.

We claim that $(\varphi_1 \circ \text{Ind}(\varphi'))_0$ equals φ_0 . To prove this, it is enough to prove $(\varphi_1 \circ \text{Ind}(\varphi'))_0 = (\varphi_1)_0 \circ \varphi'$, which can easily be obtained from (1) (the details are left to the reader). This forces $\varphi_1 \circ \text{Ind}(\varphi') = \varphi$.

So, the image of φ is a quotient of $\text{Ind}(\varphi')(J_{11}) = \chi_{V,\psi}\zeta_1 \otimes \chi_{V,\psi}\zeta_2 \times \cdots \times \omega_0 \otimes \zeta_1^{-1} \times \zeta_2 \times \cdots \times \zeta_n \rtimes 1$ (we are using the fact $\text{Im}\varphi_0 \cong \text{Im}\varphi'$ here). But, since $\text{Im}\varphi = \chi_{V,\psi}\zeta_1 \otimes \chi_{V,\psi}\zeta_2 \times \cdots \times \omega_0 \otimes \Theta(\chi_{V,\psi}\zeta_1 \otimes \cdots \times \omega_0, J_{11})$, this implies that there is an epimorphism $\zeta_1^{-1} \times \zeta_2 \times \cdots \times \zeta_n \rtimes 1 \cong \zeta_1 \times \zeta_2 \times \cdots \times \zeta_n \rtimes 1 \rightarrow \Theta(\chi_{V,\psi}\zeta_1 \otimes \cdots \times \omega_0, J_{11})$. This ends the proof of this proposition. \square

For the proof of the Theorem 3.1 we need the following lemma:

Lemma 3.5. *There is an epimorphism*

$$\Theta(\zeta_1 \times \cdots \times \zeta_n \rtimes 1, n) \rightarrow \chi_{V,\psi}\zeta_1 \times \chi_{V,\psi}\zeta_2 \times \cdots \times \chi_{V,\psi}\zeta_n \rtimes \omega_0.$$

Proof. There is an isomorphism of the vector spaces (moreover, of $\widetilde{Sp}(n)$ -modules)

$$\text{Hom}_{O(V_n)}(\omega_{n,n}, \zeta_1^{-1} \times \cdots \times \zeta_n \rtimes 1) \cong \text{Hom}_{M_1}(R_{P_1}(\omega_{n,n}), \zeta_1^{-1} \otimes \zeta_2 \times \cdots \times \zeta_n \rtimes 1).$$

From this isomorphism we obtain $\Theta(\zeta_1^{-1} \times \cdots \times \zeta_n \rtimes 1, n) \cong \Theta(\zeta_1^{-1} \otimes \zeta_2 \times \cdots \times \zeta_n \rtimes 1, R_{P_1}(\omega_{n,n}))$. As in the Proposition 3.3, we can prove $\Theta(\zeta_1^{-1} \otimes \zeta_2 \times \cdots \times \zeta_n \rtimes 1, R_{P_1}(\omega_{n,n})) \cong \Theta(\zeta_1^{-1} \otimes \zeta_2 \times \cdots \times \zeta_n \rtimes 1, I_{11})$, where I_{11} is a subrepresentation in Kudla's filtration of $R_{P_1}(\omega_{n,n})$ ($I_{11} = \text{Ind}_{\widetilde{P}_1 \times GL(1) \times O(V_{n-1})}^{M_1 \times \widetilde{Sp}(n)}(\chi_{V,\psi}\Sigma'_1 \otimes \omega_{n-1,n-1})$).

Using the inductive assumption of the Theorem 3.1, we obtain the epimorphisms $\chi_{V,\psi}\Sigma'_1 \rightarrow \chi_{V,\psi}\zeta_1 \otimes \zeta_1^{-1}$ and $\omega_{n-1,n-1} \rightarrow \zeta_2 \times \cdots \times \zeta_n \rtimes 1 \otimes \chi_{V,\psi}\zeta_2 \times \cdots \times \chi_{V,\psi}\zeta_n \rtimes \omega_0$, which give the epimorphism $I_{11} \rightarrow \zeta_1^{-1} \otimes \zeta_2 \times \cdots \times \zeta_n \rtimes 1 \otimes \chi_{V,\psi}\zeta_1 \times \chi_{V,\psi}\zeta_2 \times \cdots \times \chi_{V,\psi}\zeta_n \rtimes \omega_0$. Clearly, this gives us the epimorphism $\Theta(\zeta_1^{-1} \otimes \zeta_2 \times \cdots \times \zeta_n \rtimes 1, I_{11}) \rightarrow \chi_{V,\psi}\zeta_1 \times \chi_{V,\psi}\zeta_2 \times \cdots \times \chi_{V,\psi}\zeta_n \rtimes \omega_0$ and finally the desired epimorphism $\Theta(\zeta_1 \times \zeta_2 \times \cdots \times \zeta_n \rtimes 1, n) \rightarrow \chi_{V,\psi}\zeta_1 \times \chi_{V,\psi}\zeta_2 \times \cdots \times \chi_{V,\psi}\zeta_n \rtimes \omega_0$. \square

Now we are able to finish the proof of the Theorem 3.1.

Suppose that the representation Π reduces and let $\Pi = \pi_1 \oplus \cdots \oplus \pi_k$, $k \geq 2$. Then we also have $R_{\bar{P}_1}(\Pi) = R_{\bar{P}_1}(\pi_1) \oplus \cdots \oplus R_{\bar{P}_1}(\pi_k)$. According to the previous lemma, there is a surjection $\omega_{n,n} \rightarrow \zeta_1 \times \cdots \times \zeta_n \rtimes 1 \otimes \chi_{V,\psi}\zeta_1 \times \cdots \times \chi_{V,\psi}\zeta_n \rtimes \omega_0$ which gives a surjection $R_{\bar{P}_1}(\omega_{n,n}) \rightarrow \zeta_1 \times \cdots \times \zeta_n \rtimes 1 \otimes (R_{\bar{P}_1}(\pi_1) \oplus \cdots \oplus R_{\bar{P}_1}(\pi_k))$. For each $\pi_i \hookrightarrow \Pi$ there is an epimorphism $R_{\bar{P}_1}(\pi_i) \rightarrow \chi_{V,\psi}\zeta_1 \otimes \chi_{V,\psi}\zeta_2 \times \cdots \times \chi_{V,\psi}\zeta_n \rtimes \omega_0$. This gives us an epimorphism $R_{\bar{P}_1}(\omega_{n,n}) \rightarrow \zeta_1 \times \cdots \times \zeta_n \rtimes 1 \otimes (\chi_{V,\psi}\zeta_1 \otimes \chi_{V,\psi}\zeta_2 \times \cdots \times \chi_{V,\psi}\zeta_n \rtimes \omega_0 \oplus \cdots \oplus \chi_{V,\psi}\zeta_1 \otimes \chi_{V,\psi}\zeta_2 \times \cdots \times \chi_{V,\psi}\zeta_n \rtimes \omega_0) \cong \chi_{V,\psi}\zeta_1 \otimes \chi_{V,\psi}\zeta_2 \times \cdots \times \chi_{V,\psi}\zeta_n \rtimes \omega_0 \otimes (\zeta_1 \times \cdots \times \zeta_n \rtimes 1 \oplus \cdots \oplus \zeta_1 \times \cdots \times \zeta_n \rtimes 1)$, where the second factors have k summands each. Because of $k \geq 2$, we get a contradiction with the Propositions 3.3 and 3.4, which proves Theorem 3.1.

References

- [1] W. D. BANKS, *Heredity of Whittaker models on the metaplectic group*, Pacific J. Math., 185 (1998), pp. 89–96.
- [2] J. BERNSTEIN, *Draft of: Representations of p -adic groups.*, Lectures at Harvard University, 1992, written by Karl E. Rumelhart.
- [3] ———, *Second adjointness for representations of p -adic reductive groups*, preprint.
- [4] D. GOLDBERG, *Reducibility of induced representations for $Sp(2n)$ and $SO(n)$* , Amer. J. Math., 116 (1994), pp. 1101–1151.
- [5] M. HANZER AND I. MATIĆ, *Unitary dual of p -adic $\widetilde{Sp}(2)$* , preprint.

- [6] M. HANZER AND G. MUIĆ, *Parabolic induction and Jacquet functors for metaplectic groups*, J. Algebra (to appear).
- [7] ———, *Rank one reducibility for metaplectic groups via theta correspondence*, preprint.
- [8] S. S. KUDLA, *On the local theta-correspondence*, Invent. Math., 83 (1986), pp. 229–255.
- [9] ———, *Notes on the local theta correspondence (lectures at the European School in Group Theory)*, (1996).
- [10] C. MØGLIN, M.-F. VIGNÉRAS, AND J.-L. WALDSPURGER, *Correspondances de Howe sur un corps p -adique*, vol. 1291 of Lecture Notes in Mathematics, Springer-Verlag, Berlin, 1987.
- [11] G. MUIĆ, *On the structure of the full lift for the howe correspondence of $(Sp(n), O(V))$ for rank-one reducibilities*, Canad. Math. Bull., 49 (2006), pp. 578–591.
- [12] R. RANGA RAO, *On some explicit formulas in the theory of Weil representation*, Pacific J. Math., 157 (1993), pp. 335–371.
- [13] D. SZPRUCH, *Uniqueness of Whittaker model for the metaplectic group*, Pacific J. Math., 232 (2007), pp. 453–469.
- [14] M. TADIĆ, *Representations of p -adic symplectic groups*, Compositio Math., 90 (1994), pp. 123–181.
- [15] A. V. ZELEVINSKY, *Induced representations of reductive p -adic groups. II. On irreducible representations of $GL(n)$* , Ann. Sci. École Norm. Sup. (4), 13 (1980), pp. 165–210.

Marcela Hanzer
 Department of Mathematics, University of Zagreb
 Zagreb, Croatia
 E-mail: hanmar@math.hr

Ivan Matic
 Department of Mathematics, University of Osijek
 Trg Ljudevita Gaja 6, Osijek, Croatia
 E-mail: imatic@mathos.hr