The conservation relation for discrete series representations of metaplectic groups

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Abstract

Let F denote a non-archimedean local field of characteristic zero with odd residual characteristic and let $\widetilde{Sp(n)}$ denote the rank n metaplectic group over F. If $r^{\pm}(\sigma)$ denotes the first occurrence index of the irreducible genuine representation σ of $\widetilde{Sp(n)}$ in the theta correspondence for the dual pair $(\widetilde{Sp(n)}, O(V^{\pm}))$, the conservation relation, conjectured by Kudla and Rallis, states that $r^{+}(\sigma) + r^{-}(\sigma) = 2n$. A purpose of this paper is to prove this conjecture for discrete series which appear as subquotients of generalized principle series where the representation on the metaplectic part is strongly positive. Assuming the basic assumption, we also prove the conservation relation for general discrete series of metaplectic groups by explicitly determining the first occurrence indices.

1 Introduction

This paper is concerned with the determination of the first occurrence indices for certain classes of irreducible genuine representations of metaplectic groups in the local theta correspondence. It presents a continuation of our previous work [10, 9, 11] on the strongly positive representations of metaplectic groups, which serve as a cornerstone in the known constructions of discrete series. Our approach was motivated by the paper of Muić ([16]), who used an inductive procedure to determine the first occurrence indices for discrete series of symplectic groups, i.e., for the reductive dual pair $(Sp(n), O(V^{\pm}))$.

His work is based on the classification of discrete series for classical p-adic groups given by the work of Mœglin and Tadić ([12, 13]).

Although very elegant, the Mœglin-Tadić classification relies on a certain conjecture, called the basic assumption, which will be recalled in Section 6. It is important to note that Arthur has recently announced a proof of his conjectures about the stable transfer coming from the twisted endoscopy, which should imply the basic assumption. We have recently classified the strongly positive discrete series of metaplectic groups in a purely algebraic way and this classification, given in [9], is also valid in a classical group case. For that reason we start our inspection of the first occurrence indices building inductively from the strongly positive discrete series.

To a fixed quadratic character χ of $F^\times$, where F denotes a non-archimedean local field, one can attach two odd orthogonal towers, obtained by adding hyperbolic planes to an anisotropic quadratic space V_0 over F of odd dimension. These towers are commonly denoted by $+$–tower (if the space V_0 is 1-dimensional) and $-$–tower (if the space V_0 is 3-dimensional), while the corresponding orthogonal groups of the spaces obtained by adding r hyperbolic planes to the space V_0 are denoted by $O(V_r^+)$ and $O(V_r^-)$.

The first occurrence index $r^\pm(\sigma)$ is the smallest non-negative integer r for which the irreducible genuine representation σ of the metaplectic group $\widetilde{Sp(n)}$ occurs in the local theta correspondence for the dual pair $(\widetilde{Sp(n)}, O(V_r^\pm))$.

In their paper [8], Kudla and Rallis posed a very interesting conjecture which states that the equality $r^+(\sigma) + r^-(\sigma) = 2n$ holds for an irreducible admissible genuine representation of $\widetilde{Sp(n)}$. They proved this conjecture, known as the conservation relation, in a good deal of cases, in particular, for supercuspidal representations. We prove their conjecture for many other representations of metaplectic groups over non-archimedean local fields of characteristic zero with odd residual characteristic, explicitly determining the first occurrence indices.

Recently, B. Sun and C.B. Zhu have announced a proof of the conservation relation in full generality [20]. Their approach uses entirely different methods.

The first purpose of this paper is to sharpen the results from [11], where the lower first occurrence indices of strongly positive discrete series have been determined in an essentially combinatorial way. Using the inductive method introduced by Muić, we obtain explicit expressions for the first occurrence indices of strongly positive representations in terms of those of their partial cuspidal supports. The conservation relation for strongly positive discrete series is then a direct consequence of the results of Kudla and Rallis.

Secondly, we study the first occurrence indices of irreducible subquotients of the induced representation of the form

$$\delta([\nu^a \chi_{V,\psi}\rho, \chi_{V,\psi}\nu^b\rho]) \rtimes \sigma_{sp}$$

where the representation σ_{sp} is strongly positive (the notation is explained in more detail in Section 2). Non-strongly positive discrete series subquotients of the aforementioned representation are of special importance because they present the first inductive step in the construction of general discrete series representations. Composition series of induced representations of this type have been described in [15], but since we, actually, only need appropriate embeddings of irreducible subquotients, we obtain such embeddings independently. For certain classes of representations this can be achieved using the arguments based on the Jacquet modules method ([21]) and Bernstein-Zelevinsky theory ([2, 23]), which have been extended to the metaplectic case in [4].

However, this techniques happen to be insufficient in many cases of discrete series. Appropriate embeddings of such representations have been constructed in the classical group case in [12] and rely on the theory of L-functions, which we do not have at the disposal in its full generality. Instead of extending this theory to the metaplectic case, we use recent results of Gan and Savin ([3]) which provide a correspondence between discrete series of metaplectic groups and those of the orthogonal ones, given by the theta correspondence. This puts us in position to use ideas from [12] to first obtain the appropriate embeddings on the classical group side and then transfer them to the metaplectic group side using simple inductive arguments. Combining the embeddings of discrete series of metaplectic groups obtained in that way with the description of theta lifts of strongly positive discrete series, we determine the first occurrence indices using case-by-case consideration together with the same inductive method as for the strongly positive representations.

The full strength of this approach is presented in the determination of the first occurrence indices of the general discrete series of metaplectic groups, under the natural hypotheses. We start with the Mœglin-Tadić classification for the orthogonal groups and again use the correspondence proved by Gan and Savin to get appropriate embeddings of discrete series of metaplectic groups. This reduces the determination of the first occurrence indices of general discrete series to rather easy inductive argument which relates them to the already obtained first occurrence indices of discrete series subquotients of generalized principle series. Examining several possibilities we directly verify the conservation relation for discrete series. We choose to start our inductive determination of the first occurrence indices from discrete series subquotients of generalized principle series rather then from the strongly positive discrete series because the first case consists of several subcases and each of them has to be treated in a different way. On the other hand, after such inductive bases has been settled the inductive step can be handled in a fairly uniform way.

Since we do not use the Mœglin-Tadić classification in its full generality, the precise definition of Jordan triples is not recalled in the paper. We rather state only the main properties of embeddings of discrete series given in this classification, that turns out to be more convenient for our purposes.

However, our method of determining the first occurrence indices does not work for all irreducible subquotients of generalized principle series, since in some cases we are not able to use our basic principle for pushing down the lifts, given by Lemma 3.4.

Now we briefly describe the contents of the paper. The next section reviews notation and some background results. In the third section we recall some of the standard facts on the theta correspondence and state main techniques for determining the first occurrence indices. Section 4 studies the first occurrence indices of strongly positive discrete series, while the more difficult case of discrete series which are not strongly positive and appear as subquotients of the generalized discrete series (where the representation on the metaplectic part is strongly positive) is discussed in Section 5. The purpose of the sixth section is to prove the conservation relation for general discrete series, starting from the Mœglin-Tadić classification and following the ideas introduced in the previous two sections.

The author wishes to express his thanks to Goran Muić for drawing the author's attention to the results of Gan and Savin ([3]) and to the recent work of Arthur ([1]). The author's thanks also go to Marko Tadić for useful discussions on the discrete series of the odd-orthogonal groups. The author would also like to thank the referee for helping to greatly improve the style of the presentation. The author lectured on results of the present paper in the workshop "Automorphic forms: Arithmetic and Geometry" held at the Erwin Schrödinger Institute in Vienna in January and February of 2012. The

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2 Preliminaries

Through this paper F denotes a non-archimedean local field of characteristic zero with odd residual characteristic.

First we discuss the groups that are the object of our study.

Let $\widetilde{Sp(n)}$ be the metaplectic group of rank n, the unique non-trivial two-fold central extension of symplectic group $Sp(n, F)$. In other words, the following holds:

$$1 \to \mu_2 \to \widetilde{Sp(n)} \to Sp(n, F) \to 1,$$

where $\mu_2 = \{1, -1\}$. The multiplication in $\widetilde{Sp(n)}$ is given by the Rao's cocycle ([18]). The more thorough description of the structural theory of metaplectic groups can be found in [4], [7] and [18].

In this paper we are interested only in genuine representations of $\widetilde{Sp(n)}$ (i.e., those which do not factor through μ_2). So, let $Irr(\widetilde{Sp(n)})$ stand for the set of isomorphism classes of irreducible admissible genuine representations of group $\widetilde{Sp(n)}$. Further, let $\mathcal{S}(\widetilde{Sp(n, F)})$ denote the Grothendieck group of the category of all admissible genuine representations of finite length of $\widetilde{Sp(n)}$ (i.e., a free abelian group over the set of all irreducible genuine representations of $\widetilde{Sp(n)}$) and define $\mathcal{S} = \bigoplus_{n \geq 0} \mathcal{S}(\widetilde{Sp(n, F)})$.

Let V_0 be an anisotropic quadratic space over F of odd dimension. Then its dimension can only be 1 or 3. For more details about the invariants of this space, such as the quadratic character χ_{V_0} related to the quadratic form on V_0, we refer the reader to [6] and [8]. In each step we add a hyperbolic plane and obtain an enlarged quadratic space, a tower of quadratic spaces and a tower of corresponding orthogonal groups. In the case when r hyperbolic planes are added to the anisotropic space, enlarged quadratic space will be denoted by V_r, while a corresponding orthogonal group will be denoted by $O(V_r)$. Set $m_r = \frac{1}{2}\mathrm{dim}V_r$.

To a fixed quadratic character χ_{V_0} one can attach two odd orthogonal towers, one with $\mathrm{dim}V_0 = 1$ (+–tower) and the other with $\mathrm{dim}V_0 = 3$ (––tower), as in Chapter V of [7]. In that case, for corresponding orthogonal groups of the spaces obtained by adding r hyperbolic planes we write $O(V_r^+)$ and $O(V_r^-)$.

Similarly as before, let $Irr(O(V_r))$ denote the set of isomorphism classes of irreducible admissible representations of the orthogonal group $O(V_r)$.

The pair $(Sp(n), O(V_r))$ is a reductive dual pair in $Sp(n \cdot \mathrm{dim}V_r)$. Since the dimension of the space V_r is odd, the group $Sp(n)$ does not split in $\widetilde{Sp(n \cdot \mathrm{dim}V_r)}$, so the theta correspondence relates the representations of the metaplectic group $\widetilde{Sp(n)}$ and those of the orthogonal group $O(V_r)$. For abbreviation, we write n_1 instead of $n \cdot \mathrm{dim}V_r$. We fix a non-trivial additive character ψ of F and let $\omega_{n,r}$ stand for the pull-back of the Weil representation $\omega_{n_1,\psi}$ of the group $\widetilde{Sp(n_1)}$, restricted to the dual pair $\widetilde{Sp(n)} \times O(V_r)$ (as in [7], Chapter II).

Let $\widetilde{GL(n,F)}$ denote a double cover of $GL(n,F)$, where the multiplication is given by $(g_1, \epsilon_1)(g_2, \epsilon_2) = (g_1 g_2, \epsilon_1 \epsilon_2 (\mathrm{det}g_1, \mathrm{det}g_2)_F)$. Here $\epsilon_i \in \mu_2,\ i = 1, 2$ and $(\cdot, \cdot)_F$ denotes the Hilbert symbol of field F.

We fix a character $\chi_{V,\psi}$ of $\widetilde{GL(n,F)}$ given by $\chi_{V,\psi}(g, \epsilon) = \chi_V(\mathrm{det}g)\epsilon\gamma(\mathrm{det}g, \frac{1}{2}\psi)^{-1}$. Here γ denotes the Weil index ([7], p. 13), while χ_V is a character related to the orthogonal tower. We write $\alpha = \chi_{V,\psi}^2$ and observe that α is a quadratic character on $GL(n,F)$.

Let us define $\mathcal{R}^{gen} = \oplus_n \mathcal{R}(\widetilde{GL(n,F)})_{gen}$, where $\mathcal{R}(\widetilde{GL(n,F)})_{gen}$ denotes the Grothendieck group of the category of all admissible genuine representations of finite length of $\widetilde{GL(n,F)}$.

From now on, ν stands for the character of $GL(n,F)$ defined by $|\mathrm{det}|_F$.

If ρ is an irreducible cuspidal representation of $GL(n_\rho, F)$ (this defines n_ρ), or such genuine representation of $\widetilde{GL(n_\rho,F)}$, we call the set $\Delta = \{\nu^a\rho, \nu^{a+1}\rho, \ldots, \nu^{a+k}\rho\}$ a segment, where $a \in \mathbb{R}$ and $k \in \mathbb{Z}_{\geq 0}$. In the sequel, we abbreviate $\{\nu^a\rho,\ \nu^{a+1}\rho,\ \ldots,\ \nu^{a+k}\rho\}$ to $[\nu^a\rho, \nu^{a+k}\rho]$. We denote by $\delta(\Delta)$ the unique irreducible subrepresentation of $\nu^{a+k}\rho \times \nu^{a+k-1}\rho \times \cdots \times \nu^a\rho$. $\delta(\Delta)$ is an essentially square-integrable representation attached to the segment Δ. If ρ is a genuine representation, then so is $\delta(\Delta)$ (by [4], Proposition 4.2).

To simplify notation, we write ν^x (respectively, $\nu^x\chi_{V,\psi}$) instead of $\nu^x 1_{F^\times}$ (respectively, $\nu^x\chi_{V,\psi}1_{F^\times}$), where $1_{F^\times}$ stands for the trivial representation of the group $F^\times$.

For an ordered partition $s = (n_1, n_2, \ldots, n_i)$ of some $m \leq n$, we denote by P_s a standard parabolic subgroup of $Sp(n,F)$ (consisting of block upper-triangular matrices), whose Levi subgroup equals $GL(n_1,F) \times GL(n_2,F) \times \cdots \times GL(n_i,F) \times Sp(n-m,F)$. Then the standard parabolic subgroup $\widetilde{P_s}$

of $\widetilde{Sp(n)}$ is the preimage of P_s in $\widetilde{Sp(n)}$. There is an analogous notation for the Levi subgroups of metaplectic groups, described in more detail in [4], Section 2.2. The representation of $\widetilde{Sp(n)}$ that is parabolically induced from the representation $\pi_1 \otimes \pi_2 \otimes \cdots \otimes \pi_i \otimes \sigma$ of the Levi subgroup of $\widetilde{P_s}$ will be denoted by $\pi_1 \times \pi_2 \times \cdots \times \pi_i \rtimes \sigma$. The standard parabolic subgroups (those containing the upper triangular Borel subgroup) of $O(V_r)$ have an analogous description as the standard parabolic subgroups of $Sp(n, F)$. If $s = (k)$, for some $0 \leq k \leq n$, we denote P_s (resp., $\widetilde{P_s}$) briefly by P_k (resp., $\widetilde{P_k}$). The normalized Jacquet module of a smooth representation σ of $\widetilde{Sp(n)}$ (resp., of $O(V_r)$) with respect to the standard parabolic subgroup $\widetilde{P_s}$(resp., P_s) will be denoted by $R_{\widetilde{P_s}}(\sigma)$ (resp., $R_{P_s}(\sigma)$). For an irreducible cuspidal representation ρ of $\widetilde{GL(n_\rho, F)}$ (resp., $GL(n_\rho, F)$), we write $R_{\widetilde{P}_{n_\rho}}(\sigma)(\rho)$ (resp., $R_{P_{n_\rho}}(\sigma)(\rho)$) for the maximal ρ–isotypic quotient of $R_{\widetilde{P}_{n_\rho}}(\sigma)$ (resp., of $R_{P_{n_\rho}}(\sigma)$).

When dealing with the Jacquet modules of the representation $\omega_{n,r}$, we write $R_{\widetilde{P_1}}(\omega_{n,r})$ (resp., $R_{P_1}(\omega_{n,r})$) for $R_{\widetilde{P_1} \times O(V_m)}(\omega_{n,r})$ (resp., $R_{\widetilde{Sp(n)} \times P_1}(\omega_{n,r})$), following the notation introduced in [5].

An irreducible representation $\sigma \in \mathcal{S}$ is called strongly positive if for every embedding

$$\sigma \hookrightarrow \nu^{s_1}\rho_1 \times \nu^{s_2}\rho_2 \times \cdots \times \nu^{s_k}\rho_k \rtimes \sigma_{cusp},$$

where $\rho_i \in \mathcal{R}^{gen}$, $i = 1, 2, \ldots, k$, are irreducible cuspidal unitary representations and $\sigma_{cusp} \in \mathcal{S}$ is an irreducible cuspidal representation, we have $s_i > 0$ for each i.

Irreducible strongly positive representations are called strongly positive discrete series. Strongly positive discrete series of classical groups are defined in the completely analogous way. Note that every supercuspidal representation is strongly positive.

Non-supercuspidal strongly positive discrete series of metaplectic groups have been classified in [9]. In the following theorem we recall an inductive description of such representations. We remark that an analogous description holds in the classical group case.

Theorem 2.1. *Let $\sigma \in Irr(\widetilde{Sp(n)})$ denote a non-supercuspidal strongly positive discrete series and let $\rho \in Irr(\widetilde{Gl(n_\rho, F)})$ be a cuspidal representation such that some twist of ρ appears in the cuspidal support of σ. Also, let σ_{cusp} denote a partial cuspidal support of σ and let $s > 0$ denote the point of rank* *one reducibility. Then there exist unique $a, b \in \mathbb{R}$, $a > 0$, $b > 0$, $b - a \in \mathbb{Z}_{\geq 0}$, and a unique strongly positive discrete series τ such that σ can be characterized as the unique irreducible subrepresentation of $\delta([\nu^a\rho, \nu^b\rho]) \rtimes \tau$. Also, a and b are both strictly smaller with respect to those used to construct τ and there is a non–negative integer k such that $a + k = s$. If $k > 0$, then there exist a unique $b' > b$ and a unique strongly positive discrete series τ' such that τ is the unique irreducible subrepresentation of $\delta([\nu^{a+1}\rho, \nu^{b'}\rho]) \rtimes \tau'$. If $k = 0$ then there are no twists of representation ρ appearing in the cuspidal support of τ.*

Moreover, if τ_1 is an irreducible representation such that $\delta([\nu^a\rho, \nu^b\rho]) \otimes \tau_1$ is contained in the Jacquet module of σ with respect to the appropriate parabolic subgroup, then $\tau_1 \cong \tau$. The representation $\delta([\nu^a\rho, \nu^b\rho]) \otimes \tau$ appears with the multiplicity one in the Jacquet module of σ with respect to the appropriate parabolic subgroup.

In the rest of this section we recall some results related to calculations with Jacquet modules. Let σ denote an irreducible genuine representation of $\widetilde{Sp(n)}$. Then $R_{\widetilde{P}_k}(\sigma)$, for $0 \leq k \leq n$, can be interpreted as a genuine representation of $\widetilde{GL(k, F)} \times \widetilde{Sp(n-k)}$, i.e., is an element of $\mathcal{R}^{gen} \otimes \mathcal{S}$. For such σ we can introduce $\mu^*(\sigma) \in \mathcal{R}^{gen} \otimes \mathcal{S}$ by

$$\mu^*(\sigma) = \sum_{k=0}^{n} \text{s.s.}(R_{\widetilde{P}_k}(\sigma))$$

(s.s. denotes the semisimplification) and extend μ^* linearly to the whole of $\mathcal{S}$. In the same way μ^* can be defined for irreducible representations of classical groups.

The basic result of the paper [4] is the following metaplectic version of structure formula due to Tadić in the classical group case ([21]).

Lemma 2.2. *Let $\rho \in \mathcal{R}^{gen}$ be an irreducible cuspidal representation and $a, b \in \mathbb{R}$ such that $a + b \in \mathbb{Z}_{\geq 0}$. Let σ be an admissible genuine representation of finite length of $\widetilde{Sp(n)}$. Write $\mu^*(\sigma) = \sum_{\pi,\sigma'} \pi \otimes \sigma'$. Then the following holds:*

$$\begin{aligned} \mu^*(\delta([\nu^{-a}\rho, \nu^b\rho]) \rtimes \sigma) \;&=\; \sum_{i=-a-1}^{b} \sum_{j=i}^{b} \sum_{\pi,\sigma'} \delta([\nu^{-i}\alpha\widetilde{\rho}, \nu^a\alpha\widetilde{\rho}]) \times \delta([\nu^{j+1}\rho, \nu^b\rho]) \times \pi \\ &\otimes\; \delta([\nu^{i+1}\rho, \nu^j\rho]) \rtimes \sigma'. \end{aligned}$$

We omit $\delta([\nu^x\rho, \nu^y\rho])$ *if* $x > y$.

3 Some preliminary results on theta correspondence

In this section we summarize the relevant material on the theta correspondence which will be used afterwards in the paper.

Let σ denote an irreducible genuine smooth representation of $\widetilde{Sp(n)}$, and let $\Theta(\sigma, r)$ be a smooth representation of $O(V_r)$ given as the full lift of σ to the r-th level of the orthogonal tower, i.e., the biggest quotient of $\omega_{n,r}$ on which $\widetilde{Sp(n)}$ acts as a multiple of σ. We write $\Theta^+(\sigma, r)$ and $\Theta^-(\sigma, r)$ when emphasizing the tower.

Similarly, if τ is an irreducible smooth representation of $O(V_r)$, then one has its full lift $\Theta(\tau, n)$, which is a smooth genuine representation of $\widetilde{Sp(n)}$.

In the following theorem we summarize some basic results about the theta correspondence, which can be found in [7] and [14]. Note that we assume that the residual characteristic of the field F is different than 2.

Theorem 3.1. *Let* σ *denote an irreducible genuine smooth representation of* $\widetilde{Sp(n)}$. *Then there exists a non-negative integer* r *such that* $\Theta(\sigma, r) \neq 0$. *The smallest such* r *is called the first occurrence index of* σ *in the orthogonal tower, we denote it by* $r(\sigma)$. *Also,* $r(\sigma) \leq 2n$ *and* $\Theta(\sigma, r') \neq 0$ *for* $r' \geq r(\sigma)$.

The representation $\Theta(\sigma, r)$ *is either zero or it has a unique irreducible quotient. Following [16], we write* $\sigma(r)$ *for this unique irreducible quotient. If* σ *is an irreducible cuspidal representation of* $\widetilde{Sp(n)}$ *then* $\sigma(r(\sigma))$ *is an irreducible cuspidal representation of* $O(V_{r(\sigma)})$.

The analogous statements hold for $\Theta(\tau, n)$ *if* τ *is an irreducible smooth representation of* $O(V_r)$.

In the sequel we write $r^+(\sigma)$ (resp., $r^-(\sigma)$) for the first occurrence index of irreducible genuine smooth representation σ of $\widetilde{Sp(n)}$ in the orthogonal $+$–tower (resp., orthogonal $-$–tower). Also, we write $\sigma^+(r)$ (resp., $\sigma^-(r)$) for the unique irreducible quotient of the representation $\Theta(\sigma, r)$ in the orthogonal $+$–tower (resp., orthogonal $-$–tower).

The first occurrence indices are related by the following theorem of Kudla and Rallis ([8]):

Theorem 3.2. *If σ is an irreducible genuine smooth representation of $\widetilde{Sp(n)}$, then the inequality*

$$r^+(\sigma) + r^-(\sigma) \geq 2n \qquad (1)$$

holds. Further, if σ is a supercuspidal representation then the equality holds in (1).

They also conjectured equality in (1) and this equality is known as the conservation relation.

Let ρ denote an irreducible self-contragredient cuspidal representation of $GL(m, F)$ and σ denote an irreducible cuspidal representation of $\widetilde{Sp(n)}$. It is well-known that there exists a unique non-negative real number s_1 such that $\nu^{s_1}\rho \rtimes \sigma(r(\sigma))$ reduces ([19]). It is proved in [5], and also independently in [3], that there is a unique non-negative real number s_2 such that $\nu^{s_2}\chi_{V,\psi}\rho \rtimes \sigma$ reduces. If ρ is not a trivial representation of $F^\times$, then $s_1 = s_2$. Otherwise, $s_1 = |n - m_{r(\sigma)}|$ and $s_2 = |m_{r(\sigma)} - n - 1|$.

Now we state the results of Gan and Savin which play a central role in our determination of the first occurrence indices (Section 6 and Theorem 8.1 of [3]).

Theorem 3.3. *Let F be a non-archimedean local field of characteristic 0 with odd residual characteristic. For each non-trivial additive character ψ of F, there is an injection*

$$\Theta_\psi : Irr(\widetilde{Sp(n)}) \to Irr(O(V_n^+)) \sqcup Irr(O(V_{n-1}^-))$$

given by the theta correspondence (with respect to ψ). Suppose that $\sigma \in Irr(\widetilde{Sp(n)})$ and $\tau \in Irr(O(V))$ correspond under Θ_ψ. Then σ is a discrete series representation if and only if τ is a discrete series representation.

By abuse of notation, we write

$$t_\epsilon = \begin{cases} 0, & \text{if } \epsilon = + \\ 1, & \text{if } \epsilon = -. \end{cases}$$

The following lemma, which is proved in Section 5 of [11] and completely relies on Kudla's filtration given in [6], presents a fundamental criterion for pushing down the lifts of irreducible representations of metaplectic groups.

Lemma 3.4. *Suppose that σ is an irreducible genuine representation of $\widetilde{Sp(n)}$.*

(i) Then $\Theta(\sigma, r) \neq 0$ implies $R_{P_1}(\Theta(\sigma, r+1))(\nu^{-(m_{r+1}-n-1)}) \neq 0$.

(ii) Further, if $R_{\widetilde{P_1}}(\sigma)(\nu^{-(m_{r+1}-n-1)}\chi_{V,\psi}) = 0$, then $\Theta(\sigma, r) \neq 0$ if and only if $R_{P_1}(\Theta(\sigma, r+1))(\nu^{-(m_{r+1}-n-1)}) \neq 0$. In that case, $\sigma(r+1) \hookrightarrow \nu^{-(m_{r+1}-n-1)} \rtimes \sigma(r)$.

The following proposition is well-known:

Proposition 3.5. *Let σ denote an irreducible genuine cuspidal representation of $\widetilde{Sp(n)}$. If $k > r(\sigma)$, then $\sigma(k)$ is an irreducible subrepresentation of the induced representation*

$$\nu^{n-m_k+1} \times \nu^{n-m_k+2} \times \cdots \times \nu^{n-m_{r(\sigma)}} \rtimes \sigma(r(\sigma)).$$

Similarly as in [3], one obtains the following metaplectic version of Theorem 6.1 from [17]:

Proposition 3.6. *Let $\sigma \in Irr(\widetilde{Sp(n)})$ denote a discrete series representation and let ϵ denote an arbitrary element of $\{+, -\}$. Let $k = n - t_\epsilon$ if $r^\epsilon(\sigma) \leq n - t_\epsilon$ and $k = r^\epsilon(\sigma)$ otherwise. If l satisfies $l \geq k$ then $\sigma^\epsilon(l)$ is an irreducible subrepresentation of the induced representation*

$$\nu^{n-m_l+1} \times \nu^{n-m_l+2} \times \cdots \times \nu^{n-m_k} \rtimes \sigma^\epsilon(k).$$

Now we state, without proof, two propositions which present a core of the inductive approach that we use for determining the first occurrence indices. These propositions can be proved in completely analogous way as Remark 5.2. and Lemma 5.2. of [16].

Proposition 3.7. *Suppose that the representation $\sigma \in Irr(\widetilde{Sp(n)})$ may be written as an irreducible subrepresentation of the induced representation of the form $\delta([\nu^a\chi_{V,\psi}\rho, \nu^b\chi_{V,\psi}\rho]) \rtimes \sigma'$, where ρ is an irreducible cuspidal representation, $\sigma' \in Irr(\widetilde{Sp(n')})$ and $b - a \geq 0$. Let $\Theta(\sigma, r) \neq 0$. Then one of the following holds:*

- *There is an irreducible representation τ of some $O(V_{r'})$ such that $\sigma(r)$ is a subrepresentation of $\delta([\nu^a\rho, \nu^b\rho]) \rtimes \tau$.*

- *There is an irreducible representation* τ *of some* $O(V_{r'})$ *such that* $\sigma(r)$ *is a subrepresentation of* $\delta([\nu^{a+1}\rho, \nu^b\rho]) \rtimes \tau$.

The latter situation is impossible unless $(a, \rho) = (m_r - n, 1_{F^\times})$.

Suppose that $\delta([\nu^a\chi_{V,\psi}\rho, \nu^b\chi_{V,\psi}\rho])$ *is a representation of* $\widetilde{GL(l, F)}$ *and* $(a, \rho) \neq (m_r - n, 1_{F^\times})$. *Also, suppose that if* $\mu^*(\sigma)$ *contains the representation* $\delta([\nu^a\chi_{V,\psi}\rho, \nu^b\chi_{V,\psi}\rho]) \otimes \sigma''$ *for some irreducible genuine representation* σ'' *of* $\widetilde{Sp(n - l)}$, *then* $\sigma'' \cong \sigma'$. *Then* $\sigma(r)$ *is a subrepresentation of*

$$\delta([\nu^a\rho, \nu^b\rho]) \rtimes \sigma'(r - l).$$

Proposition 3.8. *Suppose that the representation* $\tau \in Irr(O(V_r))$ *may be written as an irreducible subrepresentation of the induced representation of the form* $\delta([\nu^a\rho, \nu^b\rho]) \rtimes \tau'$, *where* ρ *is an irreducible cuspidal representation,* $\tau' \in Irr(O(V_{r'}))$ *and* $b - a \geq 0$. *Let* $\Theta(\tau, n) \neq 0$. *Then one of the following hold:*

- *There is an irreducible representation* σ *of some* $\widetilde{Sp(n')}$ *such that* $\tau(n)$ *is a subrepresentation of* $\delta([\nu^a\chi_{V,\psi}\rho, \nu^b\chi_{V,\psi}\rho]) \rtimes \sigma$.

- *There is an irreducible representation* σ *of some* $\widetilde{Sp(n')}$ *such that* $\tau(n)$ *is a subrepresentation of* $\delta([\nu^{a+1}\chi_{V,\psi}\rho, \nu^b\chi_{V,\psi}\rho]) \rtimes \sigma$.

The latter situation is impossible unless $(a, \rho) = (n - m_r + 1, 1_{F^\times})$.

Suppose that $\delta([\nu^a\rho, \nu^b\rho])$ *is a representation of* $GL(l, F)$ *and* $(a, \rho) \neq (n - m_r + 1, 1_{F^\times})$. *Further, suppose that if* $\mu^*(\tau) \geq \delta([\nu^a\rho, \nu^b\rho]) \otimes \tau''$, *for some irreducible genuine representation* τ'' *of* $O(V_{r-l})$, *then* $\tau'' \cong \tau'$. *Then* $\tau(n)$ *is a subrepresentation of*

$$\delta([\nu^a\chi_{V,\psi}\rho, \nu^b\chi_{V,\psi}\rho]) \rtimes \tau'(n - l).$$

We take a moment to state the explicit description of the cuspidal support of theta lifts, given in Section 2 of the paper [6], which appears to be very useful for obtaining precise information about the lifts of discrete series. Following [6], we denote the cuspidal support of representation σ by $[\sigma]$.

Theorem 3.9. *Let* τ *denote an irreducible representation of* $O(V_r)$ *and suppose* $[\tau] = [\rho_1, \rho_2, \ldots, \rho_k; \tau_{cusp}]$, *with* τ_{cusp} *being an irreducible cuspidal representation of* $O(V_{r'})$. *Let* $\sigma_{cusp} = \tau_{cusp}(n')$ *be the first non-zero lift of the representation* τ_{cusp}. *Let* σ *denote an irreducible subquotient of* $\Theta(\tau, n)$. *We have the following possibilities:*

- *If* $n \geq n'+r-r'$, *then* $[\sigma] = [\chi_{V,\psi}\nu^{m_r-n}, \chi_{V,\psi}\nu^{m_r-n+1}, \ldots, \chi_{V,\psi}\nu^{m_{r'}-n'-1}, \chi_{V,\psi}\rho_1, \chi_{V,\psi}\rho_2, \ldots, \sigma_{cusp}]$,

- *If* $n < n'+r-r'$, *set* $t = r-r'-n+n'$. *Then there exist* $i_1, i_2, \ldots, i_t \in \{1, 2, \ldots, k\}$ *such that* $\rho_{i_j} = \nu^{m_r-n-j}$ *for* $j = 1, 2, \ldots, t$ *and* $[\sigma] = [\chi_{V,\psi}\rho_1, \ldots, \widehat{\chi_{V,\psi}\rho_{i_1}}, \ldots, \widehat{\chi_{V,\psi}\rho_{i_t}}, \ldots, \chi_{V,\psi}\rho_k; \sigma_{cusp}]$, *where* $\widehat{\chi_{V,\psi}\rho_i}$ *means that we omit* $\chi_{V,\psi}\rho_i$.

Similarly, let σ *denote an irreducible genuine representation of* $\widetilde{Sp(n)}$ *and suppose* $[\sigma] = [\chi_{V,\psi}\rho_1, \chi_{V,\psi}\rho_2, \ldots, \chi_{V,\psi}\rho_k; \sigma_{cusp}]$, *with* σ_{cusp} *being an irreducible genuine cuspidal representation of* $\widetilde{Sp(n')}$. *Let* $\tau_{cusp} = \sigma_{cusp}(r')$ *be the first non-zero lift of the representation* σ_{cusp}. *Let* τ *denote an irreducible subquotient of* $\Theta(\sigma, r)$. *We have the following possibilities:*

- *If* $r \geq r'+n-n'$, *then* $[\tau] = [\nu^{m_r-n-1}, \nu^{m_r-n-2}, \ldots, \nu^{m_{r'}-n'}, \rho_1, \rho_2, \ldots, \rho_k; \tau_{cusp}]$,

- *If* $r < r'+n-n'$, *set* $t = r'-n'+n-r$. *Then there exist* $i_1, i_2, \ldots, i_t \in \{1, 2, \ldots, k\}$ *such that* $\rho_{i_j} = \nu^{m_r-n+j-1}$ *for* $j = 1, 2, \ldots, t$ *and* $[\tau] = [\rho_1, \ldots, \widehat{\rho_{i_1}}, \ldots, \widehat{\rho_{i_t}}, \ldots, \rho_k; \tau_{cusp}]$, *where* $\widehat{\rho_i}$ *means that we omit* ρ_i.

4 The conservation relation for strongly positive representations

This section is devoted to the proof of the conservation relation for strongly positive discrete series.

Let σ denote an irreducible genuine representation of $\widetilde{Sp(n)}$ and let $\sigma_{cusp} \in Irr(\widetilde{Sp(n')})$ denote the partial cuspidal support of σ.

Several cases, depending on the structure of the cuspidal support of the representation σ, shall be considered separately.

Let us first assume that the representation $\nu^x\chi_{V,\psi}$ does not appear in the cuspidal support of σ, for $x \in \mathbb{R}$.

The following theorem, together with Theorem 3.2, establishes the conservation relation in this case.

Theorem 4.1. *If there are no twists of the representation* $\chi_{V,\psi}1_{F^\times}$ *appearing in the cuspidal support of* σ, *then* $r^\epsilon(\sigma) = n - n' + r^\epsilon(\sigma_{cusp})$ *for* $\epsilon \in \{+, -\}$.

Further, the first non-zero lift of σ is strongly positive representation whose cuspidal support contains no twists of the trivial representation $1_{F^\times}$.

Proof. Theorem obviously holds if σ is a cuspidal representation, i.e., if $\sigma \cong \sigma_{cusp}$. We prove the theorem for non-cuspidal strongly positive representation σ using induction over the number of segments needed to obtain the representation σ starting from its partial cuspidal support.

According to Theorem 2.1, σ can be written as a unique irreducible subrepresentation of the induced representation of the form $\delta([\nu^a\chi_{V,\psi}\rho, \nu^b\chi_{V,\psi}\rho]) \rtimes \sigma_1$, where $\sigma_1 \in Irr(\widetilde{Sp(l)})$ is a strongly positive discrete series. Also, if $\mu^*(\sigma) \geq \delta([\nu^a\chi_{V,\psi}\rho, \nu^b\chi_{V,\psi}\rho]) \otimes \sigma'$ for some $\sigma' \in Irr(\widetilde{Sp(l)})$, then $\sigma' \cong \sigma_1$. We assume that theorem holds for σ_1 and prove it for σ.

Since $\rho \neq 1_{F^\times}$, Proposition 3.7 implies

$$\sigma(r) \hookrightarrow \delta([\nu^a\rho, \nu^b\rho]) \rtimes \sigma_1(r-n+l),$$

for r such that $r-n+l \geq r(\sigma_1)$.

There exists some r such that $\Theta(\sigma, r) \neq 0$. If $r-n+l > r(\sigma_1)$, since $R_{\widetilde{P_1}}(\sigma_1)(\nu^x\chi_{V,\psi}) = 0$ for all x, Lemma 3.4 (ii) provides an embedding $\sigma_1(r-n+l) \hookrightarrow \nu^{l-m_{r-n+l-1}} \rtimes \sigma_1(r-n+l-1)$.

This gives

$$\begin{aligned}\sigma(r) &\hookrightarrow \delta([\nu^a\rho, \nu^b\rho]) \times \nu^{l-m_{r-n+l-1}} \rtimes \sigma_1(r-n+l-1)\\ &\cong \nu^{l-m_{r-n+l-1}} \times \delta([\nu^a\rho, \nu^b\rho]) \rtimes \sigma_1(r-n+l-1).\end{aligned}$$

Since $l - m_{r-n+l-1} = n - m_{r-1}$, Lemma 3.4 (ii) yields $\sigma(r-1) \neq 0$.

We continue in this fashion to obtain $r(\sigma) \leq n - n' + r(\sigma_{cusp})$. Also, Proposition 3.7 shows

$$\sigma(n-n'+r(\sigma_{cusp})) \hookrightarrow \delta([\nu^a\rho, \nu^b\rho]) \rtimes \sigma_1(l-n'+r(\sigma_{cusp})). \tag{2}$$

Since by the inductive assumption $r(\sigma_1) = l - n' + r(\sigma_{cusp})$ and there are no twists of $1_{F^\times}$ appearing in the cuspidal support of $\sigma_1(l-n'+r(\sigma_{cusp}))$, Lemma 3.4 (i) shows $r(\sigma) = n - n' + r(\sigma_{cusp})$.

Clearly, there are no twists of $1_{F^\times}$ appearing in the cuspidal support of $\sigma(r(\sigma))$, while the strong positivity of $\sigma(r(\sigma))$ follows directly from (2) and the description of strongly positive representations given in [9].

□

If we denote by ϵ_1 an element of $\{+,-\}$ such that $r^{\epsilon_1}(\sigma_{cusp}) \leq n'-t_{\epsilon_1}$, by ϵ_2 an element of $\{+,-\}$ different than ϵ_1 and suppose that $\nu^s\chi_{V,\psi} \rtimes \sigma_{cusp}$ reduces for $s \geq 0$, then the previous result can be rewritten as $r^{\epsilon_1}(\sigma) = n-s+\frac{1}{2}-t_{\epsilon_1}$ and $r^{\epsilon_2}(\sigma) = n+s+\frac{1}{2}-t_{\epsilon_2}$.

In the rest of this section we suppose that there is some representation of the form $\nu^x\chi_{V,\psi}$ appearing in the cuspidal support of σ. Let a_1 denote minimal $x > 0$ such that $\nu^x\chi_{V,\psi}$ appears in $[\sigma]$. By Theorem 2.1 there is a non–negative integer k such that $\nu^{a_1+k}\chi_{V,\psi} \rtimes \sigma_{cusp}$ reduces. It follows from Section 4 of [5] that a_1+k is half integral. Consequently, a_1 is also half integral.

Thus, σ can be written as the unique irreducible subrepresentation of the induced representation of the form $\delta([\nu^{a_1}\chi_{V,\psi}, \nu^{b_1}\chi_{V,\psi}]) \rtimes \sigma_1$, where $\sigma_1 \in Irr(\widetilde{Sp(l)})$ is a strongly positive representation and $\nu^{a_1}\chi_{V,\psi}$ does not appear in the cuspidal support of σ_1. Further, if $\mu^*(\sigma) \geq \delta([\nu^{a_1}\chi_{V,\psi}, \nu^{b_1}\chi_{V,\psi}]) \otimes \sigma_2$ for some $\sigma_2 \in Irr(\widetilde{Sp(l)})$, then $\sigma_2 \cong \sigma_1$.

Theorem 6.1 from [10] shows $R_{\widetilde{P_1}}(\sigma)(\nu^x\chi_{V,\psi}) = 0$ for $x < b_1$.

It is a direct consequence of Theorem 3.2 that there is exactly one $\epsilon \in \{+,-\}$ such that $r^{\epsilon}(\sigma_{cusp}) \leq n'-t_{\epsilon}$. In the rest of this section we denote such ϵ by ϵ_1 and let ϵ_2 denote an element of $\{+,-\}$ different than ϵ_1.

There are two possibilities to consider:

- $a_1 > \frac{1}{2}$.

The first occurrence indices are given by the following theorem:

Theorem 4.2. *If $a_1 > \frac{1}{2}$ then $r^{\epsilon_1}(\sigma) = n-t_{\epsilon_1}-a_1+\frac{3}{2}$ and $r^{\epsilon_2}(\sigma) = n-t_{\epsilon_2}+a_1-\frac{1}{2}$. Both representations $\sigma^{\epsilon_1}(r^{\epsilon_1}(\sigma))$ and $\sigma^{\epsilon_2}(r^{\epsilon_2}(\sigma))$ are strongly positive. Also, if we denote by x_i the minimal $x > 0$ such that ν^x appears in $[\sigma^{\epsilon_i}(r^{\epsilon_i}(\sigma))]$, for $i = 1, 2$, then $x_1 = a_1 - 1$ and $x_2 = a_1 + 1$.*

Proof. Let k denote a non–negative integer such that $a_1 + k = s$ where s is a positive real number such that $\nu^s\chi_{V,\psi} \rtimes \sigma_{cusp}$ reduces. We prove the theorem using induction on k.

First we assume $k = 0$. We begin by determining the first occurrence index in ϵ_2–tower.

It is shown in Theorem 4.1 that $r^{\epsilon_2}(\sigma_1) = l-n'+r^{\epsilon_2}(\sigma_{cusp})$. Since $a_1 > 0$, it is a direct consequence of Proposition 3.7 that $r^{\epsilon_2}(\sigma) > n-t_{\epsilon_2}$.

For $r > n - l + r^{\epsilon_2}(\sigma_1) = n - n' + r^{\epsilon_2}(\sigma_{cusp})$ we have

$$\begin{aligned}\sigma^{\epsilon_2}(r) &\hookrightarrow \delta([\nu^{a_1}, \nu^{b_1}]) \rtimes \sigma_1^{\epsilon_2}(r - n + l)\\ &\hookrightarrow \delta([\nu^{a_1}, \nu^{b_1}]) \times \nu^{n-m_{r-1}} \rtimes \sigma_1^{\epsilon_2}(r - n + l - 1)\\ &\cong \nu^{n-m_{r-1}} \times \delta([\nu^{a_1}, \nu^{b_1}]) \rtimes \sigma_1^{\epsilon_2}(r - n + l - 1)\end{aligned}$$

(note that we are in the first case of the Proposition 3.7).

Repeating this procedure and using Lemma 3.4 (ii) we get $r^{\epsilon_2}(\sigma) \leq n - n' + r^{\epsilon_2}(\sigma_{cusp})$.

We denote by $\tau^{(2)}_{cusp}$ the first non–zero lift of σ_{cusp} in the ϵ_2–tower. Note that $m_{r^{\epsilon_2}(\sigma_{cusp})} - n' = a_1 + 1$ (this also gives $r^{\epsilon_2}(\sigma_{cusp}) = a_1 + \frac{1}{2} + n' - t_{\epsilon_2}$) and $\nu^{s'} \rtimes \tau^{(2)}_{cusp}$ reduces for $s' = a_1 + 1$.

By Theorem 4.1, the first non–zero lift of the representation σ_1 is the strongly positive representation and it can be written as an irreducible subrepresentation of the induced representation of the form $\pi \rtimes \tau^{(2)}_{cusp}$, where π is a product of essentially square-integrable representations attached to the segments not containing representation of the form $\nu^x 1_{F^\times}$, $x \in \mathbb{R}$.

We have the following embeddings and isomorphisms:

$$\begin{aligned}\sigma^{\epsilon_2}(n - n' + r^{\epsilon_2}(\sigma_{cusp})) &\hookrightarrow \delta([\nu^{a_1}, \nu^{b_1}]) \times \pi \rtimes \tau^{(2)}_{cusp}\\ &\hookrightarrow \delta([\nu^{a_1+1}, \nu^{b_1}]) \times \nu^{a_1} \times \pi \rtimes \tau^{(2)}_{cusp}\\ &\cong \delta([\nu^{a_1+1}, \nu^{b_1}]) \times \pi \times \nu^{a_1} \rtimes \tau^{(2)}_{cusp}\\ &\cong \delta([\nu^{a_1+1}, \nu^{b_1}]) \times \pi \times \nu^{-a_1} \rtimes \tau^{(2)}_{cusp}\\ &\cong \nu^{-a_1} \times \delta([\nu^{a_1+1}, \nu^{b_1}]) \times \pi \rtimes \tau^{(2)}_{cusp}.\end{aligned}$$

Since $n - m_{n-n'+r^{\epsilon_2}(\sigma_{cusp})-1} = -a_1$, using Lemma 3.4 (ii) we get $r^{\epsilon_2}(\sigma) \leq n - n' + r^{\epsilon_2}(\sigma_{cusp}) - 1 = n + a_1 - \frac{1}{2} - t_{\epsilon_2}$.

It is a direct consequence of Theorem 3.9 that the representation ν^{a_1} does not appear in the cuspidal support of $\Theta^{\epsilon_2}(\sigma, n + a_1 - \frac{1}{2} - t_{\epsilon_2})$. So, Proposition 3.7 shows that $\sigma^{\epsilon_2}(n + a_1 - \frac{1}{2} - t_{\epsilon_2})$ is a subrepresentation of the induced representation of the form $\delta([\nu^{a_1+1}, \nu^{b_1}]) \rtimes \tau$, where τ has the same cuspidal support as $\pi \rtimes \tau^{(2)}_{cusp}$.

We will show that τ is a discrete series representation. Suppose on the contrary that τ is not square-integrable. According to Lemma 3.4 of [11], there is an embedding $\tau \hookrightarrow \delta([\nu^a\rho, \nu^b\rho]) \rtimes \tau'$, where $a + b \leq 0$, $\rho \in Irr(GL(n_\rho, F))$ and τ' is an irreducible representation of some $O(V^{\epsilon_2}_{l'})$. In that case, $\sigma^{\epsilon_2}(n + a_1 - \frac{1}{2} - t_{\epsilon_2})$ is a subrepresentation of $\delta([\nu^{a_1+1}, \nu^{b_1}]) \times \delta([\nu^a \rho, \nu^b \rho]) \rtimes \tau'$ and applying Proposition 3.8 it is a simple matter to obtain a contradiction with the strong positivity of σ.

Since τ is square-integrable, Lemma 3.6 from [10] implies that it is strongly positive. It follows that $\sigma^{\epsilon_2}(n + a_1 - \frac{1}{2} - t_{\epsilon_2})$ is an irreducible subrepresentation of $\delta([\nu^{a_1+1}, \nu^{b_1}]) \rtimes \tau$ and Theorem 5.3 from [9] shows that it is strongly positive. Thus, by Lemma 3.4 (i), $r^{\epsilon_2}(\sigma) = n + a_1 - \frac{1}{2} - t_{\epsilon_2}$.

Now we determine the first occurrence index in the ϵ_1–tower.

We have already proved $r^{\epsilon_1}(\sigma_1) = l - n' + r^{\epsilon_1}(\sigma_{cusp})$. We will denote by $\tau_{cusp}^{(1)}$ the first non–zero lift of σ_{cusp} in the ϵ_1–tower. Note that in this case $\nu^{s'} \rtimes \tau_{cusp}^{(1)}$ reduces for $s' = a_1 - 1$. Further, $r^{\epsilon_1}(\sigma_{cusp}) = n' - a_1 - t_{\epsilon_1} + \frac{1}{2}$, thus $r^{\epsilon_1}(\sigma_1) = l - a_1 - t_{\epsilon_1} + \frac{1}{2}$.

Since $r^{\epsilon_2}(\sigma) > n - t_{\epsilon_2}$, Theorem 3.3 shows that $r^{\epsilon_1}(\sigma) \leq n - t_{\epsilon_1}$.

Starting from the $n - t_{\epsilon_1}$–th level of the ϵ_1–tower, for $r > n - l + r^{\epsilon_1}(\sigma_1) + 1 = n - a_1 - t_{\epsilon_1} + \frac{3}{2}$, in each step we have

$$\begin{aligned} \sigma^{\epsilon_1}(r) &\hookrightarrow \delta([\nu^{a_1}, \nu^{b_1}]) \times \sigma_1^{\epsilon_1}(r - n + l) \\ &\hookrightarrow \nu^{n - m_{r-1}} \times \delta([\nu^{a_1}, \nu^{b_1}]) \times \sigma_1^{\epsilon_1}(r - n + l - 1). \end{aligned}$$

Observe that for r as above we have $n - m_{r-1} < a_1 - 1$ and $a_1 > m_r - n$ so we are in the first case of Proposition 3.7.

Lemma 3.4 (ii) gives $r^{\epsilon_1}(\sigma) \leq n - a_1 - t_{\epsilon_1} + \frac{3}{2}$. Since $t_{\epsilon_1} + t_{\epsilon_2} = 1$, from (1) and and already determined $r^{\epsilon_2}(\sigma)$ we obtain $r^{\epsilon_1}(\sigma) = n - a_1 - t_{\epsilon_1} + \frac{3}{2}$.

Further, we have the following embeddings and intertwining operator:

$$\begin{aligned} \sigma^{\epsilon_1}(r^{\epsilon_1}(\sigma)) &\hookrightarrow \delta([\nu^{a_1}, \nu^{b_1}]) \rtimes \sigma_1^{\epsilon_1}(r^{\epsilon_1}(\sigma_1) + 1) \\ &\hookrightarrow \delta([\nu^{a_1}, \nu^{b_1}]) \times \nu^{a_1 - 1} \rtimes \sigma_1^{\epsilon_1}(r^{\epsilon_1}(\sigma_1)) \\ &\rightarrow \nu^{a_1 - 1} \times \delta([\nu^{a_1}, \nu^{b_1}]) \rtimes \sigma_1^{\epsilon_1}(r^{\epsilon_1}(\sigma_1)). \end{aligned}$$

Part (ii) of Lemma 3.4 shows that $\sigma^{\epsilon_1}(r^{\epsilon_1}(\sigma))$ is contained in the kernel of the last intertwining operator, and this kernel is isomorphic to $\delta([\nu^{a_1 - 1}, \nu^{b_1}]) \rtimes \sigma_1^{\epsilon_1}(r^{\epsilon_1}(\sigma_1))$. We have already shown that $\sigma_1^{\epsilon_1}(r^{\epsilon_1}(\sigma_1))$ is the strongly positive representation and it is a direct consequence of the classification of strongly positive discrete series that $\delta([\nu^{a_1 - 1}, \nu^{b_1}]) \rtimes \sigma_1^{\epsilon_1}(r^{\epsilon_1}(\sigma_1))$ has the unique irreducible subrepresentation which is also strongly positive.

Suppose that $k \geq 1$ and that the claim holds for all non–negative numbers less than k. We prove it for k.

In this case, σ is subrepresentation of the induced representation of the form $\delta([\nu^{a_1}\chi_{V,\psi}, \nu^{b_1}\chi_{V,\psi}]) \rtimes \sigma_1$, where $\sigma_1 \in Irr(\widetilde{Sp(l)})$ is strongly positive and minimal $x > 0$ such that $\nu^x\chi_{V,\psi}$ appears in $[\sigma_1]$ equals $a_1 + 1$.

First we determine the first occurrence index in the ϵ_2-tower.

The inductive assumption implies $r^{\epsilon_2}(\sigma_1) = l + a_1 + \frac{1}{2} - t_{\epsilon_2}$. Also, the first non-zero lift of σ_1 in the ϵ_2–tower is the strongly positive representation that can be written as the unique irreducible subrepresentation of $\pi \rtimes \tau_{cusp}^{(2)}$, where π is a product of representations attached to the segments which do not contain ν^x for $x \leq a_1 + 1$ and $\tau_{cusp}^{(2)}$ is the first non-zero lift of σ_{cusp} in ϵ_2–tower.

In completely analogous way as before we obtain $r^{\epsilon_2}(\sigma) \leq n + a_1 + \frac{1}{2} - t_{\epsilon_2}$. Also,

$$\sigma^{\epsilon_2}(n+a_1+\frac{1}{2}-t_{\epsilon_2}) \hookrightarrow \delta([\nu^{a_1+1}, \nu^{b_1}]) \times \nu^{a_1} \times \pi \rtimes \tau_{cusp}^{(2)} \cong \nu^{-a_1} \times \delta([\nu^{a_1+1}, \nu^{b_1}]) \times \pi \rtimes \tau_{cusp}^{(2)}$$

since $\nu^{a_1} \rtimes \tau_{cusp}^{(2)}$ is irreducible. Lemma 3.4 (ii) gives $r^{\epsilon_2}(\sigma) \leq n + a_1 - \frac{1}{2} - t_{\epsilon_2}$.

Proposition 3.7 now leads to an embedding $\sigma^{\epsilon_2}(n + a_1 - \frac{1}{2} - t_{\epsilon_2}) \hookrightarrow \delta([\nu^{a_1+1}, \nu^{b_1}]) \rtimes \tau$, for some irreducible representation τ. Our next claim is that τ is a discrete series representation.

Suppose, contrary to our claim, that τ is not the discrete series representation. Then there exists an embedding of the form $\tau \hookrightarrow \delta([\nu^a\rho, \nu^b\rho]) \rtimes \tau'$, where τ' is irreducible, $a + b \leq 0$ and ρ is an irreducible cuspidal representation of some $GL(n_\rho, F)$.

Consequently, $\sigma^{\epsilon_2}(n + a_1 - \frac{1}{2} - t_{\epsilon_2})$ can be written as a subrepresentation of $\delta([\nu^{a_1+1}, \nu^{b_1}]) \times \delta([\nu^a\rho, \nu^b\rho]) \rtimes \tau'$. If ρ is not isomorphic to $1_{F^\times}$, we get

$$\sigma^{\epsilon_2}(n + a_1 - \frac{1}{2} - t_{\epsilon_2}) \hookrightarrow \delta([\nu^a\rho, \nu^b\rho]) \times \delta([\nu^{a_1+1}, \nu^{b_1}]) \rtimes \tau',$$

while in the other case we have

$$\sigma^{\epsilon_2}(n + a_1 - \frac{1}{2} - t_{\epsilon_2}) \hookrightarrow \delta([\nu^a, \nu^{b'}]) \rtimes \tau'',$$

where $b' \geq b$, and τ'' is an irreducible representation of some $O(V_{l'}^{\epsilon_2})$. Since $a \neq -a_1 + 1$, because ν^{-a_1+1} does not appear in the cuspidal support of τ, using Proposition 3.8 we get a contradiction with the strong positivity of σ.

Therefore, τ is a discrete series representation, and by [10], Lemma 3.6, it is strongly positive. It is easy to conclude that $\sigma^{\epsilon_2}(n + a_1 - \frac{1}{2} - t_{\epsilon_2})$ is also a strongly positive discrete series and minimal $x > 0$ such that ν^x appears in its cuspidal support equals $a_1 + 1$. Theorem 6.1 from [10] and Lemma 3.4 (i) imply that $n + a_1 - \frac{1}{2} - t_{\epsilon_2}$ is the first occurrence index of σ in the ϵ_2–orthogonal tower.

Let us now determine the first occurrence index in the other tower. Similarly as before, we start from $r^{\epsilon_1}(\sigma) \leq n - t_{\epsilon_1}$. Using the inductive assumption $r^{\epsilon_1}(\sigma_1) = l - a_1 - t_{\epsilon_1} + \frac{1}{2}$ and following the same lines as in the previous cases, we obtain $r^{\epsilon_1}(\sigma) \leq n - a_1 - t_{\epsilon_1} + \frac{3}{2}$. Already determined $r^{\epsilon_2}(\sigma)$ and the inequality (1) provide the equality $r^{\epsilon_1}(\sigma) = n - a_1 - t_{\epsilon_1} + \frac{3}{2}$.

Further, we have

$$\sigma^{\epsilon_1}(r^{\epsilon_1}(\sigma)) \hookrightarrow \delta([\nu^{a_1}, \nu^{b_1}]) \rtimes \sigma_1^{\epsilon_1}(r^{\epsilon_1}(\sigma_1)+1) \hookrightarrow \delta([\nu^{a_1}, \nu^{b_1}]) \times \nu^{a_1-1} \rtimes \sigma_1^{\epsilon_1}(r^{\epsilon_1}(\sigma_1))$$

and in the same way as before we see that $\sigma^{\epsilon_1}(r^{\epsilon_1}(\sigma))$ is the strongly positive subrepresentation of $\delta([\nu^{a_1-1}, \nu^{b_1}]) \rtimes \sigma_1^{\epsilon_1}(r^{\epsilon_1}(\sigma_1))$ and the proof is complete. □

- $a_1 = \frac{1}{2}$.

The following theorem, together with Theorem 4.1 and Theorem 4.2, establishes the conservation relation for the strongly positive discrete series of metaplectic groups.

Theorem 4.3. *If $a_1 = \frac{1}{2}$, then $r^{\epsilon_1}(\sigma) = n - t_{\epsilon_1} + 1$ and $r^{\epsilon_2}(\sigma) = n - t_{\epsilon_2}$. The representation $\sigma^{\epsilon_2}(r^{\epsilon_2}(\sigma))$ is strongly positive, while the representation $\sigma^{\epsilon_1}(r^{\epsilon_1}(\sigma))$ is a non-strongly positive discrete series.*

Proof. The representation σ is a subrepresentation of $\delta([\nu^{\frac{1}{2}}\chi_{V,\psi}, \nu^{b_1}\chi_{V,\psi}]) \rtimes \sigma_1$, where $\sigma_1 \in Irr(\widetilde{Sp(l)})$ is the strongly positive representation such that either there are no twists of $\chi_{V,\psi}$ appearing in $[\sigma_1]$ or the minimal $x > 0$ such that $\nu^x\chi_{V,\psi}$ appears in $[\sigma_1]$ equals $\frac{3}{2}$. In any case, $r^{\epsilon}(\sigma_1) \leq l + 1 - t_{\epsilon}$ for $\epsilon \in \{+, -\}$, by Theorems 4.1 and 4.2.

Since σ is strongly positive and $m_r - n > \frac{1}{2}$ for $r \geq n + 1 - t_{\epsilon}$, we may use an inductive procedure based on Lemma 3.4 and Proposition 3.7 to obtain $r^{\epsilon}(\sigma) \leq n + 1 - t_{\epsilon}$, for $\epsilon \in \{+, -\}$.

Let us denote by τ_{cusp} a cuspidal representation $\sigma_{cusp}^{\epsilon_2}(r^{\epsilon_2}(\sigma_{cusp}))$. Note that the representation $\nu^{\frac{1}{2}} \rtimes \tau_{cusp}$ is irreducible.

Using Theorems 4.1 and 4.2 we see that $\sigma^{\epsilon_2}(n + 1 - t_{\epsilon_2})$ can be written as an irreducible subrepresentation of the induced representation of the form

$\delta([\nu^{\frac{1}{2}}, \nu^{b_1}]) \times \pi \rtimes \tau_{cusp}$, where π is a representation induced from representations attached to the segments not containing $\nu^{\frac{3}{2}}$. It is now easy to obtain an embedding

$$\sigma^{\epsilon_2}(n+1-t_\epsilon) \hookrightarrow \nu^{-\frac{1}{2}} \times \delta([\nu^{\frac{3}{2}}, \nu^{b_1}]) \times \pi \rtimes \tau_{cusp},$$

and Lemma 3.4 (ii) yields $r^{\epsilon_2}(\sigma) \leq n - t_{\epsilon_2}$.

Inequality (1) shows $r^{\epsilon_1}(\sigma) = n - t_{\epsilon_1} + 1$ and $r^{\epsilon_2}(\sigma) = n - t_{\epsilon_2}$.

Let us now prove that $\sigma^{\epsilon_2}(r^{\epsilon_2}(\sigma))$ is the strongly positive representation. Otherwise, by Lemma 3.4 and Theorem 3.5 from [11], there are $a, b \in \mathbb{R}$, $a \leq 0$, a cuspidal representation $\rho \in GL(n_\rho, F)$ and an irreducible representation τ of the corresponding orthogonal group such that $\sigma^{\epsilon_2}(r^{\epsilon_2}(\sigma))$ is a subrepresentation of $\delta([\nu^a \rho, \nu^b \rho]) \rtimes \tau$. Using Proposition 3.8 we get a contradiction with the strong positivity of σ. Since a strongly positive discrete series representation is completely determined by its cuspidal support, it is not hard to see that $\sigma^{\epsilon_2}(r^{\epsilon_2}(\sigma))$ can be characterized as the unique irreducible subrepresentation of $\delta([\nu^{\frac{3}{2}}, \nu^{b_1}]) \rtimes \sigma_1^{\epsilon_2}(r^{\epsilon_2}(\sigma_1))$.

On the other hand, $\sigma^{\epsilon_1}(n - t_{\epsilon_1} + 1)$ is a subrepresentation of $\delta([\nu^{\frac{1}{2}}, \nu^{b_1}]) \rtimes \sigma_1^{\epsilon_1}(r^{\epsilon_1}(\sigma_1) + 1)$. Since $\sigma_1^{\epsilon_1}(r^{\epsilon_1}(\sigma_1) + 1)$ is a subrepresentation of $\nu^{-\frac{1}{2}} \rtimes \sigma_1^{\epsilon_1}(r^{\epsilon_1}(\sigma_1))$, using Lemma 3.4 (ii) we deduce

$$\sigma^{\epsilon_1}(n - t_{\epsilon_1} + 1) \hookrightarrow \delta([\nu^{-\frac{1}{2}}, \nu^{b_1}]) \rtimes \sigma_1^{\epsilon_1}(r^{\epsilon_1}(\sigma_1)).$$

Obviously, $\sigma^{\epsilon_1}(n - t_{\epsilon_1} + 1)$ is not strongly positive. We have already shown that $\sigma_1^{\epsilon_1}(r^{\epsilon_1}(\sigma_1))$ is a strongly positive representation and it is easy to check that $R_{P_1}(\sigma_1^{\epsilon_1}(r^{\epsilon_1}(\sigma_1)))(\nu^x) = 0$ holds for $x \leq b_1$. The fact that $\sigma^{\epsilon_1}(n - t_{\epsilon_1} + 1)$ is square–integrable is an integral part of the Mœglin-Tadić classification of discrete series. This proves the theorem. □

We also note the following corollary, which is a generalization of Proposition 3.5 for the strongly positive discrete series:

Corollary 4.4. *Let $\sigma \in Irr(\widetilde{Sp(n)})$ denote a strongly positive representation. If $k > r(\sigma)$, then $\sigma(k)$ is an irreducible subrepresentation of the induced representation*

$$\nu^{n-m_k+1} \times \nu^{n-m_k+2} \times \cdots \times \nu^{n-m_{r(\sigma)}} \rtimes \sigma(r(\sigma)).$$

The following theorem supplements the results of Gan and Savin [3].

Theorem 4.5. *Suppose that $\sigma \in Irr(\widetilde{Sp(n)})$ and $\tau \in Irr(O(V))$ correspond under Θ_ψ. Then σ is a strongly positive discrete series if and only if τ is a strongly positive discrete series.*

Proof. First we assumes that σ is strongly positive. If the representation $\nu^{\frac{1}{2}}\chi_{V,\psi}$ appears in $[\sigma]$, previous theorem shows that τ is also strongly positive. Otherwise, Theorem 3.9 shows that if there is some twist of the representation ρ appearing in $[\tau]$ then there exists at most one $0 < x < 1$ such that $\nu^x\rho$ appears in $[\tau]$. We already know that τ is a discrete series representation, and Theorem 3.5 of [11] implies that it is strongly positive.

Conversely, suppose that σ is not strongly positive. Lemma 3.4 and Theorem 3.5 of [11] show that then σ can be written as a subrepresentation of the induced representation of the form $\delta[\nu^a\chi_{V,\psi}\rho, \nu^b\chi_{V,\psi}\rho] \rtimes \sigma'$, where $a \leq 0$ and σ' is an irreducible representation. Applying Proposition 3.7 we obtain that τ is not strongly positive. Hence, the strong positivity of τ implies strong positivity of σ. This proves the theorem.

□

5 The conservation relation for discrete series subquotients

The objective of this section is to prove the conservation relation for discrete series which appear as irreducible subquotients of generalized principal series, where the representation on the metaplectic part is strongly positive.

Let σ denote such discrete series of $\widetilde{Sp(n)}$, i.e., $\sigma \leq \delta([\nu^x\chi_{V,\psi}\rho', \nu^y\chi_{V,\psi}\rho']) \rtimes \sigma_1$, where the representation σ_1 is strongly positive. Throughout this section we assume that σ is not strongly positive.

According to Theorem 3.5 from [11], there exists an embedding of the form

$$\sigma \hookrightarrow \delta([\nu^{-a_1}\chi_{V,\psi}\rho_1, \nu^{b_1}\chi_{V,\psi}\rho_1]) \times \cdots \times \delta([\nu^{-a_k}\chi_{V,\psi}\rho_k, \nu^{b_k}\chi_{V,\psi}\rho_k]) \rtimes \sigma_{sp},$$

where $a_i \geq 0$, $a_i + b_i > 0$ and ρ_i is an irreducible cuspidal representation of $GL(n_i, F)$ (this defines n_i) for $i = 1, 2, \ldots, k$, while σ_{sp} is a strongly positive discrete series of $\widetilde{Sp(m)}$ for some m. The assumption on σ obviously yields $k \geq 1$. Inspecting the cuspidal support of σ more closely, similarly as in Section 6 of [11], we deduce $k = 1$.

Thus, σ can be written as a subrepresentation of an induced representation of the form

$$\delta([\nu^{-a}\chi_{V,\psi}\rho, \nu^{b}\chi_{V,\psi}\rho]) \rtimes \sigma_{sp},$$

where $a \geq 0$, $b - a > 0$ and ρ is an irreducible cuspidal representation of $GL(n_\rho, F)$ (this defines n_ρ).

It is well-known that ρ has to be self-contragredient (details can be found in [22]). Let σ_{cusp} denote the partial cuspidal support of σ and let $s(\rho)$ denote a non–negative real number such that the induced representation $\nu^{s(\rho)}\chi_{V,\psi}\rho \rtimes \sigma_{cusp}$ reduces. In order to apply the inductive procedure for determining the first occurrence indices of σ, we shall consider several possibilities.

Let us first assume $2s(\rho) \notin \mathbb{Z}$.

Observe that either $a - s(\rho) \in \mathbb{Z}$ or $a + s(\rho) \in \mathbb{Z}$, since otherwise the induced representation $\delta([\nu^{-a}\chi_{V,\psi}\rho, \nu^{b}\chi_{V,\psi}\rho]) \rtimes \sigma_{sp}$ could not contain any discrete series subquotients (this also follows from [22]). In this case, the crucial requirement is given by the following lemma.

Lemma 5.1. *If $\delta([\nu^{-a}\chi_{V,\psi}\rho, \nu^{b}\chi_{V,\psi}\rho]) \otimes \sigma'$ appears in $\mu^*(\sigma)$ for some irreducible genuine representation σ', then $\sigma' \cong \sigma_{sp}$. Moreover, such representation is contained in Jacquet module of σ with the multiplicity one.*

Proof. It is the consequence of Lemma 2.2 that there exists some irreducible constituent $\pi \otimes \sigma''$ of $\mu^*(\sigma_{sp})$ and real numbers i, j satisfying the properties $i + a \in \mathbb{Z}$, $j - i \in \mathbb{Z}$, $-a - 1 \leq i \leq j \leq b$, such that

$$\delta([\nu^{-a}\chi_{V,\psi}\rho, \nu^{b}\chi_{V,\psi}\rho]) \leq \delta([\nu^{-i}\chi_{V,\psi}\rho, \nu^{a}\chi_{V,\psi}\rho]) \times \delta([\nu^{j+1}\chi_{V,\psi}\rho, \nu^{b}\chi_{V,\psi}\rho]) \times \pi$$

and $\sigma' \leq \delta([\nu^{i+1}\chi_{V,\psi}\rho, \nu^{j}\chi_{V,\psi}\rho]) \rtimes \sigma''$.

We determine all such i and j comparing the cuspidal supports. Since $a > 0$ and $2a \notin \mathbb{Z}$, it follows that $b - a$ is not an integer. Consequently, $\nu^{a}\chi_{V,\psi}\rho$ does not appear in the cuspidal support of $\delta([\nu^{-a}\chi_{V,\psi}\rho, \nu^{b}\chi_{V,\psi}\rho])$ so $i = -a - 1$.

The representation σ_{sp} is strongly positive and by Theorem 6.1 of [10] $\nu^{-a}\chi_{V,\psi}\rho$ does not appear in the cuspidal support of π. This leads to $j = i = -a - 1$. Thus, $\sigma' \cong \sigma'' \cong \sigma_{sp}$ and the lemma is proved. □

Suppose that σ_{sp} is a representation of $\widetilde{Sp(n')}$. Combining Proposition 3.7 with the previous lemma, it is not hard to see that $\Theta^{\epsilon}(\sigma, r) \neq 0$ if and only if $\Theta^{\epsilon}(\sigma_{sp}, r - n + n') \neq 0$, for $\epsilon \in \{+, -\}$. Moreover, if $\Theta^{\epsilon}(\sigma, r) \neq 0$, then

$$\sigma^{\epsilon}(r) \hookrightarrow \delta([\nu^{-a}\chi_{V,\psi}\rho, \nu^{b}\chi_{V,\psi}\rho]) \rtimes \sigma^{\epsilon}_{sp}(r - n + n').$$

So, $r^\epsilon(\sigma) = r^\epsilon(\sigma_{sp}) + n - n'$ and $r^+(\sigma) + r^-(\sigma) = r^+(\sigma_{sp}) + r^-(\sigma_{sp}) + 2n - 2n'$. Since σ_{sp} is strongly positive, using the results of the previous section we get $r^+(\sigma) + r^-(\sigma) = 2n$.

In the rest of this section we assume $2s(\rho) \in \mathbb{Z}$. First we will describe appropriate embeddings of the representation σ.

Let $\epsilon_1 \in \{+, -\}$ such that $\Theta^{\epsilon_1}(\sigma, n - t_{\epsilon_1}) \neq 0$. Theorem 3.3 implies that $\sigma^{\epsilon_1}(n - t_{\epsilon_1})$ is a discrete series and Theorem 4.5 shows that it is not strongly positive. The work of Mœglin and Tadić [12, 13], together with Theorem 3.9, enables us to obtain an embedding

$$\sigma^{\epsilon_1}(n - t_{\epsilon_1}) \hookrightarrow \delta([\nu^{-c_1}\rho, \nu^{d_1}\rho]) \times \cdots \times \delta([\nu^{-c_l}\rho, \nu^{d_l}\rho]) \rtimes \tau_{sp},$$

such that $c_i \geq 0$ and $c_i + d_i > 0$ for $i = 1, 2, \ldots, l$, τ_{sp} is a strongly positive discrete series,

$$R_{P_1}(\delta([\nu^{-c_i}\rho, \nu^{d_i}\rho]) \times \cdots \times \delta([\nu^{-c_l}\rho, \nu^{d_l}\rho]) \rtimes \tau_{sp})(\nu^x\rho) = 0,$$

for $c_{i-1} \leq x \leq d_{i-1}$ and $i = 2, 3, \ldots, l$, and $R_{P_1}(\tau_{sp})(\nu^x\rho) = 0$ for $c_l \leq x \leq d_l$. Using Theorem 3.9 again, we obtain $l \leq 2$ and $l = 1$ if ρ is not isomorphic to $1_{F^\times}$.

Let us first observe that l equals 1. Suppose, contrary to our claim, $l = 2$. This gives $\rho = 1_{F^\times}$. Using results of [13] again, we write $\sigma^{\epsilon_1}(n - t_{\epsilon_1})$ as a subrepresentation of $\delta([\nu^{-c_1}, \nu^{d_1}]) \rtimes \tau'$, where τ' is a discrete series representation, the subrepresentation of $\delta([\nu^{-c_2}, \nu^{d_2}]) \rtimes \tau_{sp}$. Obviously, $R_{P_1}(\tau')(\nu^x) = 0$ for $c_1 \leq x \leq d_1$.

The following lemma can be proved in the same way as Theorem 2.3 in [16] and uses the classical group version of Lemma 2.2.

Lemma 5.2. *If $\delta([\nu^{-c_1}, \nu^{d_1}]) \otimes \tau_1$ appears in $\mu^*(\sigma^{\epsilon_1}(n - t_{\epsilon_1}))$, for some irreducible representation τ_1, then $\tau_1 \cong \tau'$.*

It can be concluded in the same way that if $\mu^*(\tau') \geq \delta([\nu^{-c_2}, \nu^{d_2}]) \otimes \tau''$ for some irreducible representation τ'', then $\tau'' \cong \tau_{sp}$. Applying Proposition 3.8 two times, first for the representation $\sigma^{\epsilon_1}(n - t_{\epsilon_1})$ and then for the representation τ', we conclude that there is some irreducible representation σ' such that σ is a subrepresentation of the induced representation

$$\delta([\nu^{-c_1}\chi_{V,\psi}, \nu^{d_1}\chi_{V,\psi}]) \times \delta([\nu^{-c_2}\chi_{V,\psi}, \nu^{d_2}\chi_{V,\psi}]) \rtimes \sigma',$$

which is impossible since the representation $\nu^{\frac{1}{2}}\chi_{V,\psi}$ appears at most three times in the cuspidal support of σ.

Thus, $l = 1$. In this way we have shown that $\sigma^{\epsilon_1}(n - t_{\epsilon_1})$ can be written as an irreducible subrepresentation of the induced representation of the form

$$\delta([\nu^{-c}\rho, \nu^d\rho]) \rtimes \tau_{sp},$$

where $c > 0$, $-c + d > 0$ and $R_{P_1}(\tau_{sp})(\nu^x\rho) = 0$ for $c \leq x \leq d$.

Suppose that $\delta([\nu^{-c}\rho, \nu^d\rho])$ is a representation of $GL(l_1, F)$. In this notation, using Proposition 3.8 and calculations with Jacquet modules similar to those in Lemma 5.2, we obtain an embedding

$$\sigma \hookrightarrow \delta([\nu^{-c}\chi_{V,\psi}\rho, \nu^d\chi_{V,\psi}\rho]) \rtimes \tau_{sp}(n - l_1).$$

Observe that $\tau_{sp}(n - l_1)$ is also a strongly positive discrete series. For simplicity of notation we denote it by σ'_{sp}.

Description of the theta lifts of strongly positive discrete series given in the previous section shows that if there is some x such that $R_{P_1}(\tau_{sp})(\nu^x\rho) = 0$ and $R_{\widetilde{P_1}}(\sigma'_{sp})(\nu^x\chi_{V,\psi}\rho) \neq 0$ then $x < c$. We thus get $R_{\widetilde{P_1}}(\sigma'_{sp})(\nu^x\chi_{V,\psi}\rho) = 0$ for $c \leq x \leq d$.

Using the same calculations with Jacquet modules as before, we are in position to conclude that if there is some irreducible genuine representation σ' such that $\delta([\nu^{-c}\chi_{V,\psi}\rho, \nu^d\chi_{V,\psi}\rho]) \otimes \sigma'$ appears in $\mu^*(\sigma)$, then $\sigma' \cong \sigma'_{sp}$.

We have thus proved the following proposition:

Proposition 5.3. *The representation σ can be written as a subrepresentation of the induced representation of the form $\delta([\nu^{-c}\chi_{V,\psi}\rho, \nu^d\chi_{V,\psi}\rho]) \rtimes \sigma_{sp}$, where $c, d \geq 0$ and $\sigma_{sp} \in Irr(\widetilde{Sp(n')})$ is a strongly positive representation such that $R_{\widetilde{P_1}}(\sigma_{sp})(\nu^x\chi_{V,\psi}\rho) = 0$ for $c \leq x \leq d$. If $\mu^*(\sigma) \geq \delta([\nu^{-c}\chi_{V,\psi}\rho, \nu^d\chi_{V,\psi}\rho]) \otimes \sigma'$ for some irreducible genuine representation σ', then $\sigma' \cong \sigma_{sp}$.*

To each embedding $\sigma \hookrightarrow \delta([\nu^{-c}\chi_{V,\psi}\rho, \nu^d\chi_{V,\psi}\rho]) \rtimes \sigma_{sp}$ as in the previous proposition we attach a non-negative real number a_1 in the following way:

- If there are no twists of $\chi_{V,\psi}1_{F^\times}$ appearing in $[\sigma_{sp}]$, set $a_1 = 0$.
- If some twist of the representation $\chi_{V,\psi}1_{F^\times}$ appears in $[\sigma_{sp}]$, let a_1 denote the minimal x such that σ_{sp} is a subrepresentation of $\delta([\nu^x\chi_{V,\psi}, \nu^{b_1}\chi_{V,\psi}]) \rtimes \sigma'_{sp}$, where σ'_{sp} is strongly positive discrete series.

Among all such embeddings of σ we fix one with the minimal a_1 with respect to other embeddings and denote it by $\sigma \hookrightarrow \delta([\nu^{-c}\chi_{V,\psi}\rho, \nu^d\chi_{V,\psi}\rho]) \rtimes \sigma_{sp}$ again.

In the following proposition we describe the first occurrence indices of discrete series σ.

Proposition 5.4. *Suppose that* $\sigma_{sp} \in Irr(\widetilde{Sp(n')})$ *and let* $\epsilon_1 \in \{+,-\}$ *such that* $r^{\epsilon_1}(\sigma_{sp}) \leq n' - t_{\epsilon_1}$ *and* $\epsilon_2 \in \{+,-\}$ *different than* ϵ_1*. Then the following holds:*

1. *Suppose that* $\nu^{\frac{1}{2}}\chi_{V,\psi}$ *does not appear in* $[\sigma_{sp}]$ *and* $\nu^s\chi_{V,\psi} \rtimes \sigma_{cusp}$ *reduces for* $s > 0$*.*
   - *Suppose that* $a_1 = 0$*. If* $(c,\rho) = (s, 1_{F^\times})$ *and* $R_{\widetilde{P_1}}(\sigma)(\nu^s\chi_{V,\psi}) = 0$ *then* $r^{\epsilon_1}(\sigma) = n - s - \frac{1}{2} - t_{\epsilon_1}$ *and* $r^{\epsilon_2}(\sigma) = n + s + \frac{3}{2} - t_{\epsilon_2}$*. Otherwise* $r^{\epsilon_1}(\sigma) = n - s + \frac{1}{2} - t_{\epsilon_1}$ *and* $r^{\epsilon_2}(\sigma) = n + s + \frac{1}{2} - t_{\epsilon_2}$*.*
   - *Suppose that* $a_1 > 0$*. If* $(c,\rho) = (a_1 - 1, 1_{F^\times})$ *and* $a_1 > \frac{3}{2}$*, or* $(c,\rho) = (a_1 - 1, 1_{F^\times}) = (\frac{1}{2}, 1_{F^\times})$ *and* $R_{\widetilde{P_1}}(\sigma)(\nu^{\frac{1}{2}}\chi_{V,\psi}) = 0$*, then* $r^{\epsilon_1}(\sigma) = n - a_1 - t_{\epsilon_1} + \frac{1}{2}$ *and* $r^{\epsilon_2}(\sigma) = n + a_1 - t_{\epsilon_2} + \frac{1}{2}$*. Otherwise* $r^{\epsilon_1}(\sigma) = n - a_1 - t_{\epsilon_1} + \frac{3}{2}$ *and* $r^{\epsilon_2}(\sigma) = n + a_1 - t_{\epsilon_2} - \frac{1}{2}$*.*

2. *Suppose that* $\nu^{\frac{1}{2}}\chi_{V,\psi}$ *appears in* $[\sigma_{sp}]$*. If* $(c,\rho) = (\frac{1}{2}, 1_{F^\times})$ *and* $R_{\widetilde{P_1}}(\sigma)(\nu^{\frac{1}{2}}\chi_{V,\psi}) = 0$ *then* $r^{\epsilon_1}(\sigma) = n - t_{\epsilon_1} - 1$ *and* $r^{\epsilon_2}(\sigma) = n - t_{\epsilon_2} + 2$*. Otherwise* $r^{\epsilon_1}(\sigma) = n - t_{\epsilon_1}$ *and* $r^{\epsilon_2}(\sigma) = n - t_{\epsilon_2} + 1$*.*

Particularly, the conservation relation holds for σ*.*

The rest of this section will be devoted to the proof of Proposition 5.4.

First, if ρ is not equal $1_{F^\times}$, using the same reasoning as in the case of non-half integral reducibility, we obtain $\Theta^{\pm}(\sigma, r) \neq 0$ if and only if $\Theta^{\pm}(\sigma'_{sp}, r - n' + n) \neq 0$ and consequently $r^{\epsilon}(\sigma) = r^{\epsilon}(\sigma_{sp}) - n' + n$. Thus, in the rest of this section we may assume $\rho = 1_{F^\times}$.

This is the most difficult case. Observe that it is a consequence of Proposition 3.7 that $\Theta^{\epsilon_1}(\sigma, n - t_{\epsilon_1}) \neq 0$. Obviously, $\sigma^{\epsilon_1}(n - t_{\epsilon_1})$ is a discrete series representation which is not strongly positive.

In what follows the prominent role will be played by the following lemma:

Lemma 5.5. *Suppose that* σ *is a discrete series representation of* $\widetilde{Sp(n)}$*, given as a subrepresentation of* $\delta([\nu^{-a}\chi_{V,\psi}\rho, \nu^b\chi_{V,\psi}\rho]) \rtimes \sigma_1$*, where* $a \geq 0$*,* ρ *is a self-contragredient cuspidal representation and* σ_1 *is an irreducible representation. Also, assume that if* $\mu^*(\sigma)$ *contains* $\delta([\nu^{-a}\chi_{V,\psi}\rho, \nu^b\chi_{V,\psi}\rho]) \otimes \sigma_2$*, for some irreducible genuine representation* σ_2*, then* $\sigma_2 \cong \sigma_1$*. Further, let*

$(\rho, a) \neq (1_{F^\times}, n-r-\frac{1}{2}-t_\epsilon)$, $\Theta^\epsilon(\sigma, r) \neq 0$ and $R_{\widetilde{P_1}}(\sigma)(\nu^{-(r-\frac{1}{2}+t_\epsilon-n)}\chi_{V,\psi}) = 0$, for some $r > 0$ and $\epsilon \in \{+,-\}$. If $(\rho, a) \neq (1_{F^\times}, r-n-\frac{3}{2}+t_\epsilon)$ and $(\rho, b) \neq (1_{F^\times}, n-r-\frac{1}{2}-t_\epsilon)$ then $\Theta^\epsilon(\sigma_1, r-a-b-2) \neq 0$ implies $\Theta^\epsilon(\sigma, r-1) \neq 0$. If $(\rho, a) = (1_{F^\times}, r-n-\frac{3}{2}+t_\epsilon)$, then $\Theta^\epsilon(\sigma_1, r-a-b-2) \neq 0$ implies that either $\Theta^\epsilon(\sigma, r-1) \neq 0$ or $\Theta^\epsilon(\sigma_1, r-a-b-3) = 0$ holds.

Proof. We consider only the case $(\rho, a) = (1_{F^\times}, r-n-\frac{3}{2}+t_\epsilon)$, the other case can be proved in the same way. Applying Proposition 3.7, we obtain

$$\sigma^\epsilon(r) \hookrightarrow \delta([\nu^{-a}, \nu^b]) \rtimes \sigma_1^\epsilon(r-a-b-1).$$

Since $\Theta^\epsilon(\sigma_1, r-a-b-2) \neq 0$, Lemma 3.4 (ii) provides an embedding

$$\sigma^\epsilon(r) \hookrightarrow \delta([\nu^{-a}, \nu^b]) \times \nu^{-a-1} \rtimes \sigma_1^\epsilon(r-a-b-2).$$

The intertwining operator

$$\delta([\nu^{-a}, \nu^b]) \times \nu^{-a-1} \to \nu^{-a-1} \times \delta([\nu^{-a}, \nu^b]),$$

provides the following maps

$$\begin{aligned}\sigma^\epsilon(r) &\hookrightarrow \delta([\nu^{-a}, \nu^b]) \times \nu^{-a-1} \rtimes \sigma_1^\epsilon(r-a-b-2) \\ &\to \nu^{-a-1} \times \delta([\nu^{-a}, \nu^b]) \rtimes \sigma_1^\epsilon(r-a-b-2).\end{aligned}$$

If $\sigma^\epsilon(r)$ is not contained in the kernel of previous intertwining operator, then Lemma 3.4 (ii) yields $\Theta^\epsilon(\sigma, r-1) \neq 0$. Otherwise, $\sigma^\epsilon(r)$ is a subrepresentation of $\delta([\nu^{-a-1}, \nu^b]) \rtimes \sigma_1^\epsilon(r-a-b-2)$.

Suppose that $\Theta^\epsilon(\sigma_1, r-a-b-3) \neq 0$. Then, applying Lemma 3.4 (ii) again, we get the embedding $\sigma_1^\epsilon(r-a-b-2) \hookrightarrow \nu^{-a} \rtimes \sigma_1^\epsilon(r-a-b-3)$. This clearly forces

$$\sigma^\epsilon(r) \hookrightarrow \nu^{-a} \times \delta([\nu^{-a-1}, \nu^b]) \rtimes \sigma_1^\epsilon(r-a-b-3).$$

Thus, there is some irreducible representation τ such that $\sigma^\epsilon(r) \hookrightarrow \nu^{-a} \rtimes \tau$ and using Proposition 3.8 we obtain $R_{\widetilde{P_1}}(\sigma)(\nu^{-a}\chi_{V,\psi}) \neq 0$, which contradicts the square integrability of σ. This proves the lemma. □

We turn back to the investigation of the first occurrence indices. We shall consider several subcases because in each of them we have to use different methods. Also, the proof of Proposition 5.4 will be divided in the sequence of lemmas.

1. Suppose that the representation $\nu^{\frac{1}{2}}$ appears three times in the cuspidal support of $\sigma^{\epsilon_1}(n-t_{\epsilon_1})$.

   Proposition 3.7 gives

   $$\sigma^{\epsilon_1}(n-t_{\epsilon_1}) \hookrightarrow \delta([\nu^{-c},\nu^{d}]) \rtimes \sigma_{sp}^{\epsilon_1}(n'-t_{\epsilon_1})$$

   and results obtained in the previous section show that $\nu^{\frac{1}{2}}\chi_{V,\psi}$ does not appear in $[\sigma_{sp}]$ and $\nu^{s}\chi_{V,\psi} \rtimes \sigma_{cusp}$ reduces for $s > \frac{1}{2}$.

   We will denote the partial cuspidal support of $\sigma^{\epsilon_1}(n-t_{\epsilon_1})$ by τ_{cusp}.

   First, suppose $a_1 = 0$. Then $\sigma_{sp}^{\epsilon_1}(n'-t_{\epsilon_1})$ can be written as a unique irreducible subrepresentation of the induced representation of the form

   $$\nu^{\frac{1}{2}} \times \nu^{\frac{3}{2}} \times \cdots \times \nu^{s-1} \rtimes \tau_{sp_1}, \tag{3}$$

   where $\nu^{s-1} \rtimes \tau_{cusp}$ reduces and there are no twists of the trivial representation $1_{F^\times}$ appearing in the cuspidal support of strongly positive discrete series τ_{sp_1}. Using Proposition 2.1 from [13] we deduce $c \geq s$. Obviously, $d \geq s+1$.

   Since $c \geq s$, $\sigma^{\epsilon_1}(m)$ is a subrepresentation of $\delta([\nu^{-c},\nu^{d}]) \rtimes \sigma_{sp}^{\epsilon_1}(m-n+n')$ for $m \geq n-s+\frac{1}{2}-t_{\epsilon_1}$. Using Lemma 3.4, enhanced with the results of the previous section, we obtain $r^{\epsilon_1}(\sigma) \leq n-n'+r^{\epsilon_1}(\sigma_{sp})$.

   For $m \geq n-t_{\epsilon_2}$, $\sigma^{\epsilon_2}(m)$ is a subrepresentation of $\delta([\nu^{-c},\nu^{d}]) \rtimes \sigma_{sp}^{\epsilon_2}(m-n+n')$ if $\Theta^{\epsilon_2}(\sigma_{sp}, m-n+n') \neq 0$. Starting from some k such that $\Theta^{\epsilon_2}(\sigma,k) \neq 0$, using Lemma 5.5 and the same inductive procedure as before, we get $r^{\epsilon_2}(\sigma) \leq n-n'+r^{\epsilon_2}(\sigma_{sp})+1$.

   If $c \neq s$, Lemma 5.5 also leads to $r^{\epsilon_2}(\sigma) \leq n-n'+r^{\epsilon_2}(\sigma_{sp})$, and since we have already shown $r^{\epsilon_1}(\sigma_{sp})+r^{\epsilon_2}(\sigma_{sp}) = 2n'$, the inequality (1) implies $r^{\epsilon}(\sigma) = n-n'+r^{\epsilon}(\sigma_{sp})$ for $\epsilon \in \{+,-\}$.

   If $c = s$, we have $r^{\epsilon_1}(\sigma) \leq n-n'+r^{\epsilon_1}(\sigma_{sp})$ and $r^{\epsilon_2}(\sigma) \leq n-n'+r^{\epsilon_2}(\sigma_{sp})+1$. To shorten notation, write r_1 for $n-n'+r^{\epsilon_1}(\sigma_{sp})$. It is a simple matter to see that $\sigma^{\epsilon_1}(r_1)$ is a subrepresentation of $\delta([\nu^{-s},\nu^{d}]) \rtimes \tau_{sp_1}$. Note that the representation $\nu^{-s} \rtimes \tau_{cusp}$ is irreducible and τ_{sp_1} can be written as a subrepresentation of $\pi \rtimes \tau_{cusp}$, where π is induced from the essentially square integrable representations attached to the segments not containing twists of $1_{F^\times}$. Therefore, we have the following embeddings and isomorphisms:

$$\begin{aligned}\sigma^{\epsilon_1}(r_1) &\hookrightarrow \delta([\nu^{-s}, \nu^d]) \rtimes \tau_{sp_1}\\ &\hookrightarrow \delta([\nu^{-s+1}, \nu^d]) \times \nu^{-s} \times \pi \rtimes \tau_{cusp}\\ &\cong \delta([\nu^{-s+1}, \nu^d]) \times \pi \times \nu^{-s} \rtimes \tau_{cusp}\\ &\cong \nu^s \times \delta([\nu^{-s+1}, \nu^d]) \times \pi \rtimes \tau_{cusp},\end{aligned}$$

showing $R_{P_1}(\sigma^{\epsilon_1}(r_1))(\nu^s) \neq 0$.

If $R_{\widetilde{P_1}}(\sigma)(\nu^s\chi_{V,\psi}) = 0$, Lemma 3.4 (ii) implies $r^{\epsilon_1}(\sigma) \leq r_1 - 1$. Again, the inequality (1) gives the first occurrence indices.

If $R_{\widetilde{P_1}}(\sigma)(\nu^s\chi_{V,\psi}) \neq 0$, then there exists some irreducible genuine representation σ' of $\widetilde{Sp(n-1)}$ such that $\sigma \hookrightarrow \nu^s\chi_{V,\psi} \rtimes \sigma'$. Since the representation σ is square integrable, it is not hard to see that σ' has to be tempered. If σ' is not square integrable, then it has to be a subrepresentation of the induced representation of the form $\delta([\nu^{-s+1}\chi_{V,\psi}, \nu^{s-1}\chi_{V,\psi}]) \rtimes \sigma'_{sp}$ where σ'_{sp} is the strongly positive representation given as the unique irreducible subrepresentation of $\delta([\nu^s\chi_{V,\psi}, \nu^d\chi_{V,\psi}]) \rtimes \sigma_{sp}$. On the other hand, if σ' is a discrete series representation, then it has to be a subrepresentation of $\delta([\nu^{-s+1}\chi_{V,\psi}, \nu^d\chi_{V,\psi}]) \rtimes \sigma_{sp}$. In both cases, using the structure formula, we see at once that if $\mu^*(\sigma)$ contains $\nu^s\chi_{V,\psi} \otimes \sigma''$, for some irreducible genuine representation σ'', then $\sigma'' \cong \sigma'$. The analogous properties hold for listed embeddings of σ'. Applying Proposition 3.7 and the same inductive procedure as before, we obtain that one of the following holds:

$$\sigma^{\epsilon_2}(n - n' + r^{\epsilon_2}(\sigma_{sp}) + 1)) \hookrightarrow \nu^s \times \delta([\nu^{-s+1}, \nu^{s-1}]) \times \delta([\nu^s, \nu^d]) \rtimes \sigma^{\epsilon_2}_{sp}(r^{\epsilon_2}(\sigma_{sp}) + 1),$$
$$\sigma^{\epsilon_2}(n - n' + r^{\epsilon_2}(\sigma_{sp}) + 1)) \hookrightarrow \nu^s \times \delta([\nu^{-s+1}, \nu^d]) \rtimes \sigma^{\epsilon_2}_{sp}(r^{\epsilon_2}(\sigma_{sp}) + 1).$$

Lemma 3.4 (ii) shows that $\sigma^{\epsilon_2}_{sp}(r^{\epsilon_2}(\sigma_{sp}) + 1)$ is a subrepresentation of $\nu^{-s-1} \rtimes \sigma^{\epsilon_2}_{sp}(r^{\epsilon_2}(\sigma_{sp}))$, and this clearly forces $R_{P_1}(\sigma^{\epsilon_2}(n-n'+r^{\epsilon_2}(\sigma_{sp})+1))(\nu^{-s-1}) \neq 0$. Applying Lemma 3.4 (ii) one more time, we get $r^{\epsilon_2}(\sigma) \leq n - n' + r^{\epsilon_2}(\sigma_{sp})$ and the inequality (1) yields the desired first occurrence indices.

Let us now assume $a_1 > 0$. Then the results obtained in the previous section (as well as those in the sixth section of the paper [11]) show that

$\sigma_{sp}^{\epsilon_1}(n' - t_{\epsilon_1})$ is the unique irreducible subrepresentation of the induced representation of the form

$$\nu^{\frac{1}{2}} \times \cdots \times \nu^{a_1-2} \times (\prod_{j=1}^{k} \delta([\nu^{a_j-1}, \nu^{b_j}])) \rtimes \tau_{sp_1}, \tag{4}$$

where $a_j \leq b_j$ for $j = 1, 2, \ldots, k$, $a_{j-1} = a_j - 1$, $b_{j-1} < b_j$ for $j = 2, 3, \ldots, k$ and τ_{sp_1} is a strongly positive discrete series having no twists of $1_{F^\times}$ in the cuspidal support. Proposition 2.1 from [13] gives $c \geq a_1 - 1$, $b_j \neq c$ and $b_j \neq d$ for $j = 1, 2, \ldots, k$.

Also, in this case, the strongly positive representation σ_{sp} is the unique irreducible subrepresentation of

$$(\prod_{j=1}^{k} \delta([\nu^{a_j} \chi_{V,\psi}, \nu^{b_j} \chi_{V,\psi}])) \rtimes \sigma_{sp_1},$$

and there are no twists of $\chi_{V,\psi} 1_{F^\times}$ appearing in the cuspidal support of strongly positive discrete series σ_{sp_1}.

Lemma 5.6. *Assume that $(a_1, c) \neq (\frac{3}{2}, \frac{1}{2})$. If $c \neq a_1 - 1$, then $r^{\epsilon_1}(\sigma) = n - n' + r^{\epsilon_1}(\sigma_{sp})$ and $r^{\epsilon_2}(\sigma) = n - n' + r^{\epsilon_2}(\sigma_{sp})$. Otherwise, $r^{\epsilon_1}(\sigma) = n - n' + r^{\epsilon_1}(\sigma_{sp}) - 1$ and $r^{\epsilon_2}(\sigma) = n - n' + r^{\epsilon_2}(\sigma_{sp}) + 1$.*

Proof. We comment only the case $c = a_1 - 1$, the other case can be proved in the same way, but more easily. Note that in this case $b_1 > d$. In Theorem 4.2 and discussion preceding it, we have proved $r^{\epsilon_1}(\sigma_{sp}) = n' - a_1 - t_{\epsilon_1} + \frac{3}{2}$ and $r^{\epsilon_2}(\sigma_{sp}) = n' + a_1 - \frac{1}{2} - t_{\epsilon_2}$.

For $n - a_1 - t_{\epsilon_1} + \frac{5}{2} \leq m \leq n - t_{\epsilon_1}$, Proposition 3.7 shows that $\sigma^{\epsilon_1}(m)$ is a subrepresentation of $\delta([\nu^{-a_1+1}, \nu^d]) \rtimes \sigma_{sp}^{\epsilon_1}(m - n + n')$. Using Lemma 3.4 (ii) we obtain

$$\begin{aligned}\sigma^{\epsilon_1}(m) &\hookrightarrow \delta([\nu^{-a_1+1}, \nu^d]) \times \nu^{-(m-n-\frac{1}{2}+t_{\epsilon_1})} \rtimes \sigma_{sp}^{\epsilon_1}(m - n + n' - 1) \\ &\cong \nu^{-(m-n-\frac{1}{2}+t_{\epsilon_1})} \times \delta([\nu^{-a_1+1}, \nu^d]) \rtimes \sigma_{sp}^{\epsilon_1}(m - n + n' - 1),\end{aligned}$$

where the last isomorphism holds since $0 < -(m-n-\frac{1}{2}+t_{\epsilon_1}) < a_1 - 1 < d$. Using Lemma 3.4 (ii) again, we conclude $r^{\epsilon_1}(\sigma) \leq n - a_1 - t_{\epsilon_1} + \frac{3}{2}$.

Using completely analogous arguments, we conclude that $\Theta^{\epsilon_2}(\sigma, n + a_1 + \frac{1}{2} - t_{\epsilon_2}) \neq 0$ and

$$\sigma^{\epsilon_2}(n + a_1 + \frac{1}{2} - t_{\epsilon_2}) \hookrightarrow \delta([\nu^{-a_1+1}, \nu^d]) \times \nu^{-a_1} \rtimes \sigma_{sp}^{\epsilon_2}(n' + a_1 - \frac{1}{2} - t_{\epsilon_2}).$$

To obtain the first occurrence indices we use a slight variation of the method used in [16], Section 10.

Let $Jord_{1_{F^\times}} = \{\frac{1}{2}, \frac{3}{2}, \ldots, a_1 - 2, a_1 - 1, d, b_1, b_2, \ldots, b_k\}$. Further, let $Jord'_{1_{F^\times}} = \{(x, y) : x \in Jord_{1_{F^\times}}, y \in Jord_{1_{F^\times}}, x < y\}$. For $x \in Jord_{1_{F^\times}}$, $x > \frac{1}{2}$, we denote by x_- an element of $Jord_{1_{F^\times}}$ with the property that $\{z \in Jord_{1_{F^\times}} : x_- < z < x\} \cap Jord_{1_{F^\times}} = \emptyset$.

Since $\sigma^{\epsilon_1}(n - t_{\epsilon_1})$ is the discrete series representation on whose non-strongly positive part appear only twists of the trivial representation $1_{F^\times}$, we attach to $\sigma^{\epsilon_1}(n - t_{\epsilon_1})$ a function $\epsilon : Jord_{1_{F^\times}} \cup Jord'_{1_{F^\times}} \rightarrow \{1, -1\}$ defined in the following way (analogously as in the Mœglin - Tadić classification):

- Define $\epsilon(\frac{1}{2}) = 1$ if $R_{P_1}(\nu^{\frac{1}{2}})(\sigma^{\epsilon_1}(n - t_{\epsilon_1})) \neq 0$, and $\epsilon(\frac{1}{2}) = -1$ otherwise.
- For $x \in Jord_{1_{F^\times}}$, $x > \frac{1}{2}$, let $\epsilon(x_-, x) = 1$ if there is some irreducible representation τ_1 such that $\sigma^{\epsilon_1}(n - t_{\epsilon_1})$ is the subrepresentation of the induced representation of the form $\delta([\nu^{x_-+1}, \nu^x]) \rtimes \tau_1$. Further, let $\epsilon(x) = \epsilon(x_-) \cdot \epsilon(x_-, x)$, for $x > \frac{1}{2}$.
- For $x, y \in Jord_{1_{F^\times}}$ such that $x < y$ and $x \neq y_-$, let $\epsilon(x, y) = \epsilon(x) \cdot \epsilon(y)$.

Actually, in the definition given in [12, 13] the ϵ–function takes values on the elements of the form $2x + 1$ or of the form $(2x + 1, 2y + 1)$, for $x, y \in Jord_{1_{F^\times}}$, $x \neq y$, but we find this modification more appropriate to our situation.

It is proved in [12] that $\epsilon(x, y) = \epsilon(x, z)\epsilon(z, y)$ holds for all $x, y, z \in Jord_{1_{F^\times}}$, $x < z < y$.

Obviously, $\epsilon(a_1 - 1, d) = 1$. Suppose that $\epsilon(a_1 - 2, a_1 - 1) = 1$. Then, in the same way as in Lemma 5.1 of [13], we get

$$\sigma^{\epsilon_1}(n - t_{\epsilon_1}) \hookrightarrow \delta([\nu^{-a_1+2}, \nu^{a_1-1}]) \rtimes \tau',$$

where τ' is discrete series. Inspecting the cuspidal support of τ' we deduce that it is the strongly positive representation which can be written as the unique irreducible subrepresentation of the induced representation

$$\nu^{\frac{1}{2}} \times \cdots \times \nu^{a_1-3} \times \delta([\nu^{a_1-2}, \nu^{d}]) \times (\prod_{j=1}^{k} \delta([\nu^{a_j-1}, \nu^{b_j}])) \rtimes \tau_{sp_1}.$$

Observe that $R_{P_1}(\tau')(\nu^x) = 0$ for $-a_1 + 2 \leq x \leq a_1 - 1$. Using Proposition 3.8 we obtain an embedding $\sigma \hookrightarrow \delta([\nu^{-a_1+2}\chi_{V,\psi}, \nu^{a_1-1}\chi_{V,\psi}]) \rtimes \tau'(n - 2a_1 + 2)$. Results obtained in the previous section show that $\tau'(n - 2a_1 + 2)$ is a strongly positive discrete series that can be written as a subrepresentation of the induced representation of the form $\delta([\nu^{a_1-1}\chi_{V,\psi}, \nu^{d}\chi_{V,\psi}]) \rtimes \sigma'_{sp}$ for some strongly positive discrete series σ'_{sp}, contradicting the minimality of a_1. Consequently, $\epsilon(a_1-2, a_1-1) = -1$. This gives $\epsilon(a_1 - 2, d) = -1$.

It follows from [12], Sections 5 and 6, that $\epsilon(a_1-2, b_1) = -1$. Therefore, $\epsilon(d, b_1) = 1$. It is now easy to obtain an embedding

$$\sigma^{\epsilon_1}(n - t_{\epsilon_1}) \hookrightarrow \delta([\nu^{-d}, \nu^{b_1}]) \rtimes \tau'',$$

where τ'' is a strongly positive discrete series given as the unique irreducible subrepresentation of the induced representation

$$\nu^{\frac{1}{2}} \times \cdots \times \nu^{a_1-1} \times (\prod_{j=2}^{k} \delta([\nu^{a_j-1}, \nu^{b_j}])) \rtimes \tau_{sp}.$$

Theorem 6.1 from [10] leads to $R_{P_1}(\tau'')(\nu^x) = 0$ for $d \leq x \leq b_1$. Using Proposition 3.8 and results obtained in the previous section, we deduce that σ can be written as a subrepresentation of $\delta([\nu^{-d}\chi_{V,\psi}, \nu^{b_1}\chi_{V,\psi}]) \rtimes \sigma_{sp_2}$, where $\sigma_{sp_2} \in Irr(\widetilde{Sp(n'')})$ is the strongly positive subrepresentation of

$$(\prod_{j=2}^{k} \delta([\nu^{a_j}\chi_{V,\psi}, \nu^{b_j}\chi_{V,\psi}])) \rtimes \sigma_{sp_1}.$$

Since $a_2 = a_1 + 1$, it follows that $r^{\epsilon_2}(\sigma)$ equals $n + a_1 + \frac{1}{2} - t_{\epsilon_2}$, because otherwise there will be an embedding

$$\sigma^{\epsilon_2}(n + a_1 - \frac{1}{2} - t_{\epsilon_2}) \hookrightarrow \delta([\nu^{-d}, \nu^{b_1}]) \rtimes \sigma^{\epsilon_2}_{sp_2}(n'' + a_1 - \frac{1}{2} - t_{\epsilon_2}),$$

which is impossible since $r^{\epsilon_2}(\sigma_{sp_2}) = n'' + a_1 + \frac{1}{2} - t_{\epsilon_2}$ by Theorem 4.2. On the other hand, from Lemma 2.2 and [10], Theorem 6.1, we conclude $R_{\widetilde{P_1}}(\sigma)(\nu^{a_1-1}\chi_{V,\psi}) = 0$.

If $k = 1$, then τ'' is a subrepresentation of the representation of the form (3). Therefore $k \geq 2$. Further,

$$\sigma^{\epsilon_1}(n - t_{\epsilon_1} - a_1 + \frac{3}{2}) \hookrightarrow \delta([\nu^{-d}, \nu^{b_1}]) \rtimes \sigma^{\epsilon_1}_{sp_2}(n'' - t_{\epsilon_1} - a_1 + \frac{3}{2}).$$

But $r^{\epsilon_1}(\sigma_{sp_2}) = n'' - t_{\epsilon_1} - a_1 + \frac{1}{2}$, so, by Corollary 4.4, $\sigma^{\epsilon_1}_{sp_2}(n'' - t_{\epsilon_1} - a_1 + \frac{3}{2})$ is a subrepresentation of $\nu^{a_1-1} \rtimes \sigma^{\epsilon_1}_{sp_2}(n'' - t_{\epsilon_1} - a_1 + \frac{1}{2})$. Further, $\delta([\nu^{-d}, \nu^{b_1}]) \times \nu^{a_1-1}$ is isomorphic to $\nu^{a_1-1} \times \delta([\nu^{-d}, \nu^{b_1}])$, since $a_1 - 1 < d$. It follows that $R_{P_1}(\sigma^{\epsilon_1}(n - t_{\epsilon_1} - a_1 + \frac{3}{2}))(\nu^{a_1-1}) \neq 0$. Lemma 3.4 (ii) yields $r^{\epsilon_1}(\sigma) \leq n - t_{\epsilon_1} - a_1 + \frac{1}{2}$ and inequality (1) completes the proof. □

The following lemma completes the determination of the first occurrence indices in this subcase.

Lemma 5.7. *Suppose $a_1 = \frac{3}{2}$ and $c = \frac{1}{2}$. Then either $(r^{\epsilon_1}(\sigma), r^{\epsilon_2}(\sigma)) = (n - t_{\epsilon_1}, n - t_{\epsilon_2} + 1)$ or $(r^{\epsilon_1}(\sigma), r^{\epsilon_2}(\sigma)) = (n - t_{\epsilon_1} - 1, n - t_{\epsilon_2} + 2)$ holds. The second possibility occurs when $R_{\widetilde{P_1}}(\sigma)(\nu^{\frac{1}{2}}\chi_{V,\psi}) = 0$.*

Proof. It is not hard to deduce, in the same way as in the proof of the previous lemma, that $\Theta^{\epsilon_1}(\sigma, n - t_{\epsilon_1}) \neq 0$ and $\Theta^{\epsilon_2}(\sigma, n - t_{\epsilon_2} + 2) \neq 0$. We note that σ_{sp} can be written as an irreducible subrepresentation of $\delta([\nu^{\frac{3}{2}}\chi_{V,\psi}, \nu^{b_1}\chi_{V,\psi}]) \times \pi \rtimes \sigma_{cusp}$, where the representation π is induced from essentially square integrable representations attached to the segments that do not contain the representation $\nu^{\frac{3}{2}}\chi_{V,\psi}$.

We examine the following embeddings of σ (note that the representation $\nu^{\frac{1}{2}}\chi_{V,\psi} \rtimes \sigma_{cusp}$ is irreducible):

$$\begin{aligned}
\sigma &\hookrightarrow \delta([\nu^{-\frac{1}{2}}\chi_{V,\psi}, \nu^{d}\chi_{V,\psi}]) \times \delta([\nu^{\frac{3}{2}}\chi_{V,\psi}, \nu^{b_1}\chi_{V,\psi}]) \times \pi \rtimes \sigma_{cusp} \\
&\hookrightarrow \delta([\nu^{\frac{1}{2}}\chi_{V,\psi}, \nu^{d}\chi_{V,\psi}]) \times \delta([\nu^{\frac{3}{2}}\chi_{V,\psi}, \nu^{b_1}\chi_{V,\psi}]) \times \pi \times \nu^{-\frac{1}{2}}\chi_{V,\psi} \rtimes \sigma_{cusp} \\
&\cong \delta([\nu^{\frac{1}{2}}\chi_{V,\psi}, \nu^{d}\chi_{V,\psi}]) \times \delta([\nu^{\frac{3}{2}}\chi_{V,\psi}, \nu^{b_1}\chi_{V,\psi}]) \times \nu^{\frac{1}{2}}\chi_{V,\psi} \times \pi \rtimes \sigma_{cusp} \\
&\to \delta([\nu^{\frac{1}{2}}\chi_{V,\psi}, \nu^{d}\chi_{V,\psi}]) \times \nu^{\frac{1}{2}}\chi_{V,\psi} \times \delta([\nu^{\frac{3}{2}}\chi_{V,\psi}, \nu^{b_1}\chi_{V,\psi}]) \times \pi \rtimes \sigma_{cusp}.
\end{aligned}$$

If σ is contained in the kernel of the last intertwining operator then it is a subrepresentation of $\delta([\nu^{\frac{1}{2}}\chi_{V,\psi}, \nu^{b_1}\chi_{V,\psi}]) \rtimes \sigma_1$, for some irreducible genuine representation σ_1. In the same way as in Section 5 of [12] and Lemma 5.1 of [13], we obtain that there is a discrete series $\sigma_{ds} \in Irr(\widetilde{Sp(n'')})$ such that σ is a subrepresentation of the induced representation of the form $\delta([\nu^{-d}\chi_{V,\psi}, \nu^{b_1}\chi_{V,\psi}]) \rtimes \sigma_{ds}$. Inspecting the cuspidal support of σ, we deduce that σ_{ds} is a strongly positive representation and it can be characterized as the unique irreducible subrepresentation of $\pi \rtimes \sigma_{cusp}$.

Using Lemma 2.2 we conclude $R_{\widetilde{P_1}}(\sigma)(\nu^{\frac{1}{2}}\chi_{V,\psi}) = 0$ and, since $r^{\epsilon_1}(\sigma_{ds}) = n'' - t_{\epsilon_1} - 1$,

$$\sigma^{\epsilon_1}(n - t_{\epsilon_1}) \hookrightarrow \delta([\nu^{-d}, \nu^{b_1}]) \times \nu^{\frac{1}{2}} \rtimes \sigma_{ds}^{\epsilon_1}(n'' - t_{\epsilon_1} - 1).$$

Since $b_1 > \frac{1}{2}$, Lemma 3.4 (ii) and the inequality (1) show $r^{\epsilon_1}(\sigma) = n - t_{\epsilon_1} - 1$.

Now we suppose that σ is not contained in the kernel of the above intertwining operator. In that case $R_{\widetilde{P_1}}(\sigma)(\nu^{\frac{1}{2}}\chi_{V,\psi}) \neq 0$. Thus, there is an irreducible genuine representation σ_2 of $\widetilde{Sp(n-1)}$ such that $\sigma \hookrightarrow \nu^{\frac{1}{2}}\chi_{V,\psi} \rtimes \sigma_2$. We claim that σ_2 is square integrable. Otherwise, there would be an embedding $\sigma_2 \hookrightarrow \delta([\nu^x\chi_{V,\psi}\rho, \nu^y\chi_{V,\psi}\rho]) \rtimes \sigma_3$, where $x+y \leq 0$. Consequently, σ would either be a subrepresentation of $\delta([\nu^x\chi_{V,\psi}\rho, \nu^y\chi_{V,\psi}\rho]) \times \nu^{\frac{1}{2}}\chi_{V,\psi} \rtimes \sigma_3$ or a subrepresentation of $\delta([\nu^x\chi_{V,\psi}\rho, \nu^{\frac{1}{2}}\chi_{V,\psi}\rho]) \rtimes \sigma_3$, contradicting the square integrability of σ.

Looking at the cuspidal support of σ we conclude that σ_2 is strongly positive; moreover, it is a subrepresentation of $\delta([\nu^{\frac{1}{2}}\chi_{V,\psi}, \nu^d\chi_{V,\psi}]) \rtimes \sigma_{sp}$. Further, if $\mu^*(\sigma)$ contains $\nu^{\frac{1}{2}}\chi_{V,\psi} \otimes \sigma'$ for some irreducible genuine representation σ' of $\widetilde{Sp(n-1)}$, then $\sigma' \cong \sigma_2$. Since $r^{\epsilon_2}(\sigma_2) < n - t_{\epsilon_2} + 1$, Proposition 3.7 and Lemma 3.4 (ii) yield

$$\begin{aligned}\sigma^{\epsilon_2}(n - t_{\epsilon_2} + 2) &\hookrightarrow \nu^{\frac{1}{2}} \times \nu^{-\frac{3}{2}} \rtimes \sigma_2^{\epsilon_2}(n - t_{\epsilon_2}) \\ &\cong \nu^{-\frac{3}{2}} \times \nu^{\frac{1}{2}} \rtimes \sigma_2^{\epsilon_2}(n - t_{\epsilon_2}).\end{aligned}$$

Lemma 3.4 (ii) and the inequality (1) now imply $r^{\epsilon_2}(\sigma) = n - t_{\epsilon_2} + 1$, and the lemma is proved. □

2. Suppose that the representation $\nu^{\frac{1}{2}}$ appears two times in the cuspidal support of $\sigma^{\epsilon_1}(n-t_{\epsilon_1})$ and the representation $\nu^{\frac{1}{2}}\chi_{V,\psi}$ appears two times in the cuspidal support of σ.

   In this case, σ is an irreducible subrepresentation of the induced representation of the form $\delta([\nu^{-c}\chi_{V,\psi}, \nu^{d}\chi_{V,\psi}]) \rtimes \sigma_{sp}$, where σ_{sp} is a strongly positive representation without any twists of $\chi_{V,\psi}1_{F^\times}$ in its cuspidal support.

   Similarly, $\sigma^{\epsilon_1}(n-t_{\epsilon_1})$ is a subrepresentation of the induced representation of the form $\delta([\nu^{-c}, \nu^{d}]) \rtimes \tau_{sp}$, where τ_{sp} is a strongly positive discrete series such that there are no twists of the representation $1_{F^\times}$ appearing in its cuspidal support. We again denote the partial cuspidal support of $\sigma^{\epsilon_1}(n-t_{\epsilon_1})$ by τ_{cusp} and note that in this case both representations $\nu^{\frac{1}{2}}\chi_{V,\psi} \rtimes \sigma_{cusp}$ and $\nu^{\frac{1}{2}} \rtimes \tau_{cusp}$ reduce.

   There are again several cases to discuss.

   The following lemma follows directly from the results of the previous section, together with Proposition 3.7 and Lemma 5.5, the detailed verification being left to the reader.

   Lemma 5.8. *If $c \neq \frac{1}{2}$, then $r^{\epsilon_1}(\sigma) = n - t_{\epsilon_1}$ and $r^{\epsilon_2}(\sigma) = n - t_{\epsilon_2} + 1$.*

   The first exceptional case is discussed in the following lemma.

   Lemma 5.9. *If $c = \frac{1}{2}$ and $R_{\widetilde{P_1}}(\sigma)(\nu^{\frac{1}{2}}\chi_{V,\psi}) \neq 0$, then $r^{\epsilon_1}(\sigma) = n - t_{\epsilon_1}$ and $r^{\epsilon_2}(\sigma) = n - t_{\epsilon_2} + 1$.*

   Proof. Using an inductive approach, as before, we obtain $\Theta^{\epsilon_1}(\sigma, n - t_{\epsilon_1}) \neq 0$ and $\Theta^{\epsilon_2}(\sigma, n - t_{\epsilon_2} + 2) \neq 0$. Condition $R_{\widetilde{P_1}}(\sigma)(\nu^{\frac{1}{2}}\chi_{V,\psi}) \neq 0$ leads to an embedding $\sigma \hookrightarrow \nu^{\frac{1}{2}}\chi_{V,\psi} \rtimes \sigma_{sp_1}$, where σ_{sp_1} is a strongly positive representation of $\widetilde{Sp(n-1)}$, which is the unique irreducible subrepresentation of

$$\delta([\nu^{\frac{1}{2}}\chi_{V,\psi}, \nu^{d}\chi_{V,\psi}]) \rtimes \sigma_{sp}.$$

   Proposition 3.7 shows

$$\sigma^{\epsilon_2}(n - t_{\epsilon_2} + 2) \hookrightarrow \nu^{\frac{1}{2}} \rtimes \sigma_{sp_1}^{\epsilon_2}(n - t_{\epsilon_2} + 1).$$

The last case discussed in Section 4 gives $\Theta^{\epsilon_2}(\sigma_{sp_1}, n-t_{\epsilon_2}) \neq 0$. Consequently, $R_{P_1}(\sigma_{sp_1}^{\epsilon_2}(n-t_{\epsilon_2}+1))(\nu^{-\frac{3}{2}}) \neq 0$ and using Lemma 3.4 (ii) we deduce $r^{\epsilon_2}(\sigma) \leq n - t_{\epsilon_2} + 1$. The inequality (1) proves the lemma.

Observe that we have also proved $R_{P_1}(\sigma^{\epsilon_1}(n-t_{\epsilon_1}))(\nu^{\frac{1}{2}}) = 0$. Otherwise it can be seen, similarly as in Proposition 3.7 or as in Remark 5.2 from [16], that there is an embedding

$$\sigma^{\epsilon_1}(n-t_{\epsilon_1}) \hookrightarrow \nu^{\frac{1}{2}} \rtimes \sigma_{sp_1}^{\epsilon_1}(n-t_{\epsilon_1}-1),$$

but in the previous section we have seen that $r^{\epsilon_1}(\sigma_{sp_1})$ equals $n-t_{\epsilon_1}$. □

Lemma 5.10. *If $c = \frac{1}{2}$ and $R_{\widetilde{P_1}}(\sigma)(\nu^{\frac{1}{2}}\chi_{V,\psi}) = 0$, then $r^{\epsilon_1}(\sigma) = n - t_{\epsilon_1} - 1$ and $r^{\epsilon_2}(\sigma) = n - t_{\epsilon_2} + 2$.*

Proof. First, in the same way as before we conclude $r^{\epsilon_1}(\sigma) \leq n - t_{\epsilon_1}$ and $r^{\epsilon_2}(\sigma) \leq n - t_{\epsilon_2} + 2$.

By [13], or Theorem 2.1 of [15], the induced representation $\delta([\nu^{-\frac{1}{2}}, \nu^d]) \rtimes \tau_{sp}$ has exactly two irreducible subrepresentations, which are both square integrable. These representations will be denoted by τ_{ds_1} and τ_{ds_2}. By the Mœglin-Tadić classification of discrete series, there exists exactly one $i \in \{1,2\}$ such that $R_{P_1}(\tau_{ds_i})(\nu^{\frac{1}{2}}) \neq 0$.

Since the induced representation $\delta([\nu^{-\frac{1}{2}}\chi_{V,\psi}, \nu^d\chi_{V,\psi}]) \rtimes \sigma_{sp}$ also has exactly two irreducible subrepresentations, observation made in the proof of the previous lemma and injective correspondence proved in [3] and recalled in Theorem 3.3 imply $R_{P_1}(\sigma^{\epsilon_1}(n-t_{\epsilon_1}))(\nu^{\frac{1}{2}}) \neq 0$. Lemma 3.4 (ii) now gives $r^{\epsilon_1}(\sigma) < n - t_{\epsilon_1}$ and inequality (1) completes the proof. □

3. Suppose that the representation $\nu^{\frac{1}{2}}\chi_{V,\psi}$ appears three times in the cuspidal support of σ.

   We again denote the partial cuspidal support of $\sigma^{\epsilon_1}(n-t_{\epsilon_1})$ by τ_{cusp}. Results from the previous section show that the representation $\nu^{\frac{1}{2}} \rtimes \tau_{cusp}$ is irreducible.

   Under the above assumption, σ_{sp} is the unique irreducible subrepresentation of $\delta([\nu^{\frac{1}{2}}\chi_{V,\psi}, \nu^b\chi_{V,\psi}]) \rtimes \sigma_{sp_1}$, where σ_{sp_1} is a strongly positive representation and $\nu^{\frac{1}{2}}\chi_{V,\psi}$ does not appear in the cuspidal support of σ_{sp_1}.

Observe that $r^{\epsilon_1}(\sigma_{sp}) = n' - t_{\epsilon_1}$ and $r^{\epsilon_2}(\sigma_{sp}) = n' + 1 - t_{\epsilon_2}$.

We start with the lemma which can be proved by repeated application of Proposition 3.7 and Lemma 3.4, enhanced by Lemma 5.5:

Lemma 5.11. *If $c \neq \frac{1}{2}$, then $r^{\epsilon_1}(\sigma) = n - t_{\epsilon_1}$ and $r^{\epsilon_2}(\sigma) = n + 1 - t_{\epsilon_2}$.*

Remaining case is covered by the following lemma:

Lemma 5.12. *If $c = \frac{1}{2}$, then either $(r^{\epsilon_1}(\sigma), r^{\epsilon_2}(\sigma)) = (n - t_{\epsilon_1} - 1, n - t_{\epsilon_2} + 2)$ or $(r^{\epsilon_1}(\sigma), r^{\epsilon_2}(\sigma)) = (n - t_{\epsilon_1}, n - t_{\epsilon_2} + 1)$ holds. The second possibility appears when $R_{\widetilde{P_1}}(\sigma)(\nu^{\frac{1}{2}}\chi_{V,\psi}) \neq 0$.*

Proof. First we recall that $r^{\epsilon_1}(\sigma) \leq n - t_{\epsilon_1}$. Also, in the standard way one obtains $r^{\epsilon_2}(\sigma) \leq n - t_{\epsilon_2} + 2$. Suppose $R_{\widetilde{P_1}}(\sigma)(\nu^{\frac{1}{2}}\chi_{V,\psi}) \neq 0$. Then there is some irreducible genuine representation σ_{ds} such that σ is a subrepresentation of $\nu^{\frac{1}{2}}\chi_{V,\psi} \rtimes \sigma_{ds}$. As in the proof of Lemma 5.7, we deduce that σ_{ds} is discrete series representation, which is not strongly positive since it contains $\nu^{\frac{1}{2}}\chi_{V,\psi}$ two times in its cuspidal support. It is a simple combinatorial exercise to obtain that σ_{ds} is a subrepresentation of $\delta([\nu^{-d}\chi_{V,\psi}, \nu^{b}\chi_{V,\psi}]) \rtimes \sigma_{sp_1}$. Thus, we have

$$\begin{aligned}\sigma &\hookrightarrow \nu^{\frac{1}{2}}\chi_{V,\psi} \times \delta([\nu^{-d}\chi_{V,\psi}, \nu^{b}\chi_{V,\psi}]) \rtimes \sigma_{sp_1} \\ &\cong \delta([\nu^{-d}\chi_{V,\psi}, \nu^{b}\chi_{V,\psi}]) \times \nu^{\frac{1}{2}}\chi_{V,\psi} \rtimes \sigma_{sp_1}.\end{aligned}$$

Therefore, there is some irreducible genuine representation σ_{sp_2} such that $\sigma \hookrightarrow \delta([\nu^{-d}\chi_{V,\psi}, \nu^{b}\chi_{V,\psi}]) \rtimes \sigma_{sp_2}$. In the same way as in the Section 5 of [12] we see that σ_{sp_2} is square integrable. From its cuspidal support we see that σ_{sp_2} has to be strongly positive and it is a subrepresentation of

$$\nu^{\frac{1}{2}}\chi_{V,\psi} \rtimes \sigma_{sp_1}. \tag{5}$$

Note that if $\mu^*(\sigma) \geq \delta([\nu^{-d}\chi_{V,\psi}, \nu^{b}\chi_{V,\psi}]) \otimes \sigma'$ for some irreducible genuine representation σ', then $\sigma' \cong \sigma_{sp_2}$. In the same manner as before, using the first occurrence index of σ_{sp_2}, we see that there is some irreducible representation τ' of the orthogonal group such that $\sigma^{\epsilon_2}(n - t_{\epsilon_2} + 2)$ is a subrepresentation of $\delta([\nu^{-d}, \nu^{b}]) \times \nu^{-\frac{3}{2}} \rtimes \tau'$. Since

$d \geq \frac{3}{2}$, $\delta([\nu^{-d}, \nu^b]) \times \nu^{-\frac{3}{2}}$ is isomorphic to $\nu^{-\frac{3}{2}} \times \delta([\nu^{-d}, \nu^b])$ and Lemma 3.4 (ii) shows $r^{\epsilon_2}(\sigma) \leq n - t_{\epsilon_2} + 1$. The inequality (1) yields $r^{\epsilon_1}(\sigma) = n - t_{\epsilon_1}$ and $r^{\epsilon_2}(\sigma) = n - t_{\epsilon_2} + 1$.

Now suppose that $R_{\widetilde{P_1}}(\sigma)(\nu^{\frac{1}{2}}\chi_{V,\psi}) = 0$. We note that $\sigma_{sp}^{\epsilon_1}(n' - t_{\epsilon_1})$ can be written as a subrepresentation of the induced representation of the form $\delta([\nu^{\frac{3}{2}}, \nu^b]) \times \pi \rtimes \tau_{cusp}$, where π is a representation induced from essentially square integrable representations attached to the segments not containing the representation $\nu^{\frac{3}{2}}$. We have the following embeddings of the representation $\sigma^{\epsilon_1}(n - t_{\epsilon_1})$:

$$\begin{aligned}
\sigma^{\epsilon_1}(n - t_{\epsilon_1}) &\hookrightarrow \delta([\nu^{-\frac{1}{2}}, \nu^d]) \times \delta([\nu^{\frac{3}{2}}, \nu^b])) \times \pi \rtimes \tau_{cusp} \\
&\hookrightarrow \delta([\nu^{\frac{1}{2}}, \nu^d]) \times \nu^{-\frac{1}{2}} \times \delta([\nu^{\frac{3}{2}}, \nu^b])) \times \pi \rtimes \tau_{cusp} \\
&\cong \delta([\nu^{\frac{1}{2}}, \nu^d]) \times \delta([\nu^{\frac{3}{2}}, \nu^b])) \times \pi \times \nu^{-\frac{1}{2}} \rtimes \tau_{cusp} \\
&\cong \delta([\nu^{\frac{1}{2}}, \nu^d]) \times \delta([\nu^{\frac{3}{2}}, \nu^b])) \times \pi \times \nu^{\frac{1}{2}} \rtimes \tau_{cusp} \\
&\cong \delta([\nu^{\frac{1}{2}}, \nu^d]) \times \delta([\nu^{\frac{3}{2}}, \nu^b]) \times \nu^{\frac{1}{2}} \times \pi \rtimes \tau_{cusp} \\
&\rightarrow \delta([\nu^{\frac{1}{2}}, \nu^d]) \times \nu^{\frac{1}{2}} \times \delta([\nu^{\frac{3}{2}}, \nu^b]) \times \pi \rtimes \tau_{cusp}.
\end{aligned}$$

If $\sigma^{\epsilon_1}(n - t_{\epsilon_1})$ is contained in the kernel of the last intertwining operator, then there is some irreducible representation τ' such that $\sigma^{\epsilon_1}(n - t_{\epsilon_1})$ is subrepresentation of $\delta([\nu^{\frac{1}{2}}, \nu^b]) \rtimes \tau'$. As in [12, 13] we obtain an embedding

$$\sigma^{\epsilon_1}(n - t_{\epsilon_1}) \hookrightarrow \delta([\nu^{-d}, \nu^b]) \rtimes \tau_{sp},$$

where τ_{sp} is a strongly positive representation. Obviously, we may apply Proposition 3.8 to obtain

$$\sigma \hookrightarrow \delta([\nu^{-d}\chi_{V,\psi}, \nu^b\chi_{V,\psi}]) \rtimes \sigma_{sp_3},$$

where σ_{sp_3} is an irreducible subrepresentation of the representation of the form (5). But, this implies $R_{\widetilde{P_1}}(\sigma)(\nu^{\frac{1}{2}}\chi_{V,\psi}) \neq 0$, contrary to our assumption.

Therefore, $\sigma^{\epsilon_1}(n - t_{\epsilon_1})$ is not contained in the kernel of the observed intertwining operator and, in consequence, $R_{P_1}(\sigma^{\epsilon_1}(n - t_{\epsilon_1}))(\nu^{\frac{1}{2}}) \neq 0$. Lemma 3.4 (ii) implies $r^{\epsilon_1}(\sigma) \leq n - t_{\epsilon_1} - 1$ and the inequality (1) completes the proof. □

Obtained first occurrence indices, together with the results of the previous section, complete the proof of Proposition 5.4.

We close this section with the corollary that will be used afterwards in the paper. We use the same notation as in Proposition 5.4.

Corollary 5.13. *Suppose that* $r^{\epsilon_1}(\sigma) = n - n' + r^{\epsilon_1}(\sigma_{sp}) - 1$*. Then* $\sigma^{\epsilon_1}(r^{\epsilon_1}(\sigma))$ *is a discrete series subrepresentation of* $\delta([\nu^{-c+1}, \nu^d]) \rtimes \sigma_{sp}^{\epsilon_1}(r^{\epsilon_1}(\sigma_{sp}))$*. Especially, if* $c = \frac{1}{2}$ *then the first non-zero lift of* σ *in the* ϵ_1*-tower is strongly positive.*

Proof. For abbreviation, we denote $r^{\epsilon_1}(\sigma)$ by r_1. Proposition 5.4 shows $\rho = 1_{F^\times}$ and $c = -(m_{r_1+1} - n - 1)$. Further, $R_{\widetilde{P_1}}(\sigma)(\nu^c \chi_{V,\psi}) = 0$ and $\sigma^{\epsilon_1}(r_1 + 1)$ is a subrepresentation of $\delta([\nu^{-c}, \nu^d]) \rtimes \sigma_{sp}^{\epsilon_1}(r^{\epsilon_1}(\sigma_{sp}))$. Lemma 3.4 shows that $\sigma^{\epsilon_1}(r_1 + 1)$ is a subrepresentation of $\nu^c \rtimes \sigma^{\epsilon_1}(r_1)$. Frobenius reciprocity implies $R_{P_1}(\sigma^{\epsilon_1}(r_1 + 1)) \geq \nu^c \otimes \sigma^{\epsilon_1}(r_1)$ and using the structure formula for Jacquet modules we obtain that $\sigma^{\epsilon_1}(r_1)$ is a subquotient of $\delta([\nu^{-c+1}, \nu^d]) \rtimes \sigma_{sp}^{\epsilon_1}(r^{\epsilon_1}(\sigma_{sp}))$.

Results obtained in the previous section imply $R_{P_1}(\sigma_{sp}^{\epsilon_1}(r^{\epsilon_1}(\sigma_{sp})))(\nu^x) = 0$ for $c - 1 \leq x \leq d$. Let us first assume $c > \frac{1}{2}$. Then we may apply Theorem 2.1 of [15] to conclude that $\delta([\nu^{-c+1}, \nu^d]) \rtimes \sigma_{sp}^{\epsilon_1}(r^{\epsilon_1}(\sigma_{sp}))$ is a representation of the length three which has two non-isomorphic discrete series subrepresentations and a unique irreducible (Langlands) quotient. We denote this irreducible quotient by L and note that L is a subrepresentation of $\delta([\nu^{-d}, \nu^{c-1}]) \rtimes \sigma_{sp}^{\epsilon_1}(r^{\epsilon_1}(\sigma_{sp}))$. If we suppose that $\sigma^{\epsilon_1}(r_1)$ is isomorphic to L, it follows directly that there is some irreducible representation τ such that $\sigma^{\epsilon_1}(r_1 + 1)$ is either a subrepresentation of $\delta([\nu^{-d}, \nu^{c-1}]) \rtimes \tau$ or a subrepresentation of $\delta([\nu^{-d}, \nu^c]) \rtimes \tau$. In both cases, using Proposition 3.8 we get an embedding which contradicts the square integrability of σ. Thus, $\sigma^{\epsilon_1}(r_1)$ is a discrete series subrepresentation of $\delta([\nu^{-c+1}, \nu^d]) \rtimes \sigma_{sp}^{\epsilon_1}(r^{\epsilon_1}(\sigma_{sp}))$.

On the other hand, if $c = \frac{1}{2}$ then Theorem 5.1 of [15] shows that $\delta([\nu^{-c+1}, \nu^d]) \rtimes \sigma_{sp}^{\epsilon_1}(r^{\epsilon_1}(\sigma_{sp}))$ is a representation of the length two that has a unique irreducible subrepresentation (which is strongly positive) and a unique irreducible (Langlands) quotient. In the same way as in the previous case we conclude that $\sigma^{\epsilon_1}(r_1)$ is a discrete series subrepresentation of $\delta([\nu^{\frac{1}{2}}, \nu^d]) \rtimes \sigma_{sp}^{\epsilon_1}(r^{\epsilon_1}(\sigma_{sp}))$ and the corollary is proved. □

6 The conservation relation for general discrete series

Through this section we assume that the basic assumption holds. We take a moment to describe this conjecture more precisely. In fact, we will state the conjectures which are equivalent to the basic assumption by [13], Lemma 12.1, and seem to be more appropriate in our situation. In [13] this assumption has been discussed only for classical groups but results of Hanzer and Muić ([5]) extend it to the case of metaplectic groups.

Let ρ be an irreducible self-contragredient cuspidal representation of $GL(n_\rho, F)$ (this defines n_ρ) and let σ denote an irreducible cuspidal representation of classical (resp., metaplectic) group. Let s_1 denote a real number such that the induced representation $\nu^{s_1}\rho \rtimes 1$ (resp., $\nu^{s_1}\chi_{V,\psi}\rho \rtimes 1$) reduces, where 1 denotes the trivial representation of the trivial group. The results of Shahidi show that $s_1 \in \frac{1}{2}\mathbb{Z}$. Further, let us denote by s_2 a real number such that the induced representation $\nu^{s_2}\rho \rtimes \sigma$ (resp., $\nu^{s_2}\chi_{V,\psi}\rho \rtimes \sigma$) reduces. The assumption under which we work in this section states that $s_2 - s_1 \in \mathbb{Z}$ and $\nu^s\rho \rtimes \sigma$ (resp., $\nu^s\chi_{V,\psi}\rho \rtimes \sigma$) is irreducible for $s \in \mathbb{R} \setminus \{\pm s_2\}$.

The following theorem presents a metaplectic version of Mœglin-Tadić classification of discrete series.

Theorem 6.1. *Let $\sigma \in Irr(\widetilde{Sp(n)})$ denote a discrete series representation. Then there exist a positive integer k and an ordered k–tuple $S_k = (\sigma_1, \sigma_2, \ldots, \sigma_k)$ of discrete series representations with the following properties:*

- *$\sigma_i \in Irr(\widetilde{Sp(n_i)})$, $n_i < n_j$ for $i < j$;*
- *σ_1 is a strongly positive discrete series and $\sigma_k \cong \sigma$;*
- *For every $i \in \{2, 3, \ldots, k\}$ there exists a self-contragredient cuspidal representation $\rho_i \in Irr(GL(m_i, F))$ (this defines m_i) and half integers c_i, d_i such that $c_i \geq 0$ and $d_i - c_i$ is a positive integer, satisfying*

$$\sigma_i \hookrightarrow \delta([\nu^{-c_i}\chi_{V,\psi}\rho_i, \nu^{d_i}\chi_{V,\psi}\rho_i]) \rtimes \sigma_{i-1}$$

  and $R_{\widetilde{P}_{m_i}}(\sigma_{i-1})(\nu^x\chi_{V,\psi}\rho_i) = 0$ for $c_i \leq x \leq d_i$.;
- *If $\rho_i \cong \rho_j$ for $1 < i < j \leq k$ then $\rho_i \cong \rho_{j'}$ for $j' \in \{i+1, i+2, \ldots, j-1\}$;*

- *If $\rho_i \cong \rho_{i+1}$ for $i \in \{2, 3, \ldots, k-1\}$, then $c_i < c_{i+1}$;*
- *If there is some $i \in \{2, 3, \ldots, k\}$ such that $\rho_i \cong 1_{F^\times}$, then $\rho_2 \cong 1_{F^\times}$.*

Proof. Theorem obviously holds if σ is strongly positive. Thus, in the rest of the proof we suppose that σ is not strongly positive.

We will denote by ϵ_1 a unique element of the set $\{+, -\}$ with the property $\Theta^{\epsilon_1}(\sigma, n - t_{\epsilon_1}) \neq 0$. Theorem 3.3 shows that $\sigma^{\epsilon_1}(n - t_{\epsilon_1})$ is also a discrete series representation and by Theorem 4.5 it is not strongly positive. By the Mœglin-Tadić classification of discrete series, there exist a positive integer k and a k–tuple $S_k = (\tau_1, \tau_2, \ldots, \tau_k)$ of discrete series representations such that $\tau_i \in Irr(O(V_{n_i}^{\epsilon_1}))$ is a discrete series, τ_1 is strongly positive, $\tau_k \cong \sigma^{\epsilon_1}(n - t_{\epsilon_1})$ and for every $i \in \{2, 3, \ldots, k\}$ there is a self-contragredient cuspidal representation $\rho_i \in Irr(GL(m_i, F))$ and half integers c_i, d_i such that $c_i \geq 0$ and $d_i - c_i$ is a positive integer, satisfying

$$\tau_i \hookrightarrow \delta([\nu^{-c_i}\rho_i, \nu^{d_i}\rho_i]) \rtimes \tau_{i-1}$$

and $R_{P_{m_i}}(\tau_{i-1})(\nu^x \rho_i) = 0$ for $c_i \leq x \leq d_i$.

Further properties of Mœglin-Tadić classification, described in detail in [12, 13], enable us to assume that if $\rho_i \cong \rho_j$ for $i < j$ then $\rho_i \cong \rho_{j'}$ for $j' \in \{i+1, i+2, \ldots, j-1\}$ and $c_i < c_{i+1}$ if $\rho_i \cong \rho_{i+1}$. Obviously, if there is some $i \in \{2, 3, \ldots, k\}$ such that $\rho_i \cong 1_{F^\times}$, then we may take $\rho_2 \cong 1_{F^\times}$.

Note that $\tau_k(n + t_{\epsilon_1}) \neq 0$. Further, since $-c_i \leq 0$, the previously mentioned condition on the Jacquet modules of τ_i and Proposition 3.8 show that $\tau_i(n_i + t_{\epsilon_1})$ is a subrepresentation of $\delta([\nu^{-c_i}\chi_{V,\psi}\rho_i, \nu^{d_i}\chi_{V,\psi}\rho_i]) \rtimes \tau_{i-1}(n_{i-1} + t_{\epsilon_1})$ for $i \geq 2$, if $\tau_i(n_i + t_{\epsilon_1}) \neq 0$. Inductively we obtain $\tau_i(n_i + t_{\epsilon_1}) \neq 0$ for $i \in \{1, 2, \ldots, k\}$. We define $\sigma_i = \tau_i(n_i + t_{\epsilon_1})$ for $i \in \{1, 2, \ldots, k\}$. By Theorem 4.5, σ_1 is strongly positive discrete series.

If ρ_2 is not isomorphic to $1_{F^\times}$, using Lemma 2.2 we easily obtain $R_{\widetilde{P}_{m_i}}(\sigma_{i-1})(\nu^x \chi_{V,\psi}\rho_i) = 0$ for $c_i \leq x \leq d_i$ and $i \in \{1, 2, \ldots, k-1\}$.

If $\rho_2 \cong 1_{F^\times}$, using results obtained in Section 4 and Proposition 2.1 of [13], we see that if there is some half integer x such that $R_{\widetilde{P}_1}(\sigma_1)(\nu^x \chi_{V,\psi}) \neq 0$ and $R_{P_1}(\tau_1)(\nu^x) = 0$ then $x < c_1$. Consequently, $R_{\widetilde{P}_{m_i}}(\sigma_{i-1})(\nu^y \chi_{V,\psi}\rho_i) = 0$ for $c_i \leq y \leq d_i$ and $i \in \{2, 3, \ldots, k\}$ and the theorem is proved.

□

Let $\sigma \in Irr(\widetilde{Sp(n)})$ denote a discrete series representation which is not strongly positive.

An ordered k–tuple $S_k = (\sigma_1, \sigma_2, \dots, \sigma_k)$ that can be attached to σ as in the previous theorem is not unique, and to each such k–tuple we attach a non-negative real number $min_1(S_k)$ in the following way:

- If there are no representations of the form $\nu^x \chi_{V,\psi} 1_{F^\times}$, $x \in \mathbb{R}$, appearing in the cuspidal support of σ_1, set $min_1(S_k) = 0$.
- If some representation of the form $\nu^x \chi_{V,\psi} 1_{F^\times}$, $x \in \mathbb{R}$, appears in the cuspidal support of σ_1, let $min_1(S_k)$ denote the minimal a_1 such that σ_1 can be written as a subrepresentation of the induced representation of the form $\delta([\nu^{a_1}\chi_{V,\psi}, \nu^{b_1}\chi_{V,\psi}]) \rtimes \sigma_{sp}$, where σ_{sp} is a strongly positive discrete series.

Let $min(\sigma)$ denote the minimum of all $min_1(S_k)$, where S_k runs over all ordered k–tuples S_k as in Theorem 6.1. We fix an ordered k–tuple $S_k = (\sigma_1, \sigma_2, \dots, \sigma_k)$ satisfying $min_1(S_k) = min(\sigma)$ and again write $\sigma_i \hookrightarrow \delta([\nu^{-c_i}\chi_{V,\psi}\rho_i, \nu^{d_i}\chi_{V,\psi}\rho_i]) \rtimes \sigma_{i-1}$ for $i \in \{2, 3, \dots, k\}$. It is direct consequence of Proposition 3.7 and the assumption on numbers c_i, d_i that $r^\epsilon(\sigma_1) \leq n_1 - t_\epsilon$ implies $r^\epsilon(\sigma_i) \leq n_i - t_\epsilon$ for $i \in \{2, 3, \dots, k\}$ (we remind the reader that σ_1 is an irreducible representation of $\widetilde{Sp(n_1)}$).

We denote by k' the largest integer l, $3 \leq l \leq k$, such that $(c_i, \rho_i) = (c_{i-1} + 1, 1_{F^\times})$ and $R_{\widetilde{P_1}}(\sigma_i)(\nu^{c_i} 1_{F^\times}) = 0$ for $i = 3, 4, \dots, l$. If there is no such k' set $k_{ar} = 0$, otherwise set $k_{ar} = k' - 2$.

In the following proposition we determine the first occurrence indices of discrete series of metaplectic groups. For abbreviation, we denote $min(\sigma)$ by a_1.

Proposition 6.2. *Let ϵ_1 denote the unique element of the set $\{+, -\}$ such that $r^{\epsilon_1}(\sigma_1) \leq n_1 - t_{\epsilon_1}$ and let $\epsilon_2 \in \{+, -\}$ different than ϵ_1. We denote by σ_{cusp} the partial cuspidal support of σ. Then the following holds:*

1. *Suppose that $\nu^{\frac{1}{2}}\chi_{V,\psi}$ does not appear in $[\sigma_1]$ and $\nu^s\chi_{V,\psi} \rtimes \sigma_{cusp}$ reduces for $s > 0$.*
    - *Suppose that $a_1 = 0$. If $(c_2, \rho) = (s, 1_{F^\times})$ and $R_{\widetilde{P_1}}(\sigma)(\nu^s\chi_{V,\psi}) = 0$ then $r^{\epsilon_1}(\sigma) = n - s - \frac{1}{2} - k_{ar} - t_{\epsilon_1}$ and $r^{\epsilon_2}(\sigma) = n + s + \frac{3}{2} + k_{ar} - t_{\epsilon_2}$. Otherwise $r^{\epsilon_1}(\sigma) = n - s + \frac{1}{2} - t_{\epsilon_1}$ and $r^{\epsilon_2}(\sigma) = n + s + \frac{1}{2} - t_{\epsilon_2}$.*
    - *Suppose that $a_1 > 0$. If $(c_2, \rho) = (a_1 - 1, 1_{F^\times})$ and $a_1 > \frac{3}{2}$, or $(c_2, \rho) = (a_1 - 1, 1_{F^\times}) = (\frac{1}{2}, 1_{F^\times})$ and $R_{\widetilde{P_1}}(\sigma)(\nu^{\frac{1}{2}}\chi_{V,\psi}) = 0$, then*

$r^{\epsilon_1}(\sigma) = n - a_1 - k_{ar} - t_{\epsilon_1} + \frac{1}{2}$ and $r^{\epsilon_2}(\sigma) = n + a_1 + k_{ar} - t_{\epsilon_2} + \frac{1}{2}$. Otherwise $r^{\epsilon_1}(\sigma) = n - a_1 - t_{\epsilon_1} + \frac{3}{2}$ and $r^{\epsilon_2}(\sigma) = n + a_1 - t_{\epsilon_2} - \frac{1}{2}$.

2. *Suppose that $\nu^{\frac{1}{2}}\chi_{V,\psi}$ appears in $[\sigma_1]$. If $(c_2, \rho) = (\frac{1}{2}, 1_{F^\times})$ and $R_{\widetilde{P_1}}(\sigma)(\nu^{\frac{1}{2}}\chi_{V,\psi}) = 0$ then $r^{\epsilon_1}(\sigma) = n - t_{\epsilon_1} - k_{ar} - 1$ and $r^{\epsilon_2}(\sigma) = n - t_{\epsilon_2} + k_{ar} + 2$. Otherwise $r^{\epsilon_1}(\sigma) = n - t_{\epsilon_1}$ and $r^{\epsilon_2}(\sigma) = n - t_{\epsilon_2} + 1$.*

In other words, if $r^{\epsilon_1}(\sigma_2) = n_2 - n_1 + r^{\epsilon_1}(\sigma_1) - 1$, then $r^{\epsilon_1}(\sigma) = n - n_2 + r^{\epsilon_1}(\sigma_2) - k_{ar}$ and $r^{\epsilon_2}(\sigma) = n - n_2 + r^{\epsilon_2}(\sigma_2) + k_{ar}$, otherwise $r^{\epsilon}(\sigma) = n - n_1 + r^{\epsilon}(\sigma_1)$ for $\epsilon \in \{+, -\}$.

Proof. We prove this proposition using induction over k. Since σ is not strongly positive, the basis of induction is the case $k = 2$, which has been treated in the previous section. Note that in this case $k_{ar} = 0$. Thus, it remains to consider the case $k \geq 3$.

We also inductively assume that if $r^{\epsilon_1}(\sigma_i) = n_i - n_{i-1} + r^{\epsilon_1}(\sigma_{i-1}) - 1$ then $\sigma_i^{\epsilon_1}(r^{\epsilon_1}(\sigma_i))$ is a discrete series subrepresentation of $\delta([\nu^{-c_i+1}, \nu^{d_i}]) \rtimes \sigma_{i-1}^{\epsilon_1}(r^{\epsilon_1}(\sigma_{i-1}))$. For $i = 2$, this is exactly the statement of Corollary 5.13.

Assume that the claim of the proposition holds for all numbers less than i. We prove it for i. Note that it is enough to prove that $r^{\epsilon_1}(\sigma_i) = n_i - n_{i-1} + r^{\epsilon_1}(\sigma_{i-1}) - 1$ and $r^{\epsilon_2}(\sigma_i) = n_i - n_{i-1} + r^{\epsilon_2}(\sigma_{i-1}) + 1$ hold if $r^{\epsilon_1}(\sigma_{i-1}) = n_{i-1} - n_{i-2} + r^{\epsilon_1}(\sigma_{i-2}) - 1$, $(c_i, \rho) = (c_{i-1} + 1, 1_{F^\times})$ and $R_{\widetilde{P_1}}(\sigma_i)(\nu^{c_i}\chi_{V,\psi}) = 0$, while $r^{\epsilon}(\sigma_i) = n_i - n_{i-1} + r^{\epsilon}(\sigma_{i-1})$ for $\epsilon \in \{+, -\}$ holds otherwise.

Let us first assume that $r^{\epsilon_1}(\sigma_{i-1}) \geq n_{i-1} - n_{i-2} + r^{\epsilon_1}(\sigma_{i-2})$ or $(c_i, \rho) \neq (c_{i-1} + 1, 1_{F^\times})$ holds. If $\rho_i \cong 1_{F^\times}$, it is evident that $c_i \geq i - \frac{3}{2}$ holds. Thus, for $n_i - n_{i-1} + r^{\epsilon_1}(\sigma_{i-1}) < r \leq n_i - t_{\epsilon_1}$ we have $\sigma_i^{\epsilon_1}(r) \hookrightarrow \delta([\nu^{-c_i}\rho_i, \nu^{d_i}\rho_i]) \times \nu^{-(r - n_i + t_{\epsilon_1} + \frac{1}{2})} \rtimes \sigma_i^{\epsilon_1}(n_{i-1} - n_i + r - 1) \cong \nu^{-(r - n_i + t_{\epsilon_1} + \frac{1}{2})} \times \delta([\nu^{-c_i}\rho_i, \nu^{d_i}\rho_i]) \rtimes \sigma_i^{\epsilon_1}(n_{i-1} - n_i + r - 1)$, since either $\rho_i \not\cong 1_{F^\times}$ or $0 < -(r - n_i + t_{\epsilon_1} + \frac{1}{2}) < c_{i-1} < d_i$. This also implies $R_{\widetilde{P_1}}(\sigma_i)(\nu^{-(r - n_i + t_{\epsilon_1} + \frac{1}{2})}\chi_{V,\psi}) = 0$. Starting from $\sigma_i^{\epsilon_1}(n_i - t_{\epsilon_1}) \neq 0$, we inductively obtain $r^{\epsilon_1}(\sigma_i) \leq n_i - n_{i-1} + r^{\epsilon_1}(\sigma_{i-1})$.

Similarly, using the standard inductive procedure enhanced by Propositions 3.7 and 5.5, we get $r^{\epsilon_2}(\sigma_i) \leq n_i - n_{i-1} + r^{\epsilon_2}(\sigma_{i-1})$. Note that we are in the position to use Proposition 5.5 since $\sigma_{i-1}^{\epsilon_2}(r^{\epsilon_2}(\sigma_{i-1}) + 1)$ is a subrepresentation of $\nu^x \rtimes \sigma_{i-1}^{\epsilon_2}(r^{\epsilon_2}(\sigma_{i-1}))$ for $x \leq -c_i - 1$ and $x = -c_i - 1$ only if $r^{\epsilon_1}(\sigma_{i-1}) = n_{i-1} - n_{i-2} + r^{\epsilon_1}(\sigma_{i-2}) - 1$. Consequently, the inequality (1) implies the claim.

In the rest of the proof we assume $r^{\epsilon_1}(\sigma_{i-1}) = n_{i-1} - n_{i-2} + r^{\epsilon_1}(\sigma_{i-2}) - 1$ and $(c_i, \rho) = (c_{i-1} + 1, 1_{F^\times})$. The assumption on the first occurrence indices of σ_{i-1}, together with the inductive assumption, yields $c_j = c_{j-1}+1$ for $j \in \{2,3,\ldots,i-1\}$. Using the same inductive procedure for pushing down the lifts of representation σ_i as before, we obtain $r^{\epsilon_1}(\sigma_i) \leq n_i - n_{i-1} + r^{\epsilon_1}(\sigma_{i-1})$ and $r^{\epsilon_2}(\sigma_i) \leq n_i - n_{i-1} + r^{\epsilon_2}(\sigma_{i-1}) + 1$. Further, we have the following embeddings:

$$\sigma_i^{\epsilon_1}(n_i - n_{i-1} + r^{\epsilon_1}(\sigma_{i-1})) \hookrightarrow \delta([\nu^{-c_i}, \nu^{d_i}]) \rtimes \sigma_{i-1}^{\epsilon_1}(r^{\epsilon_1}(\sigma_{i-1})),$$

$$\sigma_i^{\epsilon_2}(n_i - n_{i-1} + r^{\epsilon_2}(\sigma_{i-1}) + 1) \hookrightarrow \delta([\nu^{-c_i}, \nu^{d_i}]) \times \nu^{-c_i-1} \rtimes \sigma_{i-1}^{\epsilon_2}(r^{\epsilon_2}(\sigma_{i-1})).$$

For simplicity of notation, we let r_1 stand for $n_i - n_{i-1} + r^{\epsilon_1}(\sigma_{i-1})$.

Assumption on the numbers c_i, d_i shows that $R_{P_1}(\sigma_1^{\epsilon_1}(r^{\epsilon_1}(\sigma_1)))(\nu^{c_i-1}) = 0$. In the same way as in the proof of Proposition 3.1 of [15] we deduce $\nu^{-c_i} \rtimes \sigma_1^{\epsilon_1}(r^{\epsilon_1}(\sigma_1)) \cong \nu^{c_i} \rtimes \sigma_1^{\epsilon_1}(r^{\epsilon_1}(\sigma_1))$. Since $c_j < c_i < d_j$ for $2 \leq j \leq i-1$, using the inductive assumption on the first non-zero lifts of representations σ_j for $2 \leq j \leq i-1$ (in the ϵ_1-tower), we obtain the following embeddings and isomorphisms:

$$\begin{aligned}
\sigma_i^{\epsilon_1}(r_1) &\hookrightarrow \delta([\nu^{-c_i+1}, \nu^{d_i}]) \times \nu^{-c_i} \rtimes \sigma_{i-1}^{\epsilon_1}(r^{\epsilon_1}(\sigma_{i-1})) \\
&\hookrightarrow \delta([\nu^{-c_i+1}, \nu^{d_i}]) \times \nu^{-c_i} \times \delta([\nu^{-c_{i-1}+1}, \nu^{d_{i-1}}]) \times \cdots \times \delta([\nu^{-c_2+1}, \nu^{d_2}]) \rtimes \sigma_1^{\epsilon_1}(r^{\epsilon_1}(\sigma_1)) \\
&\hookrightarrow \delta([\nu^{-c_i+1}, \nu^{d_i}]) \times \delta([\nu^{-c_{i-1}+1}, \nu^{d_{i-1}}]) \times \cdots \times \delta([\nu^{-c_2+1}, \nu^{d_2}]) \times \nu^{-c_i} \rtimes \sigma_1^{\epsilon_1}(r^{\epsilon_1}(\sigma_1)) \\
&\cong \delta([\nu^{-c_i+1}, \nu^{d_i}]) \times \delta([\nu^{-c_{i-1}+1}, \nu^{d_{i-1}}]) \times \cdots \times \delta([\nu^{-c_2+1}, \nu^{d_2}]) \times \nu^{c_i} \rtimes \sigma_1^{\epsilon_1}(r^{\epsilon_1}(\sigma_1)) \\
&\cong \nu^{c_i} \times \delta([\nu^{-c_i+1}, \nu^{d_i}]) \times \delta([\nu^{-c_{i-1}+1}, \nu^{d_{i-1}}]) \times \cdots \times \delta([\nu^{-c_2+1}, \nu^{d_2}]) \rtimes \sigma_1^{\epsilon_1}(r^{\epsilon_1}(\sigma_1)).
\end{aligned}$$

Thus, $R_{P_1}(\sigma_i^{\epsilon_1}(r_1))(\nu^{c_i}) \neq 0$. Consequently, if $R_{\widetilde{P_1}}(\sigma_i)(\nu^{c_i}\chi_{V,\psi}) = 0$ then Lemma 3.4 (ii) yields $r^{\epsilon_1}(\sigma_i) \leq n_i - n_{i-1} + r^{\epsilon_1}(\sigma_{i-1}) - 1$ and inequality (1) ends the investigation of the first occurrence indices in this case. It remains to determine the first non-zero lift of σ_i in the ϵ_1-tower. First, since $R_{P_1}(\sigma^{\epsilon_1}(r^{\epsilon_1}(\sigma_i)+1))$ contains $\nu^{c_i} \otimes \sigma^{\epsilon_1}(r^{\epsilon_1}(\sigma_i))$, standard calculation with Jacquet modules yields that $\sigma^{\epsilon_1}(r^{\epsilon_1}(\sigma_i))$ is a subquotient of $\delta([\nu^{-c_i+1}, \nu^{d_i}]) \rtimes \sigma_{i-1}^{\epsilon_1}(r^{\epsilon_1}(\sigma_{i-1}))$. Using Theorem 2.1 of [15], in the same way as in the proof of Corollary 5.13 we deduce that $\sigma^{\epsilon_1}(r^{\epsilon_1}(\sigma_i))$ is a discrete series subrepresentation of $\delta([\nu^{-c_i+1}, \nu^{d_i}]) \rtimes \sigma_{i-1}^{\epsilon_1}(r^{\epsilon_1}(\sigma_{i-1}))$.

It remains to consider the case $R_{\widetilde{P_1}}(\sigma_i)(\nu^{c_i}\chi_{V,\psi}) \neq 0$. Since $c_i \geq \frac{3}{2}$, Proposition 3.7 gives $R_{P_1}(\sigma_i^{\epsilon_1}(n_i - t_{\epsilon_1}))(\nu^{c_i}) \neq 0$. Lemma 5.1 of [13] shows that there is a discrete series τ such that $\sigma_i^{\epsilon_1}(n_i - t_{\epsilon_1})$ is a subrepresentation of $\delta([\nu^{-c_{i-1}}, \nu^{c_i}]) \rtimes \tau$. Further properties of Mœglin-Tadić classification, related to the ϵ-function defined on pairs, show that τ is a subrepresentation of $\delta([\nu^{-d_i}, \nu^{d_{i-1}}]) \rtimes \sigma_{i-2}^{\epsilon_1}(n_{i-2} - t_{\epsilon_1})$. Thus, $\sigma_i^{\epsilon_1}(n_i - t_{\epsilon_1})$ is a subrepresentation of $\delta([\nu^{-c_{i-1}}, \nu^{c_i}]) \times \delta([\nu^{-d_i}, \nu^{d_{i-1}}]) \rtimes \sigma_{i-2}^{\epsilon_1}(n_{i-2} - t_{\epsilon_1})$. It is a simple matter to see, using Proposition 3.8, that σ_i is a subrepresentation of $\delta([\nu^{-c_{i-1}}\chi_{V,\psi}, \nu^{c_i}\chi_{V,\psi}]) \times \delta([\nu^{-d_i}\chi_{V,\psi}, \nu^{d_{i-1}}\chi_{V,\psi}]) \rtimes \sigma_{i-2}$. Going back to lifts of σ_i in the ϵ_2-tower we get

$$\sigma_i^{\epsilon_2}(n_i - n_{i-1} + r^{\epsilon_2}(\sigma_{i-1}) + 1) \hookrightarrow \delta([\nu^{-c_{i-1}}, \nu^{c_i}]) \times \delta([\nu^{-d_i}, \nu^{d_{i-1}}]) \times \nu^{-c_i-1} \rtimes \tau',$$

for some irreducible representation τ'. Since $d_i \geq c_i + 1$ and $c_{i-1} = c_i - 1$, $\delta([\nu^{-c_{i-1}}, \nu^{c_i}]) \times \delta([\nu^{-d_i}, \nu^{d_{i-1}}]) \times \nu^{-c_i-1}$ is isomorphic to $\nu^{-c_i-1} \times \delta([\nu^{-c_{i-1}}, \nu^{c_i}]) \times \delta([\nu^{-d_i}, \nu^{d_{i-1}}])$. Consequently, $R_{P_1}(\sigma_i^{\epsilon_2}(n_i - n_{i-1} + r^{\epsilon_2}(\sigma_{i-1}) + 1))(\nu^{-c_i-1}) \neq 0$ and Lemma 3.4 (ii) implies $r^{\epsilon_2}(\sigma_i) \leq n_i - n_{i-1} + r^{\epsilon_2}(\sigma_{i-1})$. Using inequality (1) we get the desired conclusion. □

As a direct consequence of Proposition 6.2 and results from the Section 4, we obtain the main result of this paper.

Theorem 6.3. *The conservation relation holds for discrete series of metaplectic groups.*

We also have the following generalization of Corollary 4.4, which is a direct consequence of Proposition 6.2 and results obtained in the previous section:

Corollary 6.4. *Let $\sigma \in Irr(\widetilde{Sp(n)})$ denote a discrete series representation. If $k > r(\sigma)$, then $\sigma(k)$ is an irreducible subrepresentation of the induced representation*

$$\nu^{n-m_k+1} \times \nu^{n-m_k+2} \times \cdots \times \nu^{n-m_{r(\sigma)}} \rtimes \sigma(r(\sigma)).$$

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