Aubert duals of strongly positive discrete series and a class of unitarizable representations

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Abstract

Let G_n denote either the group $Sp(n, F)$ or $SO(2n+1, F)$ over a local non-archimedean field F. We explicitly determine the Aubert duals of strongly positive discrete series representations of the group G_n. This enables us to construct a large class of unitarizable representations of this group.

1 Introduction

Let F denote a local non-archimedean field and let G_n stand for either the group $Sp(n, F)$ or $SO(2n+1, F)$ over F. A crucial role in the classification of the unitary dual of G_n is played by the determination of unitarizable non-tempered irreducible representations in terms of the Langlands classification and tempered representations. There are very few methods known for obtaining the unitarizabile non-tempered representations, and in this paper we will construct a class of such representations using the Aubert involution.

This involution has been introduced for general reductive p-adic groups in [2] and presents a certain generalization of involutions on Grothendieck groups of smooth finite length representations of p-adic groups, studied by Zelevinsky, Schneider-Stuhler and many others.

Particulary interesting conjecture regarding the Aubert involution states that it preserves unitarity. This conjecture is still largely unsolved, but there

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are some important case that have been established. In particular, it has been proved by Hanzer in [4] that the Aubert dual of a strongly positive discrete series is unitarizable. We note that strongly positive representations present a special class of irreducible square-integrable representations and serve as a cornerstone in constructions of discrete series ([9]). An algebraic classification of strongly positive discrete series, which holds in a classical group case, is also given in [5].

Methods used in ([4]) are based on the precise analysis of the ends of the complementary series and calculation of the signature of involved hermitian forms, and no attempt to obtain an explicit description of the structure of studied Aubert duals has been made. Thus, the first purpose of this paper is to obtain such a description.

An algorithm for the determination of the Aubert duals of representations of $GL(n, F)$ is given in [10], and that algorithm might also have an application in determination of the Aubert duals of strongly positive discrete series. However, we have rather chosen an approach which is completely based on basic properties of the Aubert involution and the description of Jacquet modules of strongly positive discrete series, obtained in [6] and [7, Section 7].

The work of Hanzer and our description enable us to provide a construction of a rather large class of non-tempered unitarizable representations of G_n.

We emphasize that our construction also gives the characterization of all irreducible representations π of G_n such that for every embedding

$$\pi \hookrightarrow \nu^{a_1}\rho_1 \times \cdots \times \nu^{a_k}\rho_k \rtimes \pi_{cusp},$$

where $\nu = |\det|_F$, ρ_i is an irreducible cuspidal unitary representation of $GL(n_i, F)$ for $i = 1, \ldots, k$, and π_{cusp} is an irreducible cuspidal representation of $G_{n'}$, we have $a_i < 0$ for each i.

Let us now describe the contents of the paper in more details. In the following section we introduce some notation which will be used throughout the paper. In the third section we provide an explicit description of the Aubert duals of strongly positive discrete series, while in the fourth section we show how our description can be used to construct a class of non-tempered unitarizable representations.

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2 Preliminaries

Throughout the paper, F will denote a non-archimedean local field of characteristic different than two.

Let us first recall a definition of Aubert involution and its basic properties.

For a connected reductive p-adic group G defined over F, let Σ denote the set of roots of G with respect to fixed minimal parabolic subgroup and let Δ stand for a basis of Σ. For $\Theta \subseteq \Delta$, we let P_Θ be the standard parabolic subgroup of G corresponding to Θ and M_Θ be the standard Levi subgroup of G corresponding to Θ.

For a parabolic subgroup P of G with the Levi factor M and a representation σ of M, we denote by $i_M(\sigma)$ a normalized parabolically induced representation of G_n induced from σ. For an admissible finite length representation σ of G, the normalized Jacquet module of σ with respect to the standard parabolic subgroup having Levi factor equal to M will be denoted by $r_M(\sigma)$. We recall the following definition and results from [2, 3]:

Theorem 2.1. *Define the operator on the Grothendieck group of admissible representations of finite length of G by*

$$D_G = \sum_{\Theta \subseteq \Delta} (-1)^{|-\Theta|} i_{M_\Theta} \circ r_{M_\Theta}.$$

Operator D_G has the following properties:

1. *D_G is an involution.*

2. *D_G takes irreducible representations to irreducible ones.*

3. *If σ is irreducible cuspidal representation, then $D_G(\sigma) = (-1)^{|\Delta|}\sigma$.*

4. *For the standard Levi subgroup $M = M_\Theta$, we have*

$$r_M \circ D_G = Ad(w) \circ D_{w^{-1}(M)} \circ r_{w^{-1}(M)},$$

where w is the longest element of the set $\{w \in W : w^{-1}(\Theta) > 0\}$.

We will now describe groups that we consider.

Let $J_n = (\delta_{i,n+1-j})_{1 \leq i,j \leq n}$ denote an $n \times n$ matrix, where $\delta_{i,n+1-j}$ stands for the Kronecker symbol. For a square matrix g, we denote by g^t (resp., g^τ) the transposed matrix of g (resp., the transposed matrix of g with respect to the second diagonal). In what follows, we shall fix one of the series of classical groups

$$Sp(n, F) = \left\{ g \in GL(2n, F) : \begin{pmatrix} 0 & -J_n \\ J_n & 0 \end{pmatrix} g^t \begin{pmatrix} 0 & -J_n \\ J_n & 0 \end{pmatrix} = g^{-1} \right\},$$

$$SO(2n+1, F) = \left\{ g \in GL(2n+1, F) : g^\tau = g^{-1} \right\}$$

and denote by G_n a rank n group belonging to the series which we fixed.

If σ is an irreducible representation of G_n, we denote by $\hat{\sigma}$ the representation $\pm D_{G_n}(\sigma)$, taking the sign $+$ or $-$ such that $\hat{\sigma}$ is a positive element in the Grothendieck group of admissible representations of finite length of G_n. We call $\hat{\sigma}$ the Aubert dual of σ.

The set of standard parabolic subgroups will be fixed in a usual way, i.e., we fix a minimal F-parabolic subgroup in G_n consisting of upper-triangular matrices in the usual matrix realization of the classical group. Then the Levi factors of standard parabolic subgroups have the form $M = GL(n_1, F) \times \cdots \times GL(n_k, F) \times G_{n'}$, where $GL(m, F)$ denotes a general linear group of rank m over F. If $\delta_i, i = 1, 2, \ldots, k$ is a representation of $GL(n_i, F)$ and if τ a representation of G_m, then by $\delta_1 \times \cdots \times \delta_k \rtimes \tau$ we denote a normalized parabolically induced representation of the group G_n, induced from the representation by $\delta_1 \otimes \cdots \otimes \delta_k \otimes \tau$ of the standard parabolic subgroup with the Levi subgroup equal to $GL(n_1, F) \times \cdots \times GL(n_k, F) \times G_m$. Here n equals $n_1 + n_2 + \cdots + n_k + m$.

The set of all irreducible admissible representations of $GL(n, F)$ will be denoted by $\mathrm{Irr}(GL(n, F))$, and the set of all irreducible admissible representations of G_n will be denoted by $\mathrm{Irr}(G_n)$.

We will denote by ν a composition of the determinant mapping with the normalized absolute value on F. Let ρ denote an irreducible cuspidal representation of $GL(k, F)$. By a segment of cuspidal representations, which will be denoted by $[\rho, \nu^m \rho]$, we mean the set $\{\rho, \nu\rho, \ldots, \nu^m \rho\}$. To each such segment we attach an irreducible essentially square-integrable representation $\delta([\rho, \nu^m \rho])$ of $GL(m \cdot k, F)$, which is a unique irreducible subrepresentation of $\nu^m \rho \times \cdots \times \nu\rho \times \rho$ (here we use a well known notation for the normalized parabolic induction for the general linear groups with the usual choice of the standard parabolic subgroups). For integers x, y, $x \leq y$, we set $[x, y] = \{z \in \mathbb{Z} : x \leq z \leq y\}$. For irreducible essentially square-integrable representation

δ, there is the unique $e(\delta) \in \mathbb{R}$ such that $\nu^{-e(\delta)}\delta$ is unitarizable. Note that $e(\delta([\nu^a\rho, \nu^b\rho])) = \frac{a+b}{2}$, for irreducible cuspidal representation ρ.

In order to keep our results uniform, we put $\delta([\nu^a\rho, \nu^b\rho]) = 1$ (the one-dimensional representation of the trivial group) if $y = x-1$ and $\delta([\nu^a\rho, \nu^b\rho]) = 0$ if $y < x-1$.

Throughout the paper we prefer to use the subrepresentation version of the Langlands classification and write the non-tempered representation $\pi \in \mathrm{Irr}(G_n)$ as the unique irreducible (Langlands) subrepresentation of the induced representation of the form $\delta_1 \times \cdots \times \delta_k \rtimes \tau$, where $\tau \in \mathrm{Irr}(G_{n'})$ is a tempered representation and $\delta_1, \ldots, \delta_k$ are irreducible essentially square-integrable representations such that $e(\delta_1) \leq \cdots \leq e(\delta_k) < 0$. In this case, we write $\pi = L(\delta_1 \times \cdots \times \delta_k \rtimes \tau)$.

An irreducible representation σ of G_n is called strongly positive or a strongly positive discrete series, if for every embedding

$$\sigma \hookrightarrow \nu^{a_1}\rho_1 \times \cdots \times \nu^{a_k}\rho_k \rtimes \sigma_{cusp},$$

where $\rho_i \in \mathrm{Irr}(GL(n_i, F))$, $i = 1, \ldots, k$, are cuspidal unitary representations and $\sigma_{cusp} \in \mathrm{Irr}(G_{n'})$ is an irreducible cuspidal representation, we have $a_i > 0$ for each i.

If σ is a strongly positive discrete series, it has been proved in [4] that $\hat{\sigma}$ is unitarizable.

We have shown in [5] that every strongly positive discrete series representation can be realized in a unique way (up to a certain permutation) as a unique irreducible subrepresentation of the induced representation

$$\Big(\prod_{i=1}^{m}\prod_{j=1}^{k_i} \delta([\nu^{\alpha_i - k_i + j}\rho_i, \nu^{a_j^{(i)}}\rho_i])\Big) \rtimes \sigma_{cusp}$$

where $\rho_1, \ldots, \rho_m$ are mutually non-isomorphic irreducible self-contragredient cuspidal representations of $GL(n_1, F), \ldots, GL(n_m, F)$, σ_{cusp} is an irreducible cuspidal representation of $Sp(n')$, $\alpha_i > 0$ such that $\nu^{\alpha_i}\rho_i \rtimes \sigma_{cusp}$ reduces, $k_i = \lceil \alpha_i \rceil$, where $\lceil \alpha_i \rceil$ denotes the smallest integer which is not smaller than α_i, and, for $i = 1, \ldots, m$, we have $-1 < a_1^{(i)} < a_2^{(i)} < \cdots < a_{k_i}^{(i)}$ and $a_j^{(i)} - \alpha_i \in \mathbb{Z}$, for $j = 1, \ldots, k_i$.

We emphasize that if $\nu^x\rho$ appears in the cuspidal support of σ then ρ is selfcontragredient and $2x \in \mathbb{Z}$, by [1] and [8, Théorème 3.1.1].

Directly from Theorem 2.1 we obtain the following results, which will be frequently used in the paper.

Lemma 2.2. *Let $\sigma \in Irr(G_n)$ and suppose that the Jacquet module of $\hat{\sigma}$ with respect to the appropriate standard parabolic subgroup contains an irreducible representation $\nu^{x_1}\rho_1 \otimes \cdots \otimes \nu^{x_m}\rho_m \otimes \sigma_{cusp}$, where $\rho_1, \ldots, \rho_m, \sigma_{cusp}$ are irreducible cuspidal representations. Then the Jacquet module of $\hat{\sigma}$ with respect to the appropriate standard parabolic subgroup contains $\nu^{-x_1}\widetilde{\rho_1} \otimes \cdots \otimes \nu^{-x_m}\widetilde{\rho_m} \otimes \sigma_{cusp}$. In particular, if $\sigma \in Irr(G_n)$ is a strongly positive discrete series and the Jacquet module of $\hat{\sigma}$ with respect to the appropriate parabolic subgroup contains $\nu^{x_1}\rho_1 \otimes \cdots \otimes \nu^{x_m}\rho_m \otimes \sigma_{cusp}$, for irreducible cuspidal representations $\rho_1, \ldots, \rho_m, \sigma_{cusp}$, then $x_i < 0$ for $i = 1, \ldots, m$.*

The following proposition presents the first step in the determination of Aubert duals of strongly positive representations.

Proposition 2.3. *Let $\sigma \in Irr(G_n)$ denote a strongly positive discrete series and let σ_{cusp} denote a partial cuspidal support of σ. Then $\hat{\sigma} = L(\delta_1 \times \cdots \times \delta_m \rtimes \sigma_{cusp})$, for irreducible essentially square integrable representations $\delta_1, \ldots, \delta_m$ of general linear groups, such that $e(\delta_i) \leq e(\delta_{i+1}) < 0$ for $i = 1, \ldots, m-1$.*

Proof. By the Langlands classification, $\hat{\sigma} = L(\delta_1 \times \cdots \times \delta_m \rtimes \sigma_{cusp})$, for irreducible essentially square integrable representations $\delta_1, \ldots, \delta_m$ of general linear groups, such that $e(\delta_i) \leq e(\delta_{i+1}) < 0$ for $i = 1, \ldots, m-1$, and tempered representation $\tau \in \text{Irr}(G_{n'})$ for some $n' \leq n$.

If τ is not isomorphic to σ_{cusp}, then there is an $x \geq 0$ and a cuspidal representation $\rho \in \text{Irr}(GL(n_1, F))$ such that τ is a subrepresentation of $\nu^x\rho \rtimes \tau'$, for some $\tau' \in \text{Irr}(G_{n''})$. Using Frobenius reciprocity, together with transitivity of Jacquet modules, we get a contradiction with the previous lemma. This ends the proof. $\square$

We say that a representation $\sigma \in \text{Irr}(G_n)$ belongs to the set $D(\rho_1, \ldots, \rho_k; \sigma_{cusp})$ if every element of the cuspidal support of σ belongs to the set $\{\nu^x\rho_1, \ldots, \nu^x\rho_k, \sigma_{cusp} : x \in \mathbb{R}\}$, where $\rho_1, \ldots, \rho_k$ are mutually non-isomorphic irreducible cuspidal representations of general linear groups and σ_{cusp} is a cuspidal representation of $G_{n'}$, for some $n' \leq n$.

3 Aubert duals of strongly positive representations

In this section we determine Aubert duals of strongly positive discrete series. We will first consider the case of strongly positive representations contained in $D(\rho; \sigma_{cusp})$. Obviously, we can assume that ρ is an irreducible self-contragredient representation. Also, if $\rho \rtimes \sigma_{cusp}$ reduces then σ_{cusp} is the only strongly positive representation contained in $D(\rho; \sigma_{cusp})$, so we will assume that $\nu^{\alpha}\rho \rtimes \sigma_{cusp}$ reduces for $\alpha > 0$. We note that such α is unique by the results of [11].

Let $k = \lceil \alpha \rceil$. By [5, Section 5], the set of strongly positive discrete series in $D(\rho; \sigma_{cusp})$ is in bijection with the set of all ordered k-tuples $(a_1, \ldots, a_k)$ such that $a_i - \alpha \in \mathbb{Z}$, for $i = 1, \ldots, k$, and $-1 < a_1 < a_2 < \ldots < a_k$. Strongly positive discrete series corresponding to such k-tuple $(a_1, \ldots, a_k)$ will be denoted by $\sigma_{(a_1,\ldots,a_k)}$.

If $\sigma_{(a_1,\ldots,a_k)}$ is a cuspidal representation, we have $a_k = \alpha + 1$, $a_{i+1} = a_i + 1$ for $i = 1, \ldots, k-1$, and $\widehat{\sigma_{(a_1,\ldots,a_k)}} = \sigma_{(a_1,\ldots,a_k)}$, so in the rest of this section we will assume $a_k \geq \alpha$. Let us define $m = \min\{i : a_i \geq \alpha - k + 1\}$ and $l = k - m + 1$. Then $\sigma_{(a_1,\ldots,a_k)}$ is a unique irreducible subrepresentation of the induced representation

$$\delta([\nu^{\alpha-l+1}\rho, \nu^{a_{k-l+1}}\rho]) \times \delta([\nu^{\alpha-l+2}\rho, \nu^{a_{k-l+2}}\rho]) \times \cdots \times \delta([\nu^{\alpha}\rho, \nu^{a_k}\rho]) \rtimes \sigma_{cusp}.$$

By Proposition 2.3, Aubert dual of the representation $\sigma_{(a_1,\ldots,a_k)}$ is of the form $L(\delta_1 \times \cdots \times \delta_s \rtimes \sigma_{cusp})$ for irreducible essentially square-integrable representations $\delta_1, \ldots, \delta_s$ such that $d_i \leq d_{i+1} < 0$ for $i = 1, \ldots, s-1$. For $i = 1, \ldots, s$, we can write $\delta_i = \delta([\nu^{-x_i}\rho, \nu^{-y_i}\rho])$ for $x_i > 0$ and $y_i > 0$ such that $x_i - \alpha \in \mathbb{Z}$.

Obviously, the Jacquet module of $\sigma_{(a_1,\ldots,a_k)}$ with respect to the appropriate parabolic subgroup contains the irreducible representation

$$\nu^{-y_1}\rho \otimes \nu^{-y_1-1}\rho \otimes \cdots \otimes \nu^{-x_1}\rho \otimes \nu^{-y_2}\rho \otimes \cdots \otimes \nu^{-x_s}\rho \otimes \sigma_{cusp}.$$

Let $i \in \{1, \ldots, s\}$ be arbitrary but fixed. Using transitivity of Jacquet modules and Lemma 2.2, we deduce that there is an irreducible representation $\sigma_1 \in D(\rho; \sigma_{cusp})$ such that the Jacquet module of $\sigma_{(a_1,\ldots,a_k)}$ with respect to the appropriate parabolic subgroup contains an irreducible representation

$$\nu^{y_1}\rho \otimes \nu^{y_1+1}\rho \otimes \cdots \otimes \nu^{x_1}\rho \otimes \nu^{y_2}\rho \otimes \cdots \otimes \nu^{x_{i-1}}\rho \otimes \sigma_1.$$

It follows from [6, Lemma 3.4] that σ_1 is strongly positive discrete series and we write $\sigma_1 = \sigma_{(b_1,\ldots,b_k)}$. It can be deduced from [6, Theorem 4.6] that $b_j \leq a_j$ for $j = 1, \ldots, k$. Also, the Jacquet module of σ_1 with respect to the appropriate parabolic subgroup contains

$$\nu^{y_i}\rho \otimes \nu^{y_i+1}\rho \otimes \cdots \otimes \nu^{x_i}\rho \otimes \nu^{y_{i+1}}\rho \otimes \cdots \otimes \nu^{x_s}\rho \otimes \sigma_{cusp}.$$

Lemma 3.1. *There is $j \in \{k-l+1, \dots, k-x_i+y_i\}$ such that $y_i + r = b_{j+r}$ for $r = 0, 1, \dots, x_i - y_i$. Furthermore, if $j \geq 2$ then $b_j \geq b_{j-1} + 2$.*

Proof. By [6, Theorem 4.6], there is a $j \in \{k-l+1, k\}$ such that $y_i = b_j$ and $b_j \geq b_{j-1}+2$ if $j \geq 2$. Also, the Jacquet module of $\sigma_{(b_1,\dots,b_{i-1},b_i-1,b_{i+1},\dots,b_k)}$ with respect to the appropriate parabolic subgroup contains the irreducible representation

$$\nu^{y_i+1}\rho \otimes \cdots \otimes \nu^{x_i}\rho \otimes \nu^{y_{i+1}}\rho \otimes \cdots \otimes \nu^{x_s}\rho \otimes \sigma_{cusp}.$$

Since $b_{j+l'} \geq b_j + l'$ for $l' \geq 1$, we obtain $y_i + 1 = b_{j+1}$ and $b_{j+1} = b_j + 1$. Repeating the same arguments, we obtain $y_i + r = b_{j+r}$ for $r = 0, 1, \dots, x_i - y_i$ and $j \leq k - x_i + y_i$. This ends the proof. $\square$

Lemma 3.2. *For $t = 1, \dots, s-1$ we have $x_t > x_{t+1}$.*

Proof. Suppose that, contrary to our assumption, that there is $t \in \{1, \dots, s-1\}$ such that $x_t \leq x_{t+1}$. We denote $x_j - y_j$ by z_j for $j = 1, \dots, s$. Similarly as before, let $\sigma_t \in D(\rho; \sigma_{cusp})$ denote a strongly positive discrete series such that the Jacquet module of $\sigma_{(a_1,\dots,a_k)}$ with respect to the appropriate parabolic subgroup contains

$$\nu^{y_1}\rho \otimes \nu^{y_1+1}\rho \otimes \cdots \otimes \nu^{x_1}\rho \otimes \nu^{y_2}\rho \otimes \cdots \otimes \nu^{x_{t-1}}\rho \otimes \sigma_t,$$

and write $\sigma_t = \sigma_{(b_1,\dots,b_k)}$. By previous lemma, there is $j_1 \in \{k-l+1, \dots, k-z_t\}$ such that $y_t + r = b_{j_1+r}$ for $r = 0, 1, \dots, z_t$.

Also, we denote by $\sigma_{t+1} \in D(\rho; \sigma_{cusp})$ a strongly positive discrete series such that the Jacquet module of $\sigma_{(b_1,\dots,b_k)}$ with respect to the appropriate parabolic subgroup contains

$$\nu^{y_t}\rho \otimes \nu^{y_t+1}\rho \otimes \cdots \otimes \nu^{x_t}\rho \otimes \sigma_{t+1}.$$

Applying [6, Theorem 4.6] several times, we deduce

$$\sigma_{t+1} = \sigma_{(b_1,\dots,b_{j_1-1},b_{j_1}-1,b_{j_1+1}-1,\dots,b_{j_1+z_t}-1,b_{j_1+z_t+1}\dots,b_k)}.$$

We note that the Jacquet module of σ_{t+1} with respect to the appropriate parabolic subgroup contains

$$\nu^{y_{t+1}}\rho \otimes \nu^{y_{t+1}+1}\rho \otimes \cdots \otimes \nu^{x_{t+1}}\rho \otimes \sigma_{t+2},$$

for some irreducible representation $\sigma_{t+2} \in D(\rho; \sigma_{cusp})$.

Since $b_{j_1+z_t+1} - 1 > b_{j_1+z_t} - 1$ and $x_t \leq x_{t+1}$, using Lemma 3.1 again, we obtain $y_{t+1} \geq b_{j_1+z_t+1}$. Consequently, $y_{t+1} > x_t$ and $e(\delta_t) > e(\delta_{t+1})$, a contradiction. $\square$

Lemma 3.3. *For $t = 1, \ldots, s-1$ we have $x_{t+1} = x_t - 1$ and $y_{t+1} < y_t$. Also, $x_1 = a_k$.*

Proof. Similarly as in the proof of previous lemma, for $t = 1, \ldots, s$ we denote by σ_t a strongly positive discrete series such that the Jacquet module of $\sigma_{(a_1, \ldots, a_k)}$ with respect to the appropriate parabolic subgroup contains

$$\nu^{y_1}\rho \otimes \nu^{y_1+1}\rho \otimes \cdots \otimes \nu^{x_1}\rho \otimes \nu^{y_2}\rho \otimes \cdots \otimes \nu^{x_{t-1}}\rho \otimes \sigma_t,$$

and write $\sigma_t = \sigma_{(b_1^{(t)}, \ldots, b_k^{(t)})}$. Obviously, $b_j^{(1)} = a_j$ for $j = 1, \ldots, k$.

Since the Jacquet module of σ_t with respect to the appropriate parabolic subgroup contains an irreducible representation of the form

$$\nu^{y_t}\rho \otimes \nu^{y_t+1}\rho \otimes \cdots \otimes \nu^{x_t}\rho \otimes \sigma'_t,$$

and $x_t > x_{t+1}$ for $t = 1, \ldots, s-1$, it follows from the cuspidal support of $\sigma_{(b_1^{(t)}, \ldots, b_k^{(t)})}$ that $x_t = b_k^{(t)}$. In particular, $x_1 = a_k$. We also note that it can be easily seen $x_s = \alpha$.

For $t = 1, \ldots, s$, we define $j_t = 1$ if $b_{j-1}^{(t)} = b_j^{(t)} - 1$ for all $j = 2, \ldots, k$ and $j_t = \max\{j : b_{j-1}^{(t)} < b_j^{(t)} - 1\}$ otherwise. Lemma 3.1 gives $y_t = b_{j_t}^{(t)}$. Using [6, Theorem 4.6], we obtain that

$$(b_1^{(t+1)}, \ldots, b_k^{(t+1)}) = (b_1^{(t)}, \ldots, b_{j_t-1}^{(t)}, b_{j_t}^{(t)} - 1, \ldots, b_k^{(t)} - 1) \tag{1}$$

holds for $t = 1, \ldots, s-1$. This implies $x_{t+1} = x_t - 1$ for $t = 1, \ldots, s-1$.

From the definition of j_t and (1), we deduce $j_t \geq j_{t+1}$ for $t = 1, \ldots, s-1$. Consequently, $b_{j_{t+1}}^{(t+1)} \leq b_{j_t}^{(t)} - 1$ and $y_{t+1} < y_t$ for $t = 1, \ldots, s-1$. This ends the proof. □

We also note a direct consequence of the proof of previous lemma.

Corollary 3.4. *For $i \in \{k-l+1, \ldots, k\}$, let $t = \min\{j : x_j - y_j + 1 \geq i\}$. Then $a_i = x_t - i + 1$.*

From previous sequence of lemmas, we obtain a description of the Aubert dual of strongly positive representation $\sigma_{(a_1, \ldots, a_k)}$. We note that for i such that $l + 1 \leq i \leq k$ we have $a_{k-i+1} = a_{k-i} + 1$ and $-a_{k-i+1} > -a_{k-i} - 2$.

Theorem 3.5. *The Aubert dual of the strongly positive representation $\sigma_{(a_1,\ldots,a_k)}$ is a unique irreducible subrepresentation of the induced representation*

$$\Big(\prod_{i=1}^{k}\prod_{j=-a_{k-i+1}}^{-a_{k-i}-2}\delta([\nu^{j-i+1}\rho,\nu^{j}\rho])\Big)\rtimes\sigma_{cusp},$$

where $a_0=\alpha-\lceil\alpha\rceil-1$.

We will now describe our results in the general case. Let $\sigma\in\mathrm{Irr}(G_n)$ denote a strongly positive discrete series and suppose that σ is contained in $D(\rho_1,\ldots,\rho_m;\sigma_{cusp})$, with m minimal. Then each ρ_i is a self-contragredient representation and, for $i=1,\ldots,m$, we denote by α_i a unique non-negative real number such that $\nu^{\alpha_i}\rho_i\rtimes\sigma_{cusp}$ reduces. We note that minimality of m implies $\alpha_i>0$. Also, let $k_i=\lceil\alpha_i\rceil$ and $a_0^{(i)}=\alpha_i-\lceil\alpha_i\rceil-1$.

By [5, Section 5], for $i=1,\ldots,m$ there exist $a_1^{(i)},\ldots,a_{k_i}^{(i)}$ such that $-1<a_1^{(i)}<\cdots<a_{k_i}^{(i)}$ and $a_j^{(i)}-\alpha_i\in\mathbb{Z}$ for $j=1,\ldots,k_i$, such that σ is a unique irreducible subrepresentation of the induced representation

$$\Big(\prod_{i=1}^{m}\prod_{j=1}^{k_i}\delta([\nu^{\alpha_i-k_i+j}\rho_i,\nu^{a_j^{(i)}}\rho_i])\Big)\rtimes\sigma_{cusp}.$$

Since for non-isomorphic irreducible cuspidal representations ρ and ρ' of general linear groups and x_1,x_2,y_1,y_2 such that $x_1-y_1\in\mathbb{Z}$ and $x_2-y_2\in\mathbb{Z}$ we have $\delta([\nu^{x_1}\rho,\nu^{y_1}\rho])\times\delta([\nu^{x_2}\rho',\nu^{y_2}\rho'])\cong\delta([\nu^{x_2}\rho',\nu^{y_2}\rho'])\times\delta([\nu^{x_1}\rho,\nu^{y_1}\rho])$, we can repeat the same arguments as in $D(\rho;\sigma_{cusp})$ case to obtain a description of the Aubert dual of σ.

Theorem 3.6. *The Aubert dual of the strongly positive representation σ is a unique irreducible subrepresentation of the induced representation*

$$\left(\prod_{i=1}^{m}\prod_{l=1}^{k_i}\prod_{j=-a_{k_i-l+1}^{(i)}}^{-a_{k_i-l}^{(i)}-2}\delta([\nu^{j-l+1}\rho,\nu^{j}\rho])\right)\rtimes\sigma_{cusp}.$$

4 A class of unitarizable representations

We will now use a description obtained in the previous section and results of Hanzer ([4]) to construct a large class of non-tempered unitarizable representations of the group G_n.

Theorem 4.1. *Suppose that $\sigma_{cusp} \in Irr(G'_n)$ is a cuspidal representation and let s be a positive integer. Let $\rho_t \in Irr(GL(n_t, F))$, $t = 1, \ldots, s$, be mutually non-isomorphic cuspidal self-contragredient representations. For $t = 1, \ldots, s$, let $\alpha_t \geq 0$ be such that the induced representation $\nu^{\alpha_t}\rho_t \rtimes \sigma_{cusp}$ reduces. For $t = 1, \ldots, s$, let m_t denote a non-negative integer and suppose that $(b_1^{(t)}, \ldots, b_{m_t}^{(t)})$ is an ordered m_t-tuple of real numbers such that $-\alpha_t - m_t + 1 \leq b_1^{(t)} < \cdots < b_{m_t}^{(t)} < 0$ and $b_i^{(t)} - \alpha_t \in \mathbb{Z}$ for $i = 1, \ldots, m_t$. Then the induced representation*

$$\Big(\prod_{t=1}^{s} \prod_{i=-m_t+1}^{0} \delta([\nu^{-\alpha_t+i}\rho_t, \nu^{b_{m_t+i}^{(t)}}\rho_t])\Big) \rtimes \sigma_{cusp} \tag{2}$$

has a unique irreducible subrepresentation, which is unitarizable.

Proof. It is a direct consequence of the subrepresentation version of the Langlands classification that the induced representation (2) has a unique irreducible subrepresentation, which will be denoted by σ. For $i = 1, \ldots, s$, let $k_t = \lceil \alpha_t \rceil$.

For $t = 1, \ldots, s$, we will define $a_1^{(t)}, \ldots, a_{k_t}^{(t)}$ such that $-1 < a_1^{(t)} < \cdots < a_{k_t}^{(t)}$ and $\alpha_t - a_i^{(t)} \in \mathbb{Z}$ for $i = 1, \ldots, k_t$.

Let us write $l_t = b_{m_t}^{(t)} + \alpha_t + 1$. For i such that $1 \leq i \leq k_t - l_t$, set $a_i^{(t)} = \alpha_t - k_t + i - 1$. For i such that $k_t - l_t + 1 \leq i \leq k_t$, we define $x_i^{(t)} = \min\{j \in \{-m_t + 1, -m_t + 2, \ldots, 0\} : b_{m_t+j}^{(t)} + \alpha_t - j \geq k_t - i\}$, and set $a_i^{(t)} = \alpha_t - x_i^{(t)} - k_t + i$.

By [5, Theorem 5.3], the induced representation

$$\Big(\prod_{t=1}^{s} \prod_{i=1}^{k_t} \delta([\nu^{\alpha_t-k_t+i}\rho_t, \nu^{a_i^{(t)}}\rho_t])\Big) \rtimes \sigma_{cusp}$$

has a unique irreducible subrepresentation, which is strongly positive and will be denoted by σ_{sp}. It is not hard to see, using Lemma 3.3 and Corollary 3.4, that we have $\sigma = \widehat{\sigma_{sp}}$, and ([4]) implies that σ is unitarizable. This ends the proof. □

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