Singular two-parameter eigenvalue problems

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The two-parameter eigenvalue problem has the form
\[ \begin{align*}
A_1 x_1 &= \lambda B_1 x_1 + \mu C_1 x_1, \\
A_2 x_2 &= \lambda B_2 x_2 + \mu C_2 x_2,
\end{align*} \]  
(1)

where $A_i, B_i, \text{ and } C_i$ are given $n_i \times n_i$ complex matrices, $\lambda, \mu \in \mathbb{C}$, and $x_i \in \mathbb{C}^{n_i}$ for $i = 1, 2$. A pair $(\lambda, \mu)$ is an eigenvalue if it satisfies (1) for nonzero vectors $x_1, x_2$, and the corresponding eigenvector is $x_1 \otimes x_2$.

On the tensor product space we can define $n_1 n_2 \times n_1 n_2$ matrices
\[ \begin{align*}
\Delta_0 &= B_1 \otimes C_2 - C_1 \otimes B_2, \\
\Delta_1 &= A_1 \otimes C_2 - C_1 \otimes A_2, \\
\Delta_2 &= B_1 \otimes A_2 - A_1 \otimes B_2.
\end{align*} \]

The two-parameter eigenvalue problem (1) is nonsingular if its operator determinant $\Delta_0$ is invertible. Atkinson showed [1] that a nonsingular two-parameter eigenvalue problem is equivalent to the joint generalized eigenvalue problems
\[ \begin{align*}
\Delta_1 z &= \lambda \Delta_0 z, \\
\Delta_2 z &= \mu \Delta_0 z,
\end{align*} \]  
(2)

where $z = x_1 \otimes x_2$. Many theoretical results and numerical methods for nonsingular two-parameter eigenvalue problems are based on this relation.

However, if all linear combinations of $\Delta_0, \Delta_1$, and $\Delta_2$ are singular, then we say that (1) is singular. Just recently, some of the above relations were generalized to singular two-parameter eigenvalue problems in [3], where it is shown that the simple finite regular eigenvalues of (1) and (2) agree. We will present a numerical method from [2] that can solve a singular two-parameter eigenvalue problem by computing the common regular eigenvalues of the associated system of two singular generalized eigenvalue problems.

As possible applications that lead to singular two-parameter eigenvalue problems we will present a numerical method for the quadratic two-parameter eigenvalue problem and a numerical method for a system of two bivariate polynomials.

References

