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Particle basis of Feigin-Stoyanovsky's type subspaces of level one $\tilde{\mathfrak{sl}}_{\ell+1}(\mathbb{C})$-modules*

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Abstract. We construct a particle basis for Feigin-Stoyanovsky's type subspaces of level 1 standard $\tilde{\mathfrak{sl}}_{\ell+1}(\mathbb{C})$-modules. From the description, we obtain character formulas.

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1. Introduction

The problem of finding a monomial basis of a standard module is part of the Lepowsky-Wilson program of studying Rogers-Ramanujan type identities through representation theory of affine Lie algebras ([22, 21, 23]). A description of basis was used to obtain graded dimensions of these modules, which gave the sum side of Rogers-Ramanujan type.

B. Feigin and A. Stoyanovsky initiated another approach to Rogers-Ramanujan type identities by considering what they called principal subspace of a standard $\tilde{\mathfrak{sl}}_2(\mathbb{C})$-module ([12]). These subspaces were further studied by G. Georgiev ([15]), C. Calinescu, S. Capparelli, J. Lepowsky and A. Milas ([7, 8, 2, 3, 4, 5, 6]), C. Sadowski ([27, 28]), E. Ardonne, R. Kedem and M. Stone ([1]).

Another type of principal subspace, called Feigin-Stoyanovsky's type subspace, was introduced and studied by M. Primc who constructed a basis of this subspace and therefrom he obtained the basis of the whole standard module ([24, 25]). For $\tilde{\mathfrak{sl}}_{\ell+1}(\mathbb{C})$, these bases were parameterized by $(k, \ell+1)$-admissible configurations ([10]), combinatorial objects introduced and studied further by Feigin et al. in [10] and [11], where bosonic and fermionic formulas for characters were obtained. Using the ideas from [15, 7, 8], Primc gave another proof of linear independence of the spanning set by using intertwining operators ([26]). Primc and M. Jerković obtained fermionic formulas for characters of standard $\tilde{\mathfrak{sl}}_3(\mathbb{C})$-modules by using admissible configurations ([17]), or quasi-particle bases ([18]). In our previous paper ([30]), we have used $(1, \ell+1)$-admissible configurations to combinatorially obtain character formulas for Feigin-Stoyanovsky's type subspaces of level 1 standard $\tilde{\mathfrak{sl}}_{\ell+1}(\mathbb{C})$-modules.

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In this note, we use an approach similar to Georgiev and Jerković and Primc to construct a particle basis for Feigin-Stoyanovsky's type subspaces of level 1 standard $\tilde{\mathfrak{sl}}_{\ell+1}(\mathbb{C})$-modules. From this description, we immediately obtain character formulas.

2. Affine Lie algebra $\hat{\mathfrak{sl}}_{\ell+1}(\mathbb{C})$

Let $\mathfrak{g} = \mathfrak{sl}_{\ell+1}(\mathbb{C})$ be a simple finite-dimensional Lie algebra of type A_ℓ. Fix a Cartan subalgebra $\mathfrak{h} \subset \mathfrak{g}$ and denote by R the corresponding root system; $\mathfrak{g}$ has a root decomposition $\mathfrak{g} = \mathfrak{h} \oplus \coprod_{\alpha\in R} \mathfrak{g}_\alpha$. Let $\Pi = \{\alpha_1, \ldots, \alpha_\ell\}$ be a basis of the root system R, and let $\{\omega_1, \ldots, \omega_\ell\}$ be the corresponding set of fundamental weights, $\langle \omega_i, \alpha_j \rangle = \delta_{ij}$. Set $\omega_0 = 0$ for convenience. Let $\langle \cdot, \cdot \rangle$ be a normalized invariant bilinear form on $\mathfrak{g}$; we identify $\mathfrak{h}$ with $\mathfrak{h}^*$ via $\langle \cdot, \cdot \rangle$. Denote by Q the root lattice, and by P the weight lattice of $\mathfrak{g}$. Also, for each root $\alpha \in R$, fix a root vector $x_\alpha \in \mathfrak{g}_\alpha$.

Let $\tilde{\mathfrak{g}}$ be the associated untwisted affine Lie algebra ([19]),

$$\tilde{\mathfrak{g}} = \mathfrak{g} \otimes \mathbb{C}[t, t^{-1}] \oplus \mathbb{C}c \oplus \mathbb{C}d.$$

Denote by $x(m) = x \otimes t^m$ for $x \in \mathfrak{g}$, $m \in \mathbb{Z}$, and define formal Laurent series

$$x(z) = \sum_{m\in\mathbb{Z}} x(m) z^{-m-1}.$$

Denote by $\Lambda_0, \ldots, \Lambda_\ell$ fundamental weights for $\tilde{\mathfrak{g}}$.

Fix a minuscule weight $\omega = \omega_\ell$ and set

$$\Gamma = \{\, \alpha \in R \mid \langle \alpha, \omega \rangle = 1 \} = \{\alpha_i + \cdots + \alpha_\ell \mid i = 1, \ldots, \ell\}.$$

Denote by $\gamma_i = \alpha_i + \cdots + \alpha_\ell$. Then

$$\mathfrak{g} = \mathfrak{g}_{-1} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_1, \quad \mathfrak{g}_0 = \mathfrak{h} \oplus \sum_{\langle \alpha, \omega \rangle = 0} \mathfrak{g}_\alpha, \quad \mathfrak{g}_{\pm 1} = \sum_{\alpha \in \pm\Gamma} \mathfrak{g}_\alpha,$$

is a $\mathbb{Z}$-gradation of $\mathfrak{g}$. Subalgebras $\mathfrak{g}_1$ and $\mathfrak{g}_{-1}$ are commutative.

The $\mathbb{Z}$-gradation of $\mathfrak{g}$ gives the $\mathbb{Z}$-gradation of the affine Lie algebra $\tilde{\mathfrak{g}}$:

$$\tilde{\mathfrak{g}} = \tilde{\mathfrak{g}}_{-1} + \tilde{\mathfrak{g}}_0 + \tilde{\mathfrak{g}}_1, \quad \tilde{\mathfrak{g}}_0 = \mathfrak{g}_0 \otimes \mathbb{C}[t, t^{-1}] \oplus \mathbb{C}c \oplus \mathbb{C}d, \quad \tilde{\mathfrak{g}}_{\pm 1} = \mathfrak{g}_{\pm 1} \otimes \mathbb{C}[t, t^{-1}].$$

Again, $\tilde{\mathfrak{g}}_{-1}$ and $\tilde{\mathfrak{g}}_1$ are commutative subalgebras. We will call elements $\gamma \in \Gamma$ *colors* and say that $x_\gamma(-m)$ is an element of *color* γ and *degree* m.

Let $L(\Lambda_r)$ be a standard (i.e. integrable highest weight) $\tilde{\mathfrak{g}}$-module of level 1. Denote by v_r the highest weight vector of $L(\Lambda_r)$. Define a *Feigin-Stoyanovsky's type subspace*

$$W(\Lambda_r) = U(\tilde{\mathfrak{g}}_1) \cdot v_r \subset L(\Lambda_r).$$

By the Poincaré-Birkhoff-Witt theorem, we have a spanning set of $W(\Lambda_r)$ consisting of monomial vectors

$$\{bv_r \mid b = x_\ell(-m_{\ell,n_\ell}) \cdots x_1(-m_{1,n_1}) \cdots x_1(-m_{1,1}), m_{i,j+1} \geq m_{i,j} > 0, n_i \geq 0\}, \quad (1)$$

where we write x_i instead of x_{γ_i}, for short.

3. VOA construction

We briefly recall the vertex operator algebra construction of standard $\tilde{\mathfrak{g}}$-modules $L(\Lambda_r)$ from [13, 29]. For details and notation we turn the reader to [14, 9] and [20].

Consider tensor products $V_P = M(1)\otimes\mathbb{C}[P]$ and $V_Q = M(1)\otimes\mathbb{C}[Q]$, where $M(1)$ is the Fock space for the Heisenberg subalgebra

$$\hat{\mathfrak{h}}_{\mathbb{Z}} = \sum_{n\in\mathbb{Z}\setminus\{0\}} \mathfrak{h}\otimes t^n \oplus \mathbb{C}c,$$

and $\mathbb{C}[P]$ and $\mathbb{C}[Q]$ are group algebras of the weight and root lattice with bases consisting of $\{e^\lambda \,|\, \lambda \in P\}$, and $\{e^\alpha \,|\, \alpha \in Q\}$, respectively. We identify $\mathbb{C}[P]$ with $1\otimes\mathbb{C}[P] \subset V_P$.

Space V_Q has a structure of a vertex operator algebra and V_P is a module for this algebra

$$Y(e^\lambda, z) = E^-(-\lambda, z)E^+(-\lambda, z)\otimes e^\lambda z^\lambda \epsilon(\lambda, \cdot), \tag{2}$$

where

$$E^\pm(\lambda, z) = \exp\left(\sum_{m\geq 1} \lambda(\pm m) z^{\mp m}/\pm m\right),$$

e^λ is a multiplication operator, $z^\lambda \cdot e^\mu = e^\mu z^{\langle\lambda,\mu\rangle}$ and $\epsilon(\cdot,\cdot)$ is a 2-cocycle (cf. [14]).

By using vertex operators, one can define the structure of a $\tilde{\mathfrak{g}}$-module on V_P by setting $x_\alpha(z) = Y(e^\alpha, z)$ for $\alpha \in R$. This gives $V_Q \simeq L(\Lambda_0)$ and $V_Q e^{\omega_r} \simeq L(\Lambda_r)$, with the highest weight vectors $v_0 = 1$ and $v_r = e^{\omega_r}$, and $V_P \cong L(\Lambda_0)\oplus\cdots\oplus L(\Lambda_\ell)$.

From vertex operator formula (2) one easily obtains the following relations on $L(\Lambda_r)$

$$x_i^2(z) = 0, \quad 1 \leq i \leq \ell, \tag{3}$$

$$x_i(z)x_j(z) = 0, \quad 1 \leq i < j \leq \ell, \tag{4}$$

$$x_i(m)v_r = 0, \quad m \geq -\delta_{i\leq r}, \tag{5}$$

$$x_r(-1)v_{r-1} = Ce^{\omega_\ell+\omega_r} = Ce^{\omega_\ell}v_r, \tag{6}$$

for some $C \in \mathbb{C}^\times$. Here, $\delta_{i\leq j}$ is 1 if $i \leq j$, 0 otherwise.

For the proof of linear independence we will use certain coefficients of intertwining operators

$$\mathcal{Y}(e^\lambda, z) = Y(e^\lambda, z)e^{i\pi\lambda}c(\cdot, \lambda),$$

for $\lambda \in P$, where $c(\cdot,\lambda)$ is a commutator map (cf. [9]). Let $\lambda_i = \omega_i - \omega_{i-1}$ for $i = 1,\ldots,\ell$. From Jacobi identity ([9]), we see that operators $\mathcal{Y}(e^{\lambda_i}, z)$ commute with the action of $\tilde{\mathfrak{g}}_1$. Define the following coefficients of intertwining operators (cf. [26])

$$[i] = \operatorname{Res} z^{-1-\langle\lambda_i,\omega_{i-1}\rangle}\mathcal{Y}(e^{\lambda_i}, z),$$

for $i = 1,\ldots,\ell$. From (2), there follows (cf. [26])

$$[i]v_{i-1} = Cv_i, \tag{7}$$

for some $C \in \mathbb{C}^\times$.

We will also use simple current operators e^{ω_i}, $i = 1, \ldots, \ell$. For $\alpha \in R$ and $\lambda \in P$, from (2) we get the following commutation relation

$$x_\alpha(z)e^\lambda = \epsilon(\alpha, \lambda)z^{\langle\alpha,\lambda\rangle}e^\lambda x_\alpha(z).$$

By comparing coefficients, we get

$$x_\alpha(m)e^\lambda = \epsilon(\alpha, \lambda)e^\lambda x_\alpha(m + \langle\alpha, \lambda\rangle).$$

In particular, for $\alpha = \gamma_i$ and $\lambda = \omega_j$, we get

$$x_i(m)e^{\omega_j} = \epsilon(\gamma_i, \omega_j)e^{\omega_j} x_i(m + \delta_{i\le j}). \tag{8}$$

4. Basis of $W(\Lambda_r)$

To reduce the spanning set (1) and to prove linear independence, we need a linear order on monomials. Define a linear order $x_i(n) < x_j(m)$ if either $i > j$ or $i = j$ and $n < m$. We assume that in all monomials factors are sorted descendingly from right to left, as in (1). We compare two monomials b_1 and b_2 by comparing their factors from right to left (a reverse lexicographic order): $b_1 < b_2$ if either $b_2 = bb_1$ or $b_1 = b_1'x_i(n)b$, $b_2 = b_2'x_j(m)b$ and $x_i(n) < x_j(m)$, for some monomials b, b_1, b_2. This linear order is compatible with multiplication: if $b > c$, then $ab > ac$.

For a monomial

$$b = x_\ell(-m_{\ell,n_\ell}) \cdots x_\ell(-m_{\ell,1}) \cdots x_1(-m_{1,n_1}) \cdots x_1(-m_{1,1}),$$

define its *degree*, *weight* and *length* by

$$d(b) = m_{\ell,n_\ell} + \cdots + m_{\ell,1} + \cdots + m_{1,n_1} + \cdots + m_{1,1}, \quad w(b) = n_1\gamma_1 + \cdots + n_\ell\gamma_\ell$$

and

$$l(b) = n_1 + \cdots + n_\ell.$$

Theorem 1. *A spanning set of $W(\Lambda_r)$ is given by the set of monomial vectors* (1) *satisfying initial conditions*

$$m_{i,n} \ge 1 + \sum_{j<i} n_j + \delta_{i\le r} \tag{9}$$

and difference conditions

$$m_{i,n+1} \ge m_{i,n} + 2, \quad 1 \le n \le n_i - 1. \tag{10}$$

Proof. Difference conditions follow from (3): Assume that b does not satisfy (10). Then $b = b'x_j(-m)x_j(-m')$, for some monomial b' and $m' \le m \le m' + 1$. By (3) and (5), on $W(\Lambda_r)$ we have

$$x_j(-m)x_j(-m') = C_1 x_j(-m-1)x_j(-m'+1) + \cdots + C_{m'-1}x_j(-m-m'+1)x_j(-1),$$

for some $C_i \in \mathbb{C}^\times$. Multiply this by b' to obtain b expressed as a linear combination of greater monomials of the same degree and weight. Note also that because of (5) some of the monomials in this linear combination annihilate the highest weight vector.

Now assume that b does not satisfy (9); let $b = b_2 x_j(-m) b_1$, where b_1 contains all factors of colors $\gamma_1, \dots, \gamma_{j-1}$ and

$$m < 1 + \sum_{i<j} n_i + \delta_{j\le r}.$$

We will prove that b can be expressed in terms of greater monomials of the same degree and weight. The proof is done by induction on the length

$$l(b_1) = \sum_{i<j} n_i.$$

If $l(b_1) = 0$, then (5) gives $x_j(-m)v_r = 0$. Now, assume that all monomials with less than $l(b_1)$ factors of colors $\gamma_1, \dots, \gamma_{j-1}$, can be expressed by greater monomials of the same degree and weight. We can also assume that

$$m = \sum_{i<j} n_i + \delta_{j\le r}.$$

Let $x_k(-n)$ be the smallest factor in b_1; $b_1 = x_k(-n)b_1'$. By (4) and (5) we have

$$\begin{aligned} x_j(-m)x_k(-n) = {} & C_{1+\delta_{j\le r}} x_j(-1-\delta_{j\le r}) x_k(-n-m+1+\delta_{j\le r}) + \dots \\ & + C_{m-1} x_j(-m+1) x_k(-n-1) \\ & + C_{m+1} x_j(-m-1) x_k(-n+1) + \dots \end{aligned} \tag{11}$$

for some $C_i \in \mathbb{C}$. Multiply this by $b_2 b_1'$ and obtain

$$\begin{aligned} b = {} & b_2 x_j(-1-\delta_{j\le r}) x_k(-n-m+1+\delta_{j\le r}) b_1' + \dots \\ & + b_2 x_j(-m+1) x_k(-n-1) b_1' + b_2 x_j(-m-1) x_k(-n+1) b_1' + \dots \end{aligned} \tag{12}$$

On the right-hand side we have monomials

$$b_2 x_j(-m-1) x_k(-n+1) b_1', b_2 x_j(-m-2) x_k(-n+2) b_1', \dots$$

which are greater than b. But, we also have the first few monomials

$$b_2 x_j(-1-\delta_{j\le r}) x_k(-n-m+1+\delta_{j\le r}) b_1', \dots, b_2 x_j(-m+1) x_k(-n-1) b_1'. \tag{13}$$

Consider their factors $x_j(-1-\delta_{j\le r})b_1', \dots, x_j(-m+1)b_1'$. By the inductive assumption, they can be expressed as linear combinations of greater monomials of the same degree and weight. Then it is obvious that by multiplying these linear expressions by $b_2 x_k(-n-m+1+\delta_{j\le r}), \dots, b_2 x_k(-n-1)$ we obtain linear expressions for monomials in (13) in terms of greater monomials. Moreover, these monomials will also be greater than b. Hence, we have expressed b in terms of a greater monomial.

Since the number of monomials of the same degree and weight is finite (by (5)), in the end, we obtain monomials that satisfy relations (9) and (10). $\square$

Theorem 2. *The spanning set*

$$\mathcal{B} = \{bv_r | b \text{ satisfies } (10) \text{ and } (9)\} \tag{14}$$

is a basis of $W(\Lambda_r)$.

Proof. Let $b \in \mathcal{B}$. We first prove a particular case: if

$$Cbv_r = 0,$$

then $C = 0$. We prove this by induction on degree of b.

Obviously, if $b = 1$, then $C = 0$. Let $x_i(-n)$ be the greatest factor in b; $b = b'x_i(-n)$.

If $i \leq r$, then, since $v_r = e^{\omega_r} = e^{\omega_r}v_0$, we have

$$Cbv_r = Cbe^{\omega_r}v_0 = e^{\omega_r}C'Cb''v_0 = 0,$$

where $C' \in \mathbb{C}^\times$ and b'' is obtained from b by decreasing degrees of factors of color $\gamma_1, \ldots, \gamma_r$ by 1 (see (8)). Since e^{ω_r} is injective, $Cb''v_0 = 0$. Monomial b'' satisfies the difference and initial conditions for $W(\Lambda_0)$ and it is of a smaller degree then b. By the inductive assumption, we conclude that $C = 0$.

If $i > r$ and $n > 1$, then use operators $[i][i-1]\cdots[r+1]$ to obtain

$$Cbv_i = 0.$$

Then, by (8),

$$Cbv_i = Cbe^{\omega_i}v_0 = e^{\omega_i}C'Cb''v_0 = 0,$$

where $C' \in \mathbb{C}^\times$ and b'' is obtained from b by decreasing degrees of factors of color γ_i by 1 (see (7) and (8)). Again, b'' satisfies the difference and initial conditions for $W(\Lambda_0)$ and it is of a smaller degree than b. By induction, we conclude that $C = 0$.

If $i > r$ and $n = 1$, then use operators $[i-1]\cdots[r+1]$ to obtain

$$Cbv_{i-1} = 0,$$

(see (7)). By (6) and (8), we have

$$Cb'x_i(-1)v_{i-1} = C'Cb'e^{\omega_\ell}v_i = e^{\omega_\ell}C''Cb''v_i = 0,$$

where $C', C'' \in \mathbb{C}^\times$ and b'' is obtained from b' by decreasing degrees of all factors by 1. Monomial b'' satisfies the difference and initial conditions for $W(\Lambda_i)$ and it is of a smaller degree than b. By induction, we conclude that $C = 0$.

Now turn to the general relation of linear dependence:

$$\sum_b C_b bv_r = 0. \tag{15}$$

Let b_{min} be the smallest monimial in (15). We use the same operators as in the previous case to reduce $C_{b_{min}}b_{min}v_r$ to $CC_{b_{min}}v_j$, for some j and some $C \in \mathbb{C}^\times$. Note that operators used above at some point annihilate all other monomial vectors in (15). We will get

$$CC_{b_{min}}v_j = 0$$

and we conclude $C_{b_{min}} = 0$. We proceed inductively to conclude that all coefficients C_b in (15) are 0. $\square$

From this combinatorial description of a basis of $W(\Lambda_r)$ we immediately obtain character formulas (cf. [16], [30]): for $n_1, \ldots, n_\ell \geq 0$, set $\alpha = n_1\gamma_1 + \cdots + n_\ell\gamma_\ell$ and

$$\chi^{\alpha}_{W(\Lambda_r)}(q) = \sum_i q^i \mathrm{card}\,\{b \mid w(b) = \alpha, d(b) = i\}$$

Corollary 1.

$$\chi^{\alpha}_{W(\Lambda_r)}(q) = \frac{q^{\sum_{i=1}^{\ell} n_i^2 + \sum_{1\leq i<j\leq \ell} n_i n_j + \sum_{i=1}^{r} n_i}}{(q)_{n_1}(q)_{n_2}\cdots(q)_{n_\ell}},$$

where $(q)_n = (1-q)\cdots(1-q^n)$.

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