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On the hyper-order of solutions of nonhomogeneous linear differential equations

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Abstract. In this paper, we study the hyper-order of solutions of higher order linear differential equation

$$f^{(k)} + A_{k-1}(z)f^{(k-1)} + \ldots A_1(z)f' + A_0(z)f = H(z),$$

where $k \geq 2$ is an integer, $A_j(z)$ $(j = 0, 1, \ldots, k-1)$ and $H(z)$ $(\not\equiv 0)$ are entire functions or polynomials. We improve previous results given by Xu and Cao.

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1. Introduction and main results

We assume that the reader is familiar with usual notations and basic results of Nevanlinna theory (see [6, 11]). We also use basic results of Wiman-Valiron theory, (see [7]). In addition, we use the notation $\sigma(f)$ to denote the order of growth of a meromorphic function f, $\lambda(f)$ and $\overline{\lambda}(f)$ to denote the exponent of convergence of a sequence of zeros and a sequence of distinct zeros of f, respectively. We also by $\sigma_2(f)$ denote the hyper-order of f defined by (see [11])

$$\sigma_2(f) = \limsup_{r\to+\infty} \frac{\log\log T(r,f)}{\log r},$$

where $T(r,f)$ is the Nevanlinna characteristic function of f.
The hyper-exponent of convergence of a sequence of zeros and distinct zeros of f are defined by (see [1])

$$\lambda_2(f) = \limsup_{r\to+\infty} \frac{\log\log N(r,\frac{1}{f})}{\log r}$$

and

$$\overline{\lambda}_2(f) = \limsup_{r\to+\infty} \frac{\log\log \overline{N}(r,\frac{1}{f})}{\log r},$$

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respectively, where $N(r,\frac{1}{f})$ and $\overline{N}(r,\frac{1}{f})$ are the counting functions of zeros and distinct zeros of f, respectively.

For a set $E \subset [1,+\infty)$, let $m(E)$ and $m_l(E)$ denote the linear measure and the logarithmic measure of E, respectively. Moreover, the upper logarithmic density and lower logarithmic density of E are defined by

$$\overline{\log dens}(E) = \limsup_{r\to\infty} \frac{m_l(E\cap[1,r])}{\log r}, \ \underline{\log dens}(E) = \liminf_{r\to\infty} \frac{m_l(E\cap[1,r])}{\log r}.$$

For the second order linear differential equation

$$f^{''} + e^{-z} f^{'} + Q(z) f = 0, \tag{1}$$

where $Q(z)$ is an entire function of finite order, it is well known that every solution of equation (1) is an entire function and most solutions of (1) have an infinite order. But equation (1) with $Q(z) = -(4+2e^{-z})$ possesses a solution $f(z) = e^{2z}$ of finite order.

Thus a natural question is: what condition on $Q(z)$ will guarantee that every solution $f(\not\equiv 0)$ of equation (1) has an infinite order?

In [2], Chen studied the problem, where $Q(z) = h(z)e^{bz}$, $h(z)$ is a nonzero polynomial and b is a complex number. He proved that if $b \neq -1$, then every solution $f(\not\equiv 0)$ of equation (1) is of infinite order and $\sigma_2(f) = 1$.

In the same paper, he also considered more general equations of second order and proved the following two results:

Theorem 1 (see [2])**.** *Let $A_j(z)(\not\equiv 0)$ $(j = 0,1)$ be entire functions with $\sigma(A_j) < 1$, a, b complex constants such that $ab \neq 0$ and $a = cb$ $(c > 1)$. Then every solution $f(\not\equiv 0)$ of equation*

$$f^{''} + A_1(z)e^{az} f^{'} + A_0(z)e^{bz} f = 0, \tag{2}$$

has an infinite order.

Theorem 2 (see [2])**.** *Let $A_j(z)(\not\equiv 0)$, $D_j(z)$ $(j = 0,1)$ be entire functions with $\sigma(A_j) < 1$, $\sigma(D_j) < 1$ a, b be complex constants such that $ab \neq 0$ and $\arg a \neq \arg b$ or $a = cb$ $(0 < c < 1)$. Then every solution $f(\not\equiv 0)$ of equation*

$$f^{''} + (A_1(z)e^{az} + D_1(z)) f^{'} + (A_0(z)e^{bz} + D_0(z)) f = 0, \tag{3}$$

has an infinite order.

In 2008, Wang and Laine have investigated nonhomogeneous equations related to (2) and (3) and obtained the following two results:

Theorem 3 (see [9])**.** *Suppose that $A_0(z) \not\equiv 0$, $A_1(z) \not\equiv 0$, and H are entire functions of order less than one, and the complex constants a, b satisfy $ab \neq 0$ and $b \neq a$. Then every nontrivial solution f of equation (2) is of infinite order.*

Theorem 4 (see [9])**.** *Suppose that $A_0(z) \not\equiv 0$, $A_1(z) \not\equiv 0$, $D_0(z)$, $D_1(z)$, and H are entire functions of order less than one, and the complex numbers a, b satisfy $ab \neq 0$ and $b/a < 0$. Then every nontrivial solution f of equation (3) is of infinite order.*

In [10], Xu and Cao have studied the above problem for higher order linear differential equations and proved the following two results:

Theorem 5 (see [10]). *Let $k \geq 2$ be an integer, $P(z) = a_n z^n + \ldots + a_1 z + a_0$, and $Q(z) = b_n z^n + \ldots + b_1 z + b_0$ be nonconstant polynomials, where a_i, b_i $(i = 0, 1, \ldots, n)$ are complex numbers with $a_n b_n \neq 0$ and $a_n \neq b_n$. Suppose that $h_i(z)$ $(2 \leq i \leq k-1)$ are polynomials with degree no more than $n-1$ in z, $A_j(z) \not\equiv 0$ $(j = 0, 1)$ and H are entire functions satisfying $\sigma(A_j) < n$, $(j = 0, 1)$ and $\sigma(H) < n$, and φ is an entire function of finite order. Then every nontrivial solution f of equation*

$$f^{(k)} + h_{k-1} f^{(k-1)} + \ldots + h_2 f^{''} + A_1 e^{P(z)} f^{'} + A_0 e^{Q(z)} f = H \tag{4}$$

satisfies $\sigma(f) = +\infty$, $\sigma(f) = \lambda(f) = \overline{\lambda}(f) = \overline{\lambda}(f - \varphi) = +\infty$ and $\sigma_2(f) = \lambda_2(f) = \overline{\lambda}_2(f) = \overline{\lambda}_2(f - \varphi) \leq n$.

Theorem 6 (see [10]). *Let $k \geq 2$ be an integer. Suppose that $A_j(z) \not\equiv 0$, $D_j(z)$ $(j = 0, 1)$, and H are entire functions satisfying $\sigma(A_j) < n$, $\sigma(D_j) < n$, $(j = 0, 1)$, $\sigma(H) < n$, and $P(z)$, $Q(z)$, h_i $(2 \leq i \leq k-1)$ are as in Theorem 5 satisfying $a_n b_n \neq 0$ and $a_n / b_n < 0$. Then every nontrivial solution f of equation*

$$f^{(k)} + h_{k-1} f^{(k-1)} + \ldots + h_2 f^{''} + \left(A_1 e^{P(z)} + D_1\right) f^{'} + \left(A_0 e^{Q(z)} + D_0\right) f = H \tag{5}$$

is of infinite order.

In this paper, we investigate the hyper-order of nontrivial solutions of equations (4) and (5). We obtain the following two results:

Theorem 7. *Let $k \geq 2$ be an integer, $P(z)$, $Q(z)$, a_n, b_n, $h_i(z)$ $(2 \leq i \leq k-1)$, $A_j(z) \not\equiv 0$ $(j = 0, 1)$, $H(\equiv 0)$ and φ satisfy additional hypotheses of Theorem 5. Then every nontrivial solution f of equation (4) satisfies $\sigma_2(f) = \lambda_2(f) = \overline{\lambda}_2(f) = \overline{\lambda}_2(f - \varphi) = n$.*

Theorem 8. *Let $k \geq 2$ be an integer, $P(z)$, $Q(z)$, a_n, b_n, $h_i(z)$ $(2 \leq i \leq k-1)$, $A_j(z) \not\equiv 0$, $D_j(z)$ $(j = 0, 1)$ and $H(\equiv 0)$ satisfy additional hypotheses of Theorem 6. Then every nontrivial solution f of equation (5) satisfies $\sigma_2(f) = n$.*

2. Preliminary lemmas

Lemma 1 (see [10]). *Suppose that $k \geq 2$ is an integer, A_0, $A_1, \ldots, A_{k-1}$ and F $(\not\equiv 0)$ are entire functions of finite order. Then every solution f of infinite order of equation*

$$f^{(k)} + A_{k-1} f^{(k-1)} + \ldots + A_1 f' + A_0 f = F$$

satisfies $\sigma_2(f) \leq \max\{\sigma(A_j), \sigma(F) : j = 0, 1, \ldots, k-1\}$.

Lemma 2 (see [3]). *Let $f(z)$ be a transcendental meromorphic function and let $\alpha > 1$ and $\varepsilon > 0$ be given constants. Then there exist a set $E_1 \subset [1, +\infty)$ having finite logarithmic measure and a constant $B > 0$ that depends only on α and (i, j) (i, j positive integers with $i > j$) such that for all z satisfying $|z| = r \notin [0, 1] \cup E_1$, we have*

$$\left|\frac{f^{(i)}(z)}{f^{(j)}(z)}\right| \leq B\left[\frac{T(\alpha r, f)}{r}(\log^{\alpha} r)\log T(\alpha r, f)\right]^{i-j}.$$

Lemma 3 (see [5], p. 344). *Let $f(z) = \sum\limits_{n=0}^{+\infty} a_n z^n$ be an entire function, $\nu_f(r)$ denote the central index of f, and $\mu(r)$ denote the maximum term, $\mu(r) = \left|a_{\nu(r)}\right| r^{\nu(r)}$. Then*

$$\nu_f(r) = r\frac{d}{dr}\log\mu(r) < [\log\mu(r)]^2 \leq [\log M(r, f)]^2$$

holds outside a set $E_2 \subset (1, +\infty)$ that has finite logarithmic measure.

Lemma 4 (see [4]). *Let $P(z) = (\alpha + i\beta) z^n + \ldots (\alpha, \beta$ are real numbers, $|\alpha| + |\beta| \neq 0)$ be a polynomial with degree $n \geq 1$ and $A(z)$ a meromorphic function with $\sigma(A) < n$. Set $f(z) = A(z) e^{P(z)}$, $z = re^{i\theta}$, $\delta(P, \theta) = \alpha\cos(n\theta) - \beta\sin(n\theta)$. Then for any given $\varepsilon > 0$, there exists a set $E_3 \subset [1, +\infty)$ having finite logarithmic measure such that for any $\theta \in [0, 2\pi) \setminus H_1$ and for $|z| = r \notin [0, 1] \cup E_3$, $r \to +\infty$, we have*

(i) *if $\delta(P, \theta) > 0$, then*

$$\exp\{(1 - \varepsilon)\delta(P, \theta) r^n\} \leq \left|f\left(re^{i\theta}\right)\right| \leq \exp\{(1 + \varepsilon)\delta(P, \theta) r^n\},$$

(ii) *if $\delta(P, \theta) < 0$, then*

$$\exp\{(1 + \varepsilon)\delta(P, \theta) r^n\} \leq \left|f\left(re^{i\theta}\right)\right| \leq \exp\{(1 - \varepsilon)\delta(P, \theta) r^n\},$$

where $H_1 = \{\theta \in [0, 2\pi) : \delta(P, \theta) = 0\}$.

Lemma 5 (see [8]). *Let $f(z)$ be a transcendental entire function, and let $\nu_f(r)$ be the central index of f and δ a constant such that $0 < \delta < \frac{1}{4}$. Then there exists a set E_4 of finite logarithmic measure such that for all z satisfying $|z| = r \notin E_4$ and $|f(z)| \geq M(r, f)\,\nu_f(r)^{-\frac{1}{4}+\delta}$, we have*

$$\frac{f^{(n)}(z)}{f(z)} = \left(\frac{\nu_f(r)}{z}\right)^n (1 + o(1)) \quad (n \geq 1 \text{ is an integer}).$$

Lemma 6 (see [12]). *Let $f(z)$ be an entire function and $M(r, f) = \left|f\left(re^{i\theta_r}\right)\right|$ for every r. Set $\theta_r \to \theta_0 \in [0, 2\pi)$ as $r \to +\infty$. Then there exist a constant $l_0 > 0$ and a set E of positive lower logarithmic density such that*

$$M(r, f)^{1/5} \leq \left|f\left(re^{i\theta}\right)\right| \tag{6}$$

for all $r \in E$ large enough and all θ such that $|\theta - \theta_0| < l_0$.

Lemma 7 (see [2]). *Let $f(z)$ be an entire function of infinite order and $\sigma_2(f) = \alpha < +\infty$, and let a set $E_5 \subset (1, +\infty)$ that has finite logarithmic measure. Then there exists a sequence of points $\{z_m = r_m e^{i\theta_m}\}$ such that $|f(z_m)| = M(r_m, f)$, $\theta_m \in [0, 2\pi)$, $\lim\limits_{m \to +\infty} \theta_m = \theta_0 \in [0, 2\pi)$, $r_m \notin E_5$, $r_m \to +\infty$,*

$$\lim_{r_m \to +\infty} \frac{\log \nu_f(r_m)}{\log r_m} = +\infty$$

and for any given $\varepsilon > 0$, we have for a sufficiently large r_m

$$\exp\{r_m^{\alpha-\varepsilon}\} < \nu_f(r_m) < \exp\{r_m^{\alpha+\varepsilon}\},$$

where $\nu_f(r)$ is the central index of f.

3. Proof of Theorem 7

Proof. Assume that f is a nontrivial solution of equation (4). Then by Theorem 5, we have $\sigma(f) = +\infty$ and $\sigma_2(f) = \lambda_2(f) = \overline{\lambda}_2(f) = \overline{\lambda}_2(f - \varphi) \leq n$. We assert that $\sigma_2(f) = n$. Now we assume that $\sigma_2(f) = \alpha < n$ and we prove that $\sigma_2(f) = \alpha$ fails. By Lemma 2, there exist a constant $B > 0$ and a set $E_1 \subset [1, +\infty)$ having finite logarithmic measure such that for all z satisfying $|z| = r \notin [0, 1] \cup E_1$, we have

$$\left|\frac{f^{(j)}(z)}{f(z)}\right| \leq Br\,[T(2r, f)]^{j+1}\ (j = 1, \ldots, k) \tag{7}$$

Let $\nu_f(r)$ denote the central index of f. By Lemma 3, there is a set $E_2 \subset (1, +\infty)$ that has finite logarithmic measure such that for $|z| = r \notin [0, 1] \cup E_2$, we have

$$\nu_f(r) < [\log M(r, f)]^2. \tag{8}$$

By Lemma 4, for any given $\varepsilon > 0$, there exists a set $E_3 \subset [1, +\infty)$ having finite logarithmic measure such that for any $\theta \in [0, 2\pi) \setminus H_2$, where

$$H_2 = \{\theta \in [0, 2\pi) : \delta(P, \theta) = 0 \text{ or } \delta(Q - P, \theta) = 0 \text{ or } \delta(Q, \theta) = 0\}$$

and for $|z| = r \notin [0, 1] \cup E_3$, $r \to +\infty$, we have

- if $\delta(P, \theta) > 0$, then

$$\exp\{(1 - \varepsilon)\,\delta(P, \theta)\,r^n\} \leq \left|A_1(z)\,e^{P(z)}\right| \leq \exp\{(1 + \varepsilon)\,\delta(P, \theta)\,r^n\}, \tag{9}$$

- if $\delta(P, \theta) < 0$, then

$$\exp\{(1 + \varepsilon)\,\delta(P, \theta)\,r^n\} \leq \left|A_1(z)\,e^{P(z)}\right| \leq \exp\{(1 - \varepsilon)\,\delta(P, \theta)\,r^n\}, \tag{10}$$

- if $\delta(Q - P, \theta) > 0$, then

$$\begin{aligned} \exp\{(1 - \varepsilon)\,\delta(Q - P, \theta)\,r^n\} &\leq \left|\frac{A_0(z)}{A_1(z)} e^{Q(z) - P(z)}\right| \\ &\leq \exp\{(1 + \varepsilon)\,\delta(Q - P, \theta)\,r^n\}, \end{aligned} \tag{11}$$

- if $\delta(Q-P,\theta)<0$, then

$$\begin{aligned}\exp\{(1+\varepsilon)\,\delta(Q-P,\theta)\,r^n\} &\le \left|\frac{A_0(z)}{A_1(z)}e^{Q(z)-P(z)}\right| \\ &\le \exp\{(1-\varepsilon)\,\delta(Q-P,\theta)\,r^n\},\end{aligned} \tag{12}$$

- if $\delta(Q,\theta)>0$, then

$$\exp\{(1-\varepsilon)\,\delta(Q,\theta)\,r^n\} \le \left|A_0(z)\,e^{Q(z)}\right| \le \exp\{(1+\varepsilon)\,\delta(Q,\theta)\,r^n\}, \tag{13}$$

- if $\delta(Q,\theta)<0$, then

$$\exp\{(1+\varepsilon)\,\delta(Q,\theta)\,r^n\} \le \left|A_0(z)\,e^{Q(z)}\right| \le \exp\{(1-\varepsilon)\,\delta(Q,\theta)\,r^n\}. \tag{14}$$

By Lemma 5, for any given constant $0<\delta<\frac{1}{4}$, there exists a set E_4 of finite logarithmic measure such that for all z satisfying $|z|=r\notin E_4$ and $|f(z)|\ge M(r,f)\,\nu_f(r)^{-\frac{1}{4}+\delta}$, we have

$$\frac{f^{(j)}(z)}{f(z)}=\left(\frac{\nu_f(r)}{z}\right)^j(1+o(1))\ \ (j=1,\dots,k). \tag{15}$$

Since $m_l(E_1\cup E_2\cup E_3\cup E_4)<+\infty$, then $m_l(E\setminus([0,1]\cup E_1\cup E_2\cup E_3\cup E_4))$ is infinite, where E is the set defined in Lemma 6. Thus by Lemma 7, there exists a sequence of points $\left\{z_m=r_me^{i\theta_m}\right\}$ such that $|f(z_m)|=M(r_m,f)$, $\theta_m\in[0,2\pi)$, $\lim\limits_{m\to+\infty}\theta_m=\theta_0\in[0,2\pi)$, $r_m\in E\setminus([0,1]\cup E_1\cup E_2\cup E_3\cup E_4)$, $r_m\to+\infty$,

$$\lim_{r_m\to+\infty}\frac{\log\nu_f(r_m)}{\log r_m}=+\infty \tag{16}$$

and for any given $\varepsilon>0$, we have for a sufficiently large r_m

$$\exp\left\{r_m^{\alpha-\varepsilon}\right\}<\nu_f(r_m)<\exp\left\{r_m^{\alpha+\varepsilon}\right\}. \tag{17}$$

By (16), for any sufficiently large $A>2\sigma(H)$ and m sufficiently large, we have

$$\nu_f(r_m)>r_m^A. \tag{18}$$

By (8) and (18), for m sufficiently large we obtain

$$M(r_m,f)>\exp\left\{r_m^{A/2}\right\}. \tag{19}$$

On the other hand, for any given ε $(0<2\varepsilon<A-2\sigma(H))$ and m sufficiently large, we have

$$|H(z_m)|\le\exp\left\{r_m^{\sigma(H)+\varepsilon}\right\}. \tag{20}$$

From (19) and (20), it follows that

$$\frac{|H(z_m)|}{M(r_m, f)} \to 0 \tag{21}$$

as $r_m \to +\infty$.

For the above θ_0, there are three cases: $\delta(P,\theta_0) > 0$, $\delta(P,\theta_0) < 0$ and $\delta(P,\theta_0) = 0$.

Case 1. $\delta(P,\theta_0) > 0$. From the continuity of $\delta(P,\theta)$, we have

$$\frac{1}{2}\delta(P,\theta_0) < \delta(P,\theta_m) < \frac{3}{2}\delta(P,\theta_0) \tag{22}$$

for m sufficiently large. For any given ε $(0 < 2\varepsilon < \min\{1, n-\alpha,\ A - 2\sigma(H)\})$, from (9) and (22), we have

$$\exp\left\{\frac{(1-\varepsilon)}{2}\delta(P,\theta_0)r_m^n\right\} \le \left|A_1(z_m)e^{P(z_m)}\right| \le \exp\left\{\frac{3(1+\varepsilon)}{2}\delta(P,\theta_0)r_m^n\right\} \tag{23}$$

for m sufficiently large.

Subcase 1.1. We first assume that θ_0 satisfies $\eta := \delta(Q-P,\theta_0) > 0$. From the continuity of $\delta(Q-P,\theta)$, we have

$$\frac{1}{2}\delta(Q-P,\theta_0) < \delta(Q-P,\theta_m) < \frac{3}{2}\delta(Q-P,\theta_0). \tag{24}$$

Hence by (11) and (24), for the above ε, we have

$$\exp\left\{\frac{(1-\varepsilon)}{2}\eta r_m^n\right\} \le \left|\frac{A_0(z_m)}{A_1(z_m)}e^{Q(z_m)-P(z_m)}\right| \le \exp\left\{\frac{3(1+\varepsilon)}{2}\eta r_m^n\right\} \tag{25}$$

for m sufficiently large.

From (4) we obtain

$$\begin{aligned}\left|\frac{A_0(z)}{A_1(z)}e^{Q(z)-P(z)}\right| \le \left|\frac{f'(z)}{f(z)}\right| &+ \frac{1}{|A_1(z)e^{P(z)}|}\\ &\times\left(\left|\frac{f^{(k)}(z)}{f(z)}\right| + \sum_{j=2}^{k-1}\left|h_j(z)\frac{f^{(j)}(z)}{f(z)}\right| + \left|\frac{H(z)}{f(z)}\right|\right).\end{aligned} \tag{26}$$

Substituting (15) into (26) and from (17), (21), (23) and (25), for m sufficiently large we have

$$\begin{aligned}\exp\left\{\frac{(1-\varepsilon)}{2}\eta r_m^n\right\} \le &\exp\left\{r_m^{\alpha+\varepsilon}\right\} r_m^{-1}|1+o(1)| + \\ &+\exp\left\{-\frac{(1-\varepsilon)}{2}\delta(P,\theta_0)r_m^n\right\}\left[\exp\left\{kr_m^{\alpha+\varepsilon}\right\} r_m^{-k}|1+o(1)|\right]\\ &+\exp\left\{-\frac{(1-\varepsilon)}{2}\delta(P,\theta_0)r_m^n\right\}\\ &\times\left[M_1 r_m^{d_1}\exp\left\{(k-1)r_m^{\alpha+\varepsilon}\right\}|1+o(1)| + o(1)\right],\end{aligned} \tag{27}$$

where M_1 (>0) is a constant and d_1 is an entire number. This is a contradiction.

Subcase 1.2. $\eta := \delta(Q-P,\theta_0) < 0$. From the continuity of $\delta(Q-P,\theta_0)$ and (12), for m sufficiently large we have

$$\exp\left\{\frac{3(1+\varepsilon)}{2}\eta r_m^n\right\} \leq \left|\frac{A_0(z_m)}{A_1(z_m)}e^{Q(z_m)-P(z_m)}\right| \leq \exp\left\{\frac{(1-\varepsilon)}{2}\eta r_m^n\right\} \tag{28}$$

From (4) we obtain

$$\begin{aligned}\left|\frac{f'(z)}{f(z)}\right| \leq \left|\frac{A_0(z)}{A_1(z)}e^{Q(z)-P(z)}\right| &+ \frac{1}{\left|A_1(z)e^{P(z)}\right|}\\ &\times\left(\left|\frac{f^{(k)}(z)}{f(z)}\right| + \sum_{j=2}^{k-1}\left|h_j(z)\frac{f^{(j)}(z)}{f(z)}\right| + \left|\frac{H(z)}{f(z)}\right|\right).\end{aligned} \tag{29}$$

Substituting (15) into (29) and from (17), (21), (23) and (28), for m sufficiently large we have

$$\begin{aligned}\left(\frac{\nu_f(r_m)}{r_m}\right)|1+o(1)| \leq &\exp\left\{\frac{(1-\varepsilon)}{2}\eta r_m^n\right\}\\ &+\exp\left\{-\frac{(1-\varepsilon)}{2}\delta(P,\theta_0)r_m^n\right\}\left[\exp\left\{kr_m^{\alpha+\varepsilon}\right\}r_m^{-k}|1+o(1)|\right]\\ &+\exp\left\{-\frac{(1-\varepsilon)}{2}\delta(P,\theta_0)r_m^n\right\}\\ &\times\left[M_2 r_m^{d_2}\exp\left\{(k-1)r_m^{\alpha+\varepsilon}\right\}|1+o(1)|+o(1)\right],\end{aligned} \tag{30}$$

where M_2 (>0) is a constant and d_2 is an entire number. This implies that $\nu_f(r_m) \to 0$ as $m \to +\infty$, which is impossible.

Subcase 1.3. $\eta := \delta(Q-P,\theta_0) = 0$. Here (6) may be used to construct another sequence of points $\left\{z_m^* = r_m e^{i\theta_m^*}\right\}$ with $\lim\limits_{m\to+\infty}\theta_m^* = \theta_0^*$ such that $\eta_1 := \delta(Q-P,\theta_0^*) > 0$. Indeed, without loss of generality, we may suppose that

$$\begin{aligned}\delta(Q-P,\theta) > 0,\ \theta \in \left(\frac{\theta_0+2k\pi}{n},\frac{\theta_0+(2k+1)\pi}{n}\right),\\ \delta(Q-P,\theta) < 0,\ \theta \in \left(\frac{\theta_0+(2k-1)\pi}{n},\frac{\theta_0+2k\pi}{n}\right)\end{aligned} \tag{31}$$

with $k \in \mathbb{Z}$. When m is large enough, we have $|\theta_m - \theta_0| \leq l_0$, where l_0 is a small constant. Choose now θ_m^* such that $\frac{l_0}{2} \leq \theta_m^* - \theta_m \leq l_0$. Then $\theta_0 + \frac{l_0}{2} \leq \theta_0^* \leq \theta_0 + l_0$. For m sufficiently large, we have (6) for z_m^* and $\delta(Q-P,\theta_0^*) > 0$. Therefore

$$\left|\frac{H(z_m^*)}{f(z_m^*)}\right| \leq \frac{\exp\left\{r_m^{\sigma(H)+\varepsilon}\right\}}{(M(r_m,f))^{1/5}} \to 0,\ m \to +\infty \tag{32}$$

and

$$\exp\left\{\frac{(1-\varepsilon)}{2}\eta_1 r_m^n\right\} \leq \left|\frac{A_0\left(z_m^*\right)}{A_1\left(z_m^*\right)}e^{Q(z_m^*)-P(z_m^*)}\right| \leq \exp\left\{\frac{3\left(1+\varepsilon\right)}{2}\eta_1 r_m^n\right\} \tag{33}$$

for m sufficiently large. Taking now l_0 small enough, we have $\delta(P, \theta_0^*) > 0$ by the continuity of $\delta(P, \theta)$. Hence

$$\exp\left\{\frac{(1-\varepsilon)}{2}\delta(P,\theta_0^*)r_m^n\right\} \leq \left|A_1\left(z_m^*\right)e^{P(z_m^*)}\right| \leq \exp\left\{\frac{3\left(1+\varepsilon\right)}{2}\delta(P,\theta_0^*)r_m^n\right\} \tag{34}$$

for m sufficiently large. By (7), (26), (32)-(34), we obtain

$$\exp\left\{\frac{(1-\varepsilon)}{2}\eta_1 r_m^n\right\} \leq M_3 r_m^{d_3}[T(2r,f)]^{k+1}, \tag{35}$$

where M_3 (>0) is a constant and d_3 is an entire number. Thus $\sigma_2(f) \geq n$ and this contradicts $\sigma_2(f) < n$.

Case 2. $\delta(P, \theta_0) < 0$. From the continuity of $\delta\left(P, \theta\right)$ and (10). For any given ε $\left(0 < 2\varepsilon < \min\left\{1,\ n-\alpha,\ A-2\sigma\left(H\right)\right\}\right)$, we have

$$\exp\left\{\frac{3\left(1+\varepsilon\right)}{2}\delta(P,\theta_0)r_m^n\right\} \leq \left|A_1\left(z_m\right)e^{P(z_m)}\right| \leq \exp\left\{\frac{(1-\varepsilon)}{2}\delta(P,\theta_0)r_m^n\right\} \tag{36}$$

for m sufficiently large.

Subcase 2.1. $\delta(Q, \theta_0) > 0$. From the continuity of $\delta\left(Q, \theta\right)$ and (13), for the above ε and m sufficiently large, we have

$$\exp\left\{\frac{(1-\varepsilon)}{2}\delta(Q,\theta_0)r_m^n\right\} \leq \left|A_0\left(z_m\right)e^{Q(z_m)}\right| \leq \exp\left\{\frac{3\left(1+\varepsilon\right)}{2}\delta(Q,\theta_0)r_m^n\right\}. \tag{37}$$

From (4) we obtain

$$\begin{aligned} |A_0\left(z\right)e^{Q(z)}| \leq &\left|\frac{f^{(k)}(z)}{f(z)}\right| + \sum_{j=2}^{k-1}\left|h_j\left(z\right)\frac{f^{(j)}(z)}{f(z)}\right| \\ &+ \left|A_1(z)e^{P(z)}\right|\left|\frac{f^{'}(z)}{f(z)}\right| + \left|\frac{H\left(z\right)}{f(z)}\right|. \end{aligned} \tag{38}$$

Substituting (15) into (38) and from (17),(21),(36) and (37), for m sufficiently large we have

$$\exp\left\{\frac{(1-\varepsilon)}{2}\delta(Q,\theta_0)r_m^n\right\}\leq \exp\left\{kr_m^{\alpha+\varepsilon}\right\}r_m^{-k}\left|1+o\left(1\right)\right| + M_4 r_m^{d_4}\exp\left\{(k-1)\,r_m^{\alpha+\varepsilon}\right\}\left|1+o(1)\right| + \exp\left\{\frac{(1-\varepsilon)}{2}\delta(P,\theta_0)r_m^n\right\}r_m^{-1}\exp\left\{r_m^{\alpha+\varepsilon}\right\} \times\left|1+o(1)\right|+o(1), \quad (39)$$

where M_4 (>0) is a constant and d_4 is an entire number. This is a contradiction.

Subcase 2.2. $\delta(Q,\theta_0)<0$. From the continuity of $\delta\left(Q,\theta\right)$ and (14) for the above ε and m sufficiently large, we have

$$\exp\left\{\frac{3\left(1+\varepsilon\right)}{2}\delta\left(Q,\theta_0\right)r_m^n\right\}\leq\left|A_0\left(z_m\right)e^{Q(z_m)}\right|\leq\exp\left\{\frac{(1-\varepsilon)}{2}\delta\left(Q,\theta_0\right)r_m^n\right\}. \quad (40)$$

From (4) we obtain

$$\left|\frac{f^{(k)}(z)}{f(z)}\right|\leq\sum_{j=2}^{k-1}\left|h_j\left(z\right)\frac{f^{(j)}(z)}{f(z)}\right|+\left|A_1(z)e^{P(z)}\right|\left|\frac{f'(z)}{f(z)}\right| + \left|A_0(z)e^{Q(z)}\right|+\left|\frac{H\left(z\right)}{f(z)}\right|. \quad (41)$$

Substituting (15) into (41) and from (17) ,(21) ,(36) and (40), for m sufficiently large we have

$$\left(\nu_f(r_m)\right)^k r_m^{-k}\left|1+o(1)\right|\leq M_5 r_m^{d_5}\left(\nu_f(r_m)\right)^{k-1}\left|1+o(1)\right| + \exp\left\{\frac{(1-\varepsilon)}{2}\delta(P,\theta_0)r_m^n\right\}\exp\left\{r_m^{\alpha+\varepsilon}\right\}r_m^{-1}\left|1+o(1)\right| + \exp\left\{\frac{(1-\varepsilon)}{2}\delta(Q,\theta_0)r_m^n\right\}+o\left(1\right), \quad (42)$$

where M_5 (>0) is a constant and d_5 is an entire number. This is a contradiction.

Subcase 2.3. $\delta(Q,\theta_0)=0$. By using a similar argument as in Subcase 1.3, we may again construct another sequence of points $\left\{z_m^*=r_m e^{i\theta_m^*}\right\}$ satisfying $\frac{l_0}{2}\leq\theta_m^*-\theta_m\leq l_0$ with $\lim\limits_{m\to+\infty}\theta_m^*=\theta_0^*$ such that $\delta(P,\theta_0^*)<0<\delta(Q,\theta_0^*)$. Replace $\delta(P,\theta_0)$ with $\delta(P,\theta_0^*)$ in (36) and $\delta(Q,\theta_0)$ with $\delta(Q,\theta_0^*)$ in (37). Using the same argument as in Subcase 1.3, we also have (32) for the sequence of points z_m^*. From (38) and m sufficiently large, we have

$$\exp\left\{\frac{(1-\varepsilon)}{2}\delta(Q,\theta_0^*)r_m^n\right\}\leq M_6 r_m^{d_6}\left[T(2r,f)\right]^{k+1}, \quad (43)$$

where M_6 (>0) is a constant and d_6 is an entire number. Thus $\sigma_2(f)\geq n$ and this contradicts $\sigma_2(f)<n$.

Case 3. $\delta(P,\theta_0)=0$. We discuss three cases according to $\delta(Q,\theta_0)$ as follows:

Subcase 3.1. $\delta(Q,\theta_0)>0$. By an argument similar to Subcase 1.3, we can choose another sequence of points $\left\{z_m^*=r_m e^{i\theta_m^*}\right\}$ satisfying $\frac{l_0}{2}\le\theta_m^*-\theta_m\le l_0$ with $\lim\limits_{m\to+\infty}\theta_m^*=\theta_0^*$, such that z_m^* satisfies (32) and $\delta(P,\theta_0^*)<0<\delta(Q,\theta_0^*)$. Similarly to Subcase 2.3, a contradiction follows as m is large enough.

Subcase 3.2. $\delta(Q,\theta_0)<0$. By the definition of $\delta(P,\theta)$ in Lemma 4, we may define

$$\delta'(P,\theta)=-n\alpha\sin(n\theta)-n\beta\cos(n\theta),$$

where $a_n=\alpha+i\beta$. Since $a_n\ne0$, we have $\delta'(P,\theta_0)\ne0$. Take $z_m^{'}=r_m e^{i\theta_m^{'}}$ satisfying $0<\left|\theta_m^{'}-\theta_0\right|\le l_0$, we know that $z_m^{'}$ satisfies (32) and $\delta(P,\theta_m')\ne0$. By the continuity of $\delta(Q,\theta)$, we may assume that $\delta(Q,\theta_m')<0<\delta(P,\theta_m')$ for a suitable l_0, $0<\theta_m^{'}-\theta_0\le l_0$. Then $\delta'(P,\theta_0)>0$, which means that for a suitable l_0,

$$\frac{1}{2}\delta'(P,\theta_0)<\delta'(P,\theta)<\frac{3}{2}\delta'(P,\theta_0),\ \theta\in(\theta_0,\theta_0+l_0). \tag{44}$$

Since we have chosen z_m such that $|f(z_m)|=M(r_m,f)$ and $\theta_m\to\theta_0$ as $m\to\infty$, we have $|f(r_m e^{i\theta_0})|\ge M(r_m,f)v_f(r_m)^{-\frac{1}{4}+\delta}$ for m sufficiently large. From (4), we have

$$\begin{aligned}\left|\frac{f^{'}(z)}{f(z)}\right|\le&\left|\frac{1}{A_1(z)e^{P(z)}}\right|\left(|\frac{f^{(k)}(z)}{f(z)}|+\sum_{j=2}^{k-1}|h_j(z)||\frac{f^{(j)}(z)}{f(z)}|\right)\\&+\left|\frac{1}{A_1(z)e^{P(z)}}\right|\left(|A_0(z)e^{Q(z)}|+\left|\frac{H(z)}{f(z)}\right|\right).\end{aligned} \tag{45}$$

By (9) and (14), for the above ε and for m sufficiently large we have

$$\exp\{(1+\varepsilon)\,\delta(Q,\theta_m')r_m^n\}\le|A_0\,(z_m')\,e^{Q(z_m')}|\le\exp\{(1-\varepsilon)\,\delta(Q,\theta_m')r_m^n\} \tag{46}$$

and

$$\exp\{-\,(1+\varepsilon)\,\delta(P,\theta_m')r_m^n\}\le\left|\frac{e^{-P(z_m')}}{A_1\,(z_m')}\right|\le\exp\{-\,(1-\varepsilon)\,\delta(P,\theta_m')r_m^n\} \tag{47}$$

for m sufficiently large. From the definition of the hyper-order, it follows that

$$T(2r_m,f)\le\exp\{(2r_m)^{\alpha+\varepsilon}\} \tag{48}$$

for m sufficiently large. By (7), (32), (45) – (48), for m sufficiently large we can get

$$\left|\frac{f^{'}(z_m')}{f(z_m')}\right|\le\exp\{-(1-2\varepsilon)\delta(P,\theta_m')r_m{}^n\} \tag{49}$$

Since θ_m' is arbitrary in (θ_0,θ_0+l_0), for m sufficiently large, we can obtain

$$\left|\frac{f^{'}(r_m e^{i\theta})}{f(r_m e^{i\theta})}\right|\le\exp\{-(1-2\varepsilon)\delta(P,\theta)r_m^n\},\ \theta\in(\theta_0,\theta_0+l_0) \tag{50}$$

Therefore, for $\theta \in (\theta_0, \theta_0 + l_0)$, we have

$$\begin{aligned}\xi(r_m, \theta) =& r_m \int_{\theta_0}^{\theta} \left| \frac{f'(r_m e^{i\theta})}{f(r_m e^{i\theta})} \right| d\theta \leq r_m \int_{\theta_0}^{\theta} e^{-\eta_2(\theta) r_m^n} d\theta \\ =& \int_{\theta_0}^{\theta} \frac{1}{\eta_1(\theta) {r_m}^{n-1}} e^{-\eta_2(\theta) r_m^n} d(\eta_2(\theta) r_m^n),\end{aligned} \tag{51}$$

where $\eta_1(\theta) = (1 - 2\varepsilon)\delta'(P, \theta)$ and $\eta_2(\theta) = (1 - 2\varepsilon)\delta(P, \theta)$.
Since $\delta(P, \theta) > 0$ for all $\theta \in (\theta_0, \theta_0 + l_0)$, we can get

$$0 \leq \xi(r_m, \theta) \leq \frac{2}{(1 - 2\varepsilon)\delta'(P, \theta_0) r_m^{n-1}} (e^{-\eta_2(\theta_0) r_m^n} - e^{-\eta_2(\theta) r_m^n}).$$

Thus for m sufficiently large, we can get

$$0 \leq \xi(r_m, \theta) \leq \frac{2}{\eta_1(\theta_0)}. \tag{52}$$

By the proof of Lemma 2.4 in [9], we have

$$\log |f(r_m e^{i\theta_0})| - \xi(r_m, \theta) \leq \log |f(r_m e^{i\theta})| + 2\pi.$$

From this and (52) it follows that

$$\nu_f(r_m)^{-\frac{1}{4}+\delta'} M(r_m, f) = \exp\{-2\pi - 2/\eta_1(\theta_0)\} \nu_f(r_m)^{-\frac{1}{4}+\delta} M(r_m, f) \leq |f(r_m e^{i\theta})| \tag{53}$$

for $\theta \in (\theta_0, \theta_0 + l_0)$, where $0 < \delta' < \delta < \frac{1}{4}$. Therefore, we choose another sequence of points $z_m^* = r_m e^{i\theta_m^*}$ satisfying $\theta_m^* = \frac{l_0}{2} + \theta_0$ and (32) for z_m^*. Furthermore, from (53), we have (15) for z_m^* when m is sufficiently large. Thus, from (15) and (50), we can deduce that $\nu_f(r_m) \to 0$ as $m \to \infty$, which is impossible.
When $\delta(Q, \theta'_m) < 0 < \delta(P, \theta'_m)$ for $-l_0 < \theta'_m - \theta_0 < 0$. Clearly $\xi(r_m, \theta) \leq 0$ for all $\theta \in (\theta_0 - l_0, \theta_0)$. Similarly, we can get

$$\nu_f(r_m)^{-\frac{1}{4}+\delta'} M(r_m, f) = \exp\{-2\pi\} \nu_f(r_m)^{-\frac{1}{4}+\delta} M(r_m, f) \leq |f(r_m e^{i\theta})| \tag{54}$$

for $\theta \in (\theta_0 - l_0, \theta_0)$, where $0 < \delta' < \delta < \frac{1}{4}$. Thus we can also get a contradiction.

Subcase 3.3. Finally, suppose that $\delta(Q, \theta_0) = 0$. We now have $a_n = cb_n$, $c \in \mathbb{R} \setminus \{0, 1\}$. Then we have $P(z) = cb_n z^n + a_{n-1} z^{n-1} + \ldots + a_0$, $Q(z) - P(z) = (1 - c) b_n z^n + R_{n-1}(z)$, where $R_{n-1}(z)$ is a polynomial of degree at most $n - 1$.

If $c < 0$, we may take l_0 small enough such that $\delta(Q, \theta) < 0 < \delta(P, \theta)$, provided that either $\theta \in (\theta_0, \theta_0 + l_0)$ or $(\theta_0 - l_0, \theta_0)$. By an argument similar to that in Subcase 3.2, we can get a contradiction.

If $0 < c < 1$, we similarly obtain $\delta(Q - P, \theta) > 0$ and $\delta(P, \theta) > 0$, provided that either $\theta \in (\theta_0, \theta_0 + l_0)$ or $(\theta_0 - l_0, \theta_0)$ for l_0 small enough. By an argument similar to that in Subcase 1.3, a contradiction follows.

Finally, if $c > 1$, we obtain $\delta(Q - P, \theta) < 0$ and $\delta(P, \theta) > 0$ for either $\theta \in (\theta_0, \theta_0 + l_0)$ or $(\theta_0 - l_0, \theta_0)$. Furthermore, $z'_m = r_m e^{i\theta'_m}$ satisfies (32) provided that either $\theta'_m \in (\theta_0, \theta_0 + l_0)$ or $(\theta_0 - l_0, \theta_0)$. Similarly to Subcase 3.2, we get (50) and (53). By a standard Wiman-Valiron argument, a contradiction also follows. Therefore from these three cases it results that $\sigma_2(f) = n$. $\square$

4. Proof of Theorem 8

Proof. Assume that $f(z)$ is a non-trivial solution of (5). We know that $\sigma(f) = +\infty$. By Lemma 1, it follows that $\sigma_2(f) \leq n$. Set $\sigma_2(f) = \alpha$ and we assert that $\alpha = n$. Now we assume that $\alpha < n$. Since $\rho = \max\{\rho(D_j) : j = 0, 1\} < n$, then for any ε $(0 < 2\varepsilon < n - \rho)$, we have

$$|D_j(z)| \leq \exp\{r^{\rho+\varepsilon}\} \ \ (j = 0, 1)\,. \tag{55}$$

Similarly to the proof of Theorem 7, we can take a sequence of points $z_m = r_m e^{i\theta_m}$, $r_m \to \infty$, such that $\lim\limits_{m\to+\infty} \theta_m = \theta_0$ and $|f(z_m)| = M(r_m, f)$, $r_m \in E \setminus ([0,1] \cup E_1 \cup E_2 \cup E_3 \cup E_4)$ and the sequence of points satisfies $(15) - (17)$ and (21).

Since $a_n/b_n = c < 0$, there are three cases to be discussed, according to the signs of $\delta(P, \theta_0)$ and $\delta(Q, \theta_0)$.

Case 1. Suppose that $\delta(Q, \theta_0) < 0 < \delta(P, \theta_0)$. By (9), (14) and the continuity of $\delta(Q, \theta)$ and $\delta(P, \theta)$, for any given ε $(0 < 2\varepsilon < \min\{1, n - \alpha, A - 2\sigma(H), n - \rho\})$ we have (23) and (40) for m sufficiently large. From (5), we have

$$\begin{aligned}\left|A_1(z)e^{P(z)} + D_1(z)\right|\left|\frac{f'(z)}{f(z)}\right| \leq \left|\frac{f^{(k)}(z)}{f(z)}\right| + \sum_{j=2}^{k-1} |h_j(z)| \left|\frac{f^{(j)}(z)}{f(z)}\right| \\ + \left|(A_0 e^{Q(z)} + D_0(z))\right| + \left|\frac{H(z)}{f(z)}\right|.\end{aligned} \tag{56}$$

Combining (55) with (23) and (40), we conclude

$$|A_0\,(z_m)\, e^{Q(z_m)} + D_0(z_m)| \leq \exp\{r^{\rho+2\varepsilon}\} \tag{57}$$

and

$$|A_1(z_m)e^{P(z_m)} + D_1(z_m)| \geq \exp\left\{\frac{(1-2\varepsilon)}{2}\delta(P, \theta_0) r_m^n\right\} \tag{58}$$

for m large enough. Substituting (15), (21), (57) and (58) into (56), for m sufficiently large we get

$$\begin{aligned}\left(\frac{\nu_f(r_m)}{r_m}\right)|1 + o\,(1)| \leq \exp\left\{\frac{-(1-2\varepsilon)}{2}\delta(P, \theta_0) r_m^n\right\}\left[\exp\{k r_m^{\alpha+\varepsilon}\} r_m^{-k}\, |1 + o\,(1)|\right. \\ + M_7 r_m^{d_7} \exp\{(k-1)\, r_m^{\alpha+\varepsilon}\}\, |1 + o\,(1)| \\ \left. + \exp\{r_m^{\rho+2\varepsilon}\} + o\,(1)\right],\end{aligned} \tag{59}$$

where M_7 (> 0) is a constant and d_7 is an entire number. This implies that $\nu_f(r_m) \to 0$, $m \to +\infty$, which is impossible.

Case 2. Suppose that $\delta(P, \theta_0) < 0 < \delta(Q, \theta_0)$. By (10), (13) and the continuity of $\delta(Q, \theta)$ and $\delta(P, \theta)$, for any given ε $(0 < 2\varepsilon < \min\{1, n - \alpha, \ A - 2\sigma(H), n - \rho\})$, we have (36) and (37) for m sufficiently large. From (5), we have

$$\begin{aligned}\left|A_0(z)e^{Q(z)} + D_0(z)\right| \leq \left|\frac{f^{(k)}(z)}{f(z)}\right| + \sum_{j=2}^{k-1} |h_j(z)| \left|\frac{f^{(j)}(z)}{f(z)}\right| \\ + |(A_1(z)e^{P(z)} + D_1(z))| \left|\frac{f'(z)}{f(z)}\right| + \left|\frac{H(z)}{f(z)}\right|.\end{aligned} \tag{60}$$

Combining (55) with (36) and (37), we conclude

$$|A_1\,(z_m)\,e^{P(z_m)} + D_1(z_m)| \le \exp\{r^{\rho+2\varepsilon}\} \tag{61}$$

and

$$|A_0(z_m)e^{Q(z_m)} + D_0(z_m)| \ge \exp\left\{\frac{(1-2\varepsilon)}{2}\delta(Q,\theta_0)r_m^n\right\} \tag{62}$$

for m large enough. Substituting (15), (21),(61) and (62) into (60), for m sufficiently large we get

$$\exp\left\{\frac{(1-2\varepsilon)}{2}\delta(Q,\theta_0)r_m^n\right\} \le M_8 r_m^{d_8}\exp\{kr_m^{\alpha+\varepsilon}\}\exp\{r_m^{\rho+2\varepsilon}\}, \tag{63}$$

where M_8 (>0) is a constant and d_8 is an entire number. This is a contradiction.

Case 3. Suppose that $\delta(Q,\theta_0) = 0 = \delta(P,\theta_0)$. Similarly to Subcase 1.3 of the proof of Theorem 7, we may again construct another sequence of points $z_m^* = r_m e^{i\theta_m^*}$ with $\lim\limits_{m\to+\infty}\theta_m^* = \theta_0^*$, such that $\delta(P,\theta_0^*) < 0$ and (32) holds for z_m^*.

Without loss of generality, we can assume that

$$\delta(P,\theta) > 0,\ \theta \in \left(\frac{\theta_0 + 2q\pi}{n}, \frac{\theta_0 + (2q+1)\,\pi}{n}\right)$$

and

$$\delta(P,\theta) < 0,\ \theta \in \left(\frac{\theta_0 + (2q-1)\,\pi}{n}, \frac{\theta_0 + 2q\pi}{n}\right)$$

for all $q \in \mathbb{Z}$. Provided m is large enough, we have $|\theta - \theta_m| \le l_0$. Choosing now θ_0^* such that $\frac{l_0}{2} \le \theta_m - \theta_m^* \le l_0$, then $\theta_0 - l_0 \le \theta_0^* \le \theta_0 - \frac{l_0}{2}$ and $\delta(P,\theta_0^*) < 0$. Since $\delta(Q,\theta_0^*) > 0$, a contradiction follows as in case 2 above. □

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