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The square of the line graph and path ideals

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Abstract. For path ideals of the square of the line graph we compute the Krull dimension and characterize the linear resolution property in combinatorial terms. We bound the Castelnuovo–Mumford regularity and the projective dimension of these ideals in terms of the corresponding invariants of two certain sub-hypergraphs. Finally, we present some open questions.

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Introduction

Let G be a directed graph on the vertex set $V = \{1, \ldots, n\}$ and with the set of directed edges $E(G)$. By $S = \Bbbk[x_1, \ldots, x_n]$ we denote the polynomial ring in n variables over a field $\Bbbk$. For an integer $2 \leq t \leq n$, a sequence of vertices $i_1, \ldots, i_t$ with the property that $(i_j, i_{j+1}) \in E(G)$, for all $1 \leq j \leq t-1$, is called a path of length t in G. The t-path ideal of the graph G is the squarefree monomial ideal whose minimal generators correspond to the paths of length t in G. Path ideals were defined by A. Conca and E. De Negri [6] and they are generalizations of the edge ideals introduced by R.H. Villarreal. In the recent years, path ideals have been intensively studied; see for instance [1, 2, 3, 9, 14, 18, 20]. J. He and A. Van Tuyl [14] proved that path ideals of the rooted trees are sequentially Cohen-Macaulay and this result has been generalized to arbitrary trees [18]. Moreover, the Betti numbers of path ideals of rooted trees have been computed [3] (see also [4]) and [17]. The Betti numbers of path ideals of cycles were computed in [1] and sequentially Cohen-Macaulay path ideals of cycles have been characterized [20]. Being classes of directed graphs which arise from the Hasse diagram of posets endowed with the natural orientation induced from the poset, from the bottom to the top, path ideals of posets also have been studied [18]. Moreover, in [18] connections between path ideals of posets and two statistical ranking models have been established: the Luce-decomposable model and the ascending model, [21].

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In this paper, we are mainly interested in studying the path ideal of the square of the line graph. We focus on its algebraic and homological invariants and properties such as being Cohen-Macaulay, having a linear resolution or being a set-theoretic complete intersection. The structure of the paper is the following: in the first section we recall the notions that will be used throughout the paper and we fix the notations. In the second section, we give a lower bound for the Castelnuovo-Mumford regularity of path ideals of the square of the line graph. Moreover, we express the Castelnuovo-Mumford regularity and the projective dimension in terms of the corresponding invariants of two certain sub-hypergraphs. The third section is devoted to the study of the height of path ideals and a lower bound for their projective dimension is given. A particular class of path ideals of the square of the line graphs will be studied in the fourth section. For this particular class we describe Cohen-Macaulay path ideals and we show that, in this case, path ideals have a linear resolution. These results help us to give a complete characterization of path ideals of the square of the line graph which have a linear resolution. We show in fact that this property is equivalent to having linear first syzygies and to having linear quotients, equivalences that, in general, do not hold. In the last section, we formulate some open problems that follow from a large number of examples obtained by using Singular [12].

1. Background and notations

Throughout this section we recall the notions that will be used throughout this paper.

1.1. Algebraic and homological invariants of squarefree monomial ideals

For more details about this section, one may see for instance [5]. Let $S = \Bbbk[x_1, \ldots, x_n]$ be the polynomial ring in n variables over a field $\Bbbk$ and $I \subseteq S$ a squarefree monomial ideal. Let $\mathbb{F}$ be the graded minimal free resolution of S/I as an S-module

$$\mathbb{F} : 0 \to \bigoplus_j S(-j)^{\beta_{pj}} \to \cdots \to \bigoplus_j S(-j)^{\beta_{1j}} \to S \to S/I \to 0.$$

The numbers $\beta_i = \sum_j \beta_{ij}$ are called the Betti numbers of S/I. The Castelnuovo–Mumford regularity is defined as

$$\operatorname{reg}(S/I) = \max\{j - i : \beta_{ij} \neq 0\}$$

and the projective dimension of S/I is

$$\operatorname{pd}(S/I) = \max\{i : \beta_{ij} \neq 0\}.$$

A particular class of monomial ideals which will be used is that of ideals with linear quotients [16]:

Definition 1. *Let I be a monomial ideal of S. The ideal I has linear quotients (or it is an ideal with linear quotients) if there exists an ordering $u_1, \ldots, u_m$ of its minimal monomial generators such that, for all $2 \le i \le m$, the colon ideals $\langle u_1, \ldots, u_{i-1}\rangle : u_i$ are generated by variables.*

Let $V = \{1, \ldots, n\}$ and let $S = \Bbbk[x_1, \ldots, x_n]$ be the polynomial ring in n variables over a field $\Bbbk$. For $F \subseteq V$, we denote $\mathbf{x}_F = \prod_{i \in F} x_i$. We will also refer to F as the support of the monomial $\mathbf{x}_F$. We recall that, given a monomial $m = x_1^{a_1} \cdots x_n^{a_n}$, *the support of the monomial* m is the set $\operatorname{supp}(m) = \{j \ : \ a_j \neq 0\}$.

If an ideal with linear quotients is squarefree, then one may use the following equivalent definition [15, Corollary 8.2.4]:

Definition 2. *The ideal I is said to have linear quotients if there is an ordering of the minimal monomial generators $\mathbf{x}_{F_1} < \cdots < \mathbf{x}_{F_r}$ such that for all $i < j$ there are some $k < j$ and some $l \in [n]$ such that $l \in F_i \setminus F_j$ and $F_k \setminus F_j = \{l\}$.*

For a monomial ideal with linear quotients generated in one degree, we can compute the Betti numbers. Let I be a monomial ideal of S with $\mathcal{G}(I) = \{u_1, \ldots, u_m\}$ and assume that I has linear quotients with respect to the sequence $u_1, \ldots, u_m$. We denote $L_k = (u_1, \ldots, u_{k-1}) : u_k$, $\operatorname{set}(u_k) = \{i \in [n] \ : \ x_i \in \mathcal{G}(L_k)\}$ and $r_k = |\operatorname{set}(u_k)|$.

Proposition 1 ([15, Corollary 8.2.2])**.** *Let $I \subset S$ be a monomial ideal with linear quotients generated in one degree. Then, with the above notations, one has*

$$\beta_i(I) = \sum_{k=1}^{m} \binom{r_k}{i},$$

for all i. In particular, it follows that

$$\operatorname{pd}(I) = \max\{r_1, \ldots, r_m\}.$$

1.2. Path ideals

Let G be a finite, simple graph with the vertex set $V(G) = \{1, \ldots, n\}$ and the set of edges $E(G)$, and let $S = \Bbbk[x_1, \ldots, x_n]$ be the polynomial ring in n variables over a field $\Bbbk$. R.H. Villarreal associated to the graph G *the edge ideal*, which is the squarefree monomial ideal

$$I(G) = \langle x_i x_j \ : \ \{i, j\} \in E(G)\rangle.$$

If G is also a directed graph, then a sequence of vertices $i_1, \ldots, i_t$ from $V(G)$ with the property that $\{i_j, i_{j+1}\} \in E(G)$ for all $1 \le j \le t-1$ is called *a path of length t* in G (or simply a *t-path*). A. Conca and E. De Negri considered the path ideal of the graph G which is a generalization of the edge ideal [6]. Let $2 \le t \le n$ be a fixed integer. The squarefree monomial ideal

$$I = \langle x_{i_1} \cdots x_{i_t} \ : \ i_1, \ldots, i_t \text{ is a } t\text{-path in } G\rangle$$

is called *the t-path ideal of G*. If it is clear from the context, we will simply call it the path ideal of G. It is obvious that, for $t = 2$, the path ideal is the edge ideal of the graph G.

The square of the graph G, denoted as G^2, is the graph with the same set of vertices as G whose set of edges is

$$E(G) \cup \{\{i,j\} : \text{ there is some } k \in V(G) \text{ such that } \{i,k\},\{j,k\} \in E(G)\}.$$

We are interested in studying the properties of path ideals of the square of the line graph. In order to do this, let us fix the notations. Let $n \geq 2$ be an integer and L_n the line graph, that is, the graph with the vertex set $V = [n] := \{1, \ldots n\}$ and the set of edges $E = \{\{i, i+1\} : 1 \leq i \leq n-1\}$. The square of the line graph L_n is the graph L_n^2 with the set of vertices $V = [n]$ and the set of edges $\{\{i,j\} : j - i \leq 2, 1 \leq i < j \leq n\}$. One may view L_n^2 as a directed graph by assigning to each edge $\{i,j\}$, $i < j$ an orientation from i to j. In the next figure, one may see L_6 and L_6^2.

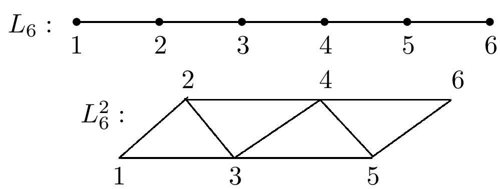

Figure 1: *The graphs L_6 and L_6^2*

The t-path ideal of L_n^2 is the squarefree monomial ideal

$$I_t(L_n^2) = \langle x_{i_1} \cdots x_{i_t} \ : 1 \leq \ i_j - i_{j-1} \leq 2,\ 2 \leq j \leq t\rangle \subseteq S = \Bbbk[x_1, \ldots, x_n]. \qquad (1)$$

Since the square of line graphs are chordal graphs, they have nice properties. Moreover, the only class of chordal graphs for which the path ideals have been studied is that of trees.

1.3. Hypergraphs and edge ideals of hypergraphs

Let V be a finite set and 2^V the set of all subsets of V. *A simple hypergraph $\mathcal{H}$* is a pair $(V, \mathcal{E}(\mathcal{H}))$, where V is the set of vertices of $\mathcal{H}$ and $\mathcal{E}(\mathcal{H}) \subseteq 2^V$ is the set of edges. The hypergraph $\mathcal{H}$ is called *d-uniform* if all its edges have the same cardinality, d. Graphs are particular classes of uniform hypergraphs. Let $V = \{1, \ldots, n\}$ and let $S = \Bbbk[x_1, \ldots, x_n]$ be the polynomial ring in n variables over a field $\Bbbk$. *The edge ideal of the simple hypergraph $\mathcal{H}$* is the squarefree monomial ideal

$$I(\mathcal{H}) = \langle \mathbf{x}_F \ : \ F \in \mathcal{E}(\mathcal{H})\rangle.$$

Moreover, if $I \subseteq S$ is a squarefree monomial ideal with the minimal set of monomial generators $\mathcal{G}(I)$, one may consider the associated hypergraph $\mathcal{H}(I)$ with the set of edges $\{F \subseteq \{1, \ldots n\} \ : \ \mathbf{x}_F \in \mathcal{G}(I)\}$. Therefore, path ideals of a graph can be seen as edge ideals of hypergraphs.

Let $2 \leq t \leq n$ be an integer and $I_t(L_n^2)$ the t-path ideal. By $\mathcal{L}_t(n)$ we denote the associated hypergraph of $I_t(L_n^2)$. Taking into account equation (1), we get that

$$\mathcal{E}(\mathcal{L}_t(n)) = \{\{i_1, \ldots, i_t\} \ : \ 1 \leq i_j - i_{j-1} \leq 2, \ 2 \leq j \leq t\}. \tag{2}$$

Definition 3 ([8])**.** *A monomial ideal I is splittable if I is the sum of two nonzero monomial ideals J and K, that is, $I = J + K$, such that*

(i) $\mathcal{G}(I)$ is the disjoint union of $\mathcal{G}(J)$ and $\mathcal{G}(K)$;

(ii) there is a splitting function

$$\begin{aligned} \mathcal{G}(J \cap K) &\longrightarrow \mathcal{G}(J) \times \mathcal{G}(K) \\ w &\mapsto (\phi(w), \psi(w)) \end{aligned}$$

satisfying

(a) for all $w \in \mathcal{G}(J \cap K)$, $w = \operatorname{lcm}(\phi(w), \psi(w))$,

(b) for every subset $S \subset \mathcal{G}(J \cap K)$, both $\operatorname{lcm}(\phi(S))$ and $\operatorname{lcm}(\psi(S))$ strictly divide $\operatorname{lcm}(S)$.

If J and K satisfy the above properties, then $I = J + K$ is a splitting of I.

Here $\mathcal{G}(I)$ denotes the set of minimal monomial generators of I.

If $\mathcal{H}$ is a hypergraph and E is an edge of $\mathcal{H}$, then E is *a splitting edge* of $\mathcal{H}$ if

$$I(\mathcal{H}) = \langle \mathbf{x}_E \rangle + I(\mathcal{H} \setminus E)$$

is a splitting of $I(\mathcal{H})$. We write $\mathcal{H} \setminus E$ for the sub-hypergraph of $\mathcal{H}$ obtained by removing the edge E (for more details, see [13]).

2. Invariants of path ideals of L_n^2

Let $\mathcal{H}$ be a d-uniform hypergraph with the set of vertices $V = \{1, \ldots, n\}$ and the set of edges $\mathcal{E}$.

Definition 4 ([13])**.** *A chain of length r in $\mathcal{H}$ is a sequence of edges and vertices $E_0, i_1, E_1, i_2, \ldots, E_{r-1}, i_r, E_r$ such that*

(i) $i_1, \ldots, i_r \in V$ are all distinct,

(ii) $E_0, \ldots, E_r \in \mathcal{E}$ are all distinct,

(iii) $i_1 \in E_0$, $i_r \in E_r$ and $i_j, i_{j+1} \in E_j$ for all $j \in \{1, \ldots, r-1\}$.

The following notions have been defined in [13] and we will use the same terminology in the sequel. For two edges $F, G \in \mathcal{E}$ the chain connecting F and G is called *proper* if $|E_i \cap E_{i+1}| = |E_{i+1}| - 1$ for all $i \in \{0, \ldots, r-1\}$, where we denote $F = E_0$

and $G = E_r$. Moreover, the chain is called *irredundant* if no proper subsequence is a chain from F to G. The *distance* between F and G is

$$\mathrm{dist}_{\mathcal{H}}(F,G) = \min\{l \ : \ F = E_0, i_1, E_1, \ldots, E_l = G \text{ is a proper irredundant chain}\}.$$

In the sequel, we show that for any two edges of $\mathcal{L}_t(n)$ there is an irredundant proper chain.

Construction 1. Let $F, G \in \mathcal{E}$ be two edges of $\mathcal{L}_t(n)$. We prove that there is a chain $F = E_0, i_1, E_1, i_2, \ldots, E_{r-1}, i_r, E_r = G$ which satisfies the conditions of Definition 4 and $|E_i \cap E_{i+1}| = |E_{i+1}| - 1$ for all $0 \leq i \leq r-1$. We consider three cases:

Case I: If $F \cap G \neq \emptyset$, then let us denote $d = t - |F \cap G|$, $F = E_0$ and $i_1 = \min(F \cap G)$. Let

$$E_1 = (F \setminus \{\min(F \setminus G)\}) \cup \{\min(G \setminus F)\}$$

and $i_2 = \min(G \setminus F)$. Clearly, $i_1 \in E_0$, $i_1, i_2 \in E_1$, $i_2 \neq i_1$ and $|E_0 \cap E_1| = |E_1| - 1$. Next, we consider

$$E_2 = (E_1 \setminus \{\min(E_1 \setminus G)\}) \cup \{\min(G \setminus E_1)\}$$

and $i_3 = \min(G \setminus E_1)$. Again, one has that $i_2, i_3 \in E_2$, $i_3 \neq i_2, i_1$ and $|E_1 \cap E_2| = |E_2| - 1$. This construction will end after d steps since there are d distinct elements in $G \setminus F$. The last edge, $E_d = G$, will be obtained as

$$\begin{aligned} E_d &= (E_{d-1} \setminus \{\min(E_{d-1} \setminus G)\}) \cup \{\min(G \setminus E_{d-1})\} \\ &= (E_{d-1} \setminus \{\max(F \setminus G)\}) \cup \{\max(G \setminus F)\}. \end{aligned}$$

It is clear that each set E_i is a t-path in L_n^2 since the sets F and G are not disjoint. Moreover, the sequence is irredundant. Since this is the minimal length that such a sequence can have, we also get that $\mathrm{dist}_{\mathcal{H}}(F,G) = d = t - |F \cap G|$.

Case II: If $F \cap G = \emptyset$ and $\min(G) \leq \max(F) + 2$. We may assume that $\min(F) < \min(G)$ since otherwise we interchange F and G. Let $E_0 = F$, $i_1 = \max(F)$ and

$$E_1 = (E_0 \setminus \{\min(E_0)\}) \cup \{\min(G)\}.$$

Since $\min(F) < \min(G) \leq \max(F) + 2$, it is clear, by using (2) that $E_1 \in \mathcal{E}$. Also, one has that i_1 is in both E_0 and E_1. Let $i_2 = \min(G)$ and

$$E_2 = (E_1 \setminus \{\min(E_1 \setminus G)\}) \cup \{\min(G \setminus E_1)\}.$$

Again, it is clear that $E_2 \in \mathcal{E}$ and that i_2 satisfies the conditions from Definition 4. Next, we continue by taking $i_3 = \min(G \setminus E_1)$ and

$$E_3 = E_2 \setminus \{\min(E_2 \setminus G)\}) \cup \{\min(G \setminus E_2)\}.$$

After t steps, the process will stop. We will have that $i_t = \min(G \setminus E_{t-2})$ and

$$E_t = E_{t-1} \setminus \{\min(E_{t-1} \setminus G)\}) \cup \{\min(G \setminus E_{t-1})\},$$

where $\min(G \setminus E_{t-1}) = \max(G)$. By the construction, we note that this is also of minimal length, so in this case, $\operatorname{dist}_{\mathcal{H}}(F, G) = t$.

Case III: If $F \cap G = \emptyset$ and $\min(G) > \max(F) + 2$. As before, let $E_0 = F$, $i_1 = \max(E_0)$ and

$$E_1 = (E_0 \setminus \{\min(E_0)\}) \cup \{\max(E_0) + 2\}.$$

Now we check whether $\min(G) > \max(E_1) + 2$. If this is not the case, then we apply the second case of this construction for E_1 and G, so we will get that $\operatorname{dist}_{\mathcal{H}}(F, G) = t + 1$. If the inequality is true, then we take $i_2 = \max(E_0) + 2$ and

$$E_2 = (E_1 \setminus \{\min(E_1)\}) \cup \{\max(E_1) + 2\}$$

and we check again if $\min(G) > \max(E_2) + 2$. By continuing the construction as described below, we get an irrendundant chain of minimal length. In particular, $\operatorname{dist}_{\mathcal{H}}(F, G) \geq t + 1$.

Definition 5 ([13])**.** *A d-uniform hypergraph $\mathcal{H}$ is called properly-connected if for any two edges F and G such that $F \cap G \neq \emptyset$ we have*

$$\operatorname{dist}_{\mathcal{H}}(F, G) = d - |F \cap G|.$$

The following result is now straightforward by the first case of Construction 1:

Proposition 2. *The hypergraph $\mathcal{L}_t(n)$ is properly-connected.*

Definition 6 ([13])**.** *Let $\mathcal{H}$ be a properly-connected d-uniform hypergraph and E an edge in $\mathcal{H}$. The vertex neighbour set of E is the set of vertices*

$$N = \bigcup_{\{H \in \mathcal{H} \ : \ \operatorname{dist}_{\mathcal{H}}(E, H) = 1\}} H \setminus E.$$

One may describe the splitting edges in terms of their vertex neighbour sets.

Theorem 1 ([13, Theorem 4.8])**.** *Let E be an edge of a d-uniform properly-connected hypergraph $\mathcal{H}$ and assume that $N(E) = \{z_1, \ldots, z_r\}$. Then E is a splitting edge if and only if there exists a vertex $z \in E$ such that $(E \setminus \{z\}) \cup \{z_i\} \in \mathcal{H}$ for each $z_i \in N(E)$.*

If we consider that $E = \{1, \ldots, t\} \in \mathcal{L}_t(n)$, then its vertex neighbour set is $\{t + 1, t + 2\}$.

Proposition 3. *The set $E = \{1, \ldots, t\}$ is a splitting edge of $\mathcal{L}_t(n)$.*

Proof. Let $i \in E$, $i \neq t$. Then it is clear by the definition of the edges of $\mathcal{L}_t(n)$ that the sets $\{1, \ldots, \hat{i}, \ldots, t, t+1\}$ and $\{1, \ldots, \hat{i}, \ldots, t, t+2\}$ are edges of $\mathcal{L}_t(n)$. The statement follows by Theorem 1. $\square$

If E is an edge of the hypergraph $\mathcal{H}$, then let $\mathcal{H}' = \{H \in \mathcal{H} : \operatorname{dist}_{\mathcal{H}}(E, H) \geq d + 1\}$. If $\mathcal{H} = \mathcal{L}_t(n)$ and $E = \{1, \ldots, t\}$, then according to Construction 1,

$$\mathcal{L}'_t(n) = \{H \in \mathcal{L}_t(n) \ : \ \min(H) \geq t + 3\}.$$

It is clear that $\mathcal{L}'_t(n)$ is the hypergraph associated to the path ideal of the square of the line graph on the vertex set $\{t+3, \ldots, n\}$.

If $\mathcal{H}$ is a d-uniform properly-connected hypergraph and E is a splitting edge of $\mathcal{H}$, then the Castelnuovo-Mumford regularity and the projective dimension of the edge ideal of $\mathcal{H}$ can be expressed in terms of the corresponding invariants of the edge ideals of the hypergraphs $\mathcal{H} \setminus E$ and $\mathcal{H}'$.

Theorem 2 ([13, Theorem 6.2])**.** *Let $\mathcal{H}$ be a d-uniform properly-connected hypergraph, E a splitting edge of $\mathcal{H}$ and $r = |N(E)|$. Then*

(a) $\operatorname{reg}(I(\mathcal{H})) = \max\{\operatorname{reg}(I(\mathcal{H} \setminus E)), \operatorname{reg}(I(\mathcal{H}')) + d - 1\}$.

(b) $\operatorname{pd}(I(\mathcal{H})) = \max\{\operatorname{pd}(I(\mathcal{H} \setminus E)), \operatorname{pd}(I(\mathcal{H}')) + r + 1\}$.

In our case, we get

Theorem 3. *Let $2 \leq t \leq n$ be an integer, $I_t(L_n^2)$ the path ideal of L_n^2 and $E = \{1, \ldots, t\}$. Then*

(a) $\operatorname{reg}(I_t(L_n^2)) = \max\{\operatorname{reg}(I(\mathcal{L}_t(n) \setminus E)), \operatorname{reg}(I(\mathcal{L}'_t(n))) + t - 1\}$.

(b) $\operatorname{pd}(I_t(L_n^2)) = \max\{\operatorname{pd}(I(\mathcal{L}_t(n) \setminus E)), \operatorname{pd}(I(\mathcal{L}'_t(n))) + 3\}$.

Proof. The statement follows directly from Theorem 2, and using the fact that $N(E) = \{t+1, t+2\}$. □

In order to define a lower bound for the Castelnuovo-Mumford regularity, we recall the following definition:

Definition 7 ([13])**.** Let $\mathcal{H}$ be a d-uniform properly connected hypergraph. Two edges E, H of $\mathcal{H}$ are *t-disjoint* if $\operatorname{dist}_{\mathcal{H}}(E, H) \geq t$. A set of edges $\mathcal{E}' \subseteq \mathcal{E}$ is called *pairwise t-disjoint* if every pair of edges of $\mathcal{E}'$ is t-disjoint.

Theorem 4 ([13, Theorem 6.5])**.** *Let $\mathcal{H}$ be a d-uniform properly-connected hypergraph. Then $\beta_{i-1,di}(I(\mathcal{H}))$ equals the number of sets of i pairwise $(d+1)$-disjoint edges of $\mathcal{H}$. In particular, if c is the maximal number of pairwise $(d+1)$-disjoint edges of $\mathcal{H}$, then*

$$\operatorname{reg}(I(\mathcal{H})) \geq (d-1)c + 1.$$

For our case, we get

Lemma 1. *The maximal number of pairwise $(t+1)$-disjoint edges of $\mathcal{L}_t(n)$ is at least $\left[\frac{n}{t+2}\right]$.*

Proof. Let us denote by $c = \left[\frac{n}{t+2}\right]$. It is clear that the set of edges $E_0 = \{1, \ldots, t\}$, $E_1 = \{t+3, \ldots, 2t+2\}, \cdots, E_{c-1} = \{(c-1)(t+2)+1, \ldots, c(t+2)-2\}$ is a pairwise $(t+1)$-disjoint set of edges. The statement follows. □

The following corollary is now straightforward due to Theorem 4:

Corollary 1. *Let $2 \leq t \leq n$ and let $I_t(L_n^2)$ be the corresponding t-path ideal of L_n^2. Then*

$$\operatorname{reg}(S/I_t(L_n^2)) \geq \left[\frac{n}{t+2}\right](t-1).$$

3. The height and the projective dimension of path ideals of L_n^2

In this section, we will give a complete description of the height of path ideals of L_n^2. In order to do this we split the proof in several cases. First, we consider the case when $n \leq 2t$ and we describe all the minimal prime ideals of height 2. By $\mathrm{Min}(I_t(L_n^2))$ we denote the set of minimal prime ideals which contain $I_t(L_n^2)$.

Proposition 4. *If $n \leq 2t$, then $\{\mathfrak{p} \in \mathrm{Min}(I_t(L_n^2)) : \mathrm{ht}\,\mathfrak{p} = 2\} = \{\langle x_i, x_{i+1}\rangle : n-t \leq i \leq t\}$.*

Proof. Assume that $\mathfrak{p} \in \mathrm{Min}(I_t(L_n^2))$ has $\mathrm{ht}\,\mathfrak{p} = 2$. Since the monomials $x_1 \cdots x_t$, $x_{n-t+1} \cdots x_n, \in \mathcal{G}(I_t(L_n^2))$, there are $1 \leq \alpha \leq t$ and $n-t+1 \leq \beta \leq n$ such that $\mathfrak{p} = \langle x_\alpha, x_\beta\rangle$. If α and β are not consecutive integers, then the monomial $m = \mathbf{x}_{[n]\setminus\{\alpha,\beta\}} \in I_t(L_n^2)$ and $m \notin \mathfrak{p}$. Therefore $\mathfrak{p} = \langle x_i, x_{i+1}\rangle$ with $n-t \leq i \leq t$.

Conversely, we prove that $\mathfrak{p} = \langle x_i, x_{i+1}\rangle \in \mathrm{Min}(I_t(L_n^2))$ for $n-t \leq i \leq t$. Let $n-t \leq i \leq t$ and assume by contradiction that there is $m \in \mathcal{G}(I_t(L_n^2))$ such that $m \notin \mathfrak{p}$. It follows that $\max(\mathrm{supp}(m)) < i$ or $\min(\mathrm{supp}(m)) > i+1$. If $\max(\mathrm{supp}(m)) < i$ and since $i \leq t$, then $\deg(m) < t$, a contradiction with $m \in \mathcal{G}(I_t(L_n^2))$. If $\min(\mathrm{supp}(m)) > i+1$ and since $n-t+1 \leq i+1$, it follows that $\min(\mathrm{supp}(m)) > n-t+2$, s contradiction with $\deg(m) = t$. $\square$

Next, we consider the case when $n = 2t+r$, with $1 \leq r \leq t-1$.

Proposition 5. *Let $n = 2t+r$, with $2 \leq r \leq t-1$. Then*

$$\mathfrak{p} = \langle x_t, x_{t+1}, x_{2t+1}, x_{2t+2}\rangle \in \mathrm{Min}(I_t(L_n^2))$$

is of minimal height. Moreover, $\mathrm{ht}(I_t(L_n^2)) = 4$. If $n = 2t+1$, then

$$\mathfrak{p} = \langle x_t, x_{t+1}, x_{2t+1}\rangle \in \mathrm{Min}(I_t(L_n^2))$$

is of minimal height and $\mathrm{ht}(I_t(L_n^2)) = 3$.

Proof. If $n = 2t+r$, then we consider the sets $A_1 = \{1, \ldots, t\}$, $A_2 = \{t+1, \ldots, 2t\}$ and $B = \{2t+1, \ldots, 2t+r\}$, where $|A_i| = t$ and $|B| = r$. It is clear that in both cases, $\mathfrak{p} \in \mathrm{Min}(I_t(L_n^2))$, since any path of length t has a corresponding vertex in $\mathfrak{p}$ and if we remove a variable from $\mathfrak{p}$, then one may construct a path of length t such that its corresponding monomial does not belong to $\mathfrak{p}$.

Assume by contradiction that there is a minimal prime ideal $\mathfrak{q}$ of $I_t(L_n^2)$ such that $\mathrm{ht}(\mathfrak{q}) < \mathrm{ht}(\mathfrak{p})$. It is clear that $\mathrm{ht}(\mathfrak{q}) \geq 2$ since the monomials $x_1 \cdots x_t,\ x_{t+1} \cdots x_{2t} \in G(I_t(L_n^2))$.

For the case when $n = 2t+1$, if we assume that $\mathfrak{q} = \langle x_i, x_j\rangle$, with $1 \leq i \leq t$ and $t+1 \leq j \leq 2t$, then we may consider the monomial $m = \frac{x_{t+1}\cdots x_{2t}}{x_j} x_{2t+1} \in \mathcal{G}(I_t(L_n^2))$ such that $m \notin \mathfrak{q}$.

For the case $n = 2t+r$, with $2 \leq r < t$, it follows as before that $\mathrm{ht}(\mathfrak{q})$ should be strictly greater than 2. Assume that $\mathrm{ht}(\mathfrak{q}) = 3$, that is $\mathfrak{q} = (x_i, x_j, x_s)$ such that $1 \leq i \leq t$ and $t+1 \leq j \leq 2t$. Then the monomial $m = \frac{x_1\cdots x_{2t+r}}{x_i x_j x_s} \in I_t(L_n^2)$ since there is at least one path of length t on the set $\{1, \ldots, 2t+r\} \setminus \{i, j, s\}$ and $m \notin \mathfrak{q}$.

In both cases we obtain a contradiction, therefore the ideal $\mathfrak{p}$ is of minimal height. $\square$

We analyze the case when $3t \leq n$, that is $n = kt + r$, $k \geq 3$ and $0 \leq r \leq t-1$. We consider the sets:

$$A_1 = \{1,\dots,t\},\ A_2 = \{t+1,\dots,2t\},\dots,\ A_k = \{(k-1)t+1,\dots,kt\},\ B = \{kt+1,\dots,n\},$$

where $|A_i| = t$ and $|B| = r$, $1 \leq i \leq k$.

Proposition 6. *Let $n = kt + r$, $3 \leq k$, $0 \leq r \leq t-1$. An ideal of the form*

$$\mathfrak{p} = \langle x_t, x_{t+1}\rangle + \langle x_{(s-1)t+1}, x_{(s-1)t+2} : 3 \leq s \leq k\rangle + \langle x_{kt+1}, x_{kt+2}\rangle,\ \text{if } r > 2,$$
$$\mathfrak{p} = \langle x_t, x_{t+1}\rangle + \langle x_{(s-1)t+1}, x_{(s-1)t+2} : 3 \leq s \leq k\rangle + \langle x_{kt+1}\rangle,\ \text{if } 1 \leq r \leq 2,$$
$$\mathfrak{p} = \langle x_t, x_{t+1}\rangle + \langle x_{(s-1)t+1}, x_{(s-1)t+2} : 3 \leq s \leq k\rangle,\ \text{if } r = 0,$$

has the property that $\mathfrak{p} \in \mathrm{Min}(I_t(L_n^2))$ and it is of minimal height.

Moreover,

$$\mathrm{ht}(I_t(L_n^2)) = \begin{cases} 2k, & r > 2 \\ 2k-1, & 1 \leq r \leq 2 \\ 2k-2, & r = 0 \end{cases}$$

Proof. We claim that $\mathfrak{p} \supset I$ since there is no path of length t which may be constructed on the vertices which are not in $\mathfrak{p}$. Furthermore, $\mathfrak{p} \in \mathrm{Min}(I_t(L_n^2))$, because if we remove any vertex from $\mathfrak{p}$, then one may find a path of length t with the vertices which are not in $\mathfrak{p}$.

Next, we consider I to be the Stanley–Reisner ideal of the simplicial complex Δ. Assume, by contradiction, that there is a minimal prime ideal $\mathfrak{q}$ of I, $\mathfrak{q} = P_{H^c} := \langle x_i : i \in H^c\rangle$, where H is a face of Δ, such that $\mathrm{ht}(\mathfrak{q}) < \mathrm{ht}(\mathfrak{p})$. We denote $H = \{j_1, j_1+1, \dots, j_1+s_1, j_2, j_2+1, \dots, j_2+s_2, \dots, j_a, j_a+1, \dots, j_a+s_a\}$, with $j_i - (j_{i-1} + s_{i-1}) > 2$, for all $2 \leq i \leq a$, since H is a face of Δ.

One may note that $\mathrm{ht}(\mathfrak{q}) \geq k$ since $\mathfrak{q}$ must contain at least one variable from each set A_i, $1 \leq i \leq k$, because the monomials $\mathbf{x}_{A_i} \in \mathcal{G}(I)$, for all i. This implies that in H we have at least k sequences of consecutive integers. Moreover, $|H| \leq n-k$, thus $(j_a + s_a) - j_1 + 1 \leq n - k$.

Case 1: We consider $n = kt + r$, with $r > 2$, and $\mathrm{ht}(\mathfrak{q}) < 2k = \mathrm{ht}(\mathfrak{p})$, that is, $|H| = s_1 + \cdots + s_a + a > n - 2k$. Since $j_i - (j_{i-1} + s_{i-1}) > 2$, for all $2 \leq i \leq a$, by summing all relations we obtain that $j_a - j_1 > 2(a-1) + (s_1 + \ldots + s_{a-1})$. Thus

$$j_a - j_1 > a - 2 + (s_1 + \cdots + s_{a-1} + a) > a - 2 + n - 2k - s_a$$

implies that $(j_a + s_a) - j_1 > n - 2k - 1 + a$.

Therefore $n - 2k + a < (j_a + s_a) - j_1 + 1 \leq n - k$, that is, $a < k$. This means that in H there are at most $k-1$ sequences of consecutive integers, a contradiction.

Case 2: We take $n = kt + r$, with $1 \leq r \leq 2$, and $\mathrm{ht}(\mathfrak{q}) < 2k - 1 = \mathrm{ht}(\mathfrak{p})$, that is, $|H| = s_1 + \cdots + s_a + a > n - 2k + 1$. As before, we get

$$j_a - j_1 > a - 2 + (s_1 + \cdots + s_{a-1} + a) > a - 2 + n - 2k + 1 - s_a,$$

that is, $(j_a + s_a) - j_1 > n - 2k + a$.

We get $n - 2k + 1 + a < (j_a + s_a) - j_1 + 1 \leq n - k$, that is, $a < k - 1$, a contradiction.

Case 3: For $n = kt$ and $|H| = s_1 + \cdots + s_a + a > n - 2k + 2$, we obtain

$$j_a - j_1 > a - 2 + (s_1 + \cdots + s_{a-1} + a) > a - 2 + n - 2k + 2 - s_a,$$

that is, $(j_a + s_a) - j_1 > n - 2k + 1 + a$.

It follows that $n - 2k + 2 + a < (j_a + s_a) - j_1 + 1 \leq n - k$, that is, $a < k - 2$. This means that in H there are at most $k - 3$ sequences of consecutive integers, a contradiction. □

Therefore, we proved the following:

Theorem 5. *Let $2 \leq t \leq n$ and let $I_t(L_n^2)$ be the t-path ideal of L_n^2. Then*

$$\dim(S/I_t(L_n^2)) = \begin{cases} n-2, & \text{if } n \leq 2t \\ n-3, & \text{if } n = 2t+1 \\ n-4, & \text{if } n = 2t+r \text{ and } 2 \leq r \leq t-1 \\ n-2k+2, & \text{if } n = kt,\ k \geq 3 \\ n-2k+1, & \text{if } n = kt+r,\ k \geq 3 \text{ and } 1 \leq r \leq 2 \\ n-2k, & \text{if } n = kt+r, k \geq 3 \text{ and } r > 2 \end{cases}.$$

Next, we pay attention to the projective dimension of path ideals of L_n^2.

Proposition 7. *Assume that $t < \left[\frac{n}{2}\right]$ and denote $n = 2tk + r$, with $1 \leq r \leq 2t - 1$. The ideals*

$$\begin{aligned} \mathfrak{p} =& \langle x_2, x_4, \ldots, x_{2tk}\rangle + \langle x_1, x_{2t+1}, x_{4t+1}, \ldots, x_{2tk+1}\rangle, \quad \text{if } 1 \leq r \leq t, \\ \mathfrak{p} =& \langle x_2, x_4, \ldots, x_{2tk}\rangle + \langle x_1, x_{2t+1}, x_{4t+1}, \ldots, x_{2tk+1}\rangle + \langle x_{2tk+2}, x_{2tk+4}, \\ & \ldots, x_{2tk+2(r-t)}\rangle, \quad \text{if } t < r \leq 2t-1, \end{aligned}$$

is a minimal prime of $I_t(L_n^2)$. In particular,

$$\operatorname{pd}(S/I_t(L_n^2)) \geq \begin{cases} (t+1)k+1 & \text{if } 1 \leq r \leq t \\ (t+1)k+1+r-t & \text{if } t < r \leq 2t-1. \end{cases}$$

Proof. If we assume that $1 \leq r \leq t$, then the variables which are not in $\mathfrak{p}$ belong to the sets:

$$\begin{aligned} &(\{i \in [2tk+1] : i \text{ odd number}\} \setminus \{1, 2t+1, 4t+1, \ldots, 2tk+1\}) \\ &\quad \cup \{2tk+2, 2tk+3, \ldots, 2tk+r\}. \end{aligned}$$

Therefore, there is no path of length t which may be constructed with these vertices.

For the second case, we obtain that the variables which are not in $\mathfrak{p}$ belong to the sets:

$$\begin{aligned} &(\{i \in \{1, \ldots, 2tk+1\} : i \text{ odd number}\} \setminus \{1, 2t+1, 4t+1, \ldots, 2tk+1\}) \\ &\quad \cup \{2tk+3, 2tk+5, \ldots, 2tk+2(r-t)-1\} \\ &\quad \cup \{2tk+2(r-t)+1, 2tk+2(r-t)+2, \ldots, 2tk+r\}. \end{aligned}$$

In either of these cases there is no path of length t on these vertices. □

4. Properties of particular classes of path ideals of L_n^2

In this section, we study the case when $t \geq \left[\frac{n}{2}\right]$. We pay attention to properties such as being Cohen-Macaulay or having a linear resolution.

Proposition 8. *If $t \geq \left[\frac{n}{2}\right]$, then $I_t(L_n^2)$ has linear quotients.*

Proof. Let us assume that $\mathcal{G}(I_t(L_n^2)) = \{\mathbf{x}_{F_1}, \ldots, \mathbf{x}_{F_r}\}$, $r \geq 2$, where $\mathbf{x}_{F_1} >_{lex} \cdots >_{lex} \mathbf{x}_{F_r}$.

Let $1 \leq \alpha < \beta \leq r$, that is, $\mathbf{x}_{F_\alpha} >_{lex} \mathbf{x}_{F_\beta}$. One has to show that there are integers $l \in [n]$ and $\gamma < \beta$ such that $l \in F_\alpha \setminus F_\beta$ and $l = F_\gamma \setminus F_\beta$. Let us assume that $F_\alpha = \{i_1, \ldots, i_t\}$ and $F_\beta = \{j_1, \ldots, j_t\}$, so $\min(F_\alpha) = i_1$, $\min(F_\beta) = j_1$, $\max(F_\alpha) = i_t$, $\max(F_\beta) = j_t$.

If $i_1 = j_1$, then, since $\mathbf{x}_{F_\alpha} >_{lex} \mathbf{x}_{F_\beta}$, there is some l such that for all $1 \leq s < l$, $i_s = j_s$ and $i_l < j_l$. By using that $\mathbf{x}_{F_\alpha}$ and $\mathbf{x}_{F_\beta}$ are in $\mathcal{G}(I_t(L_n^2))$, the following inequalities hold: $i_l - i_{l-1} \leq 2$, $j_l - j_{l-1} \leq 2$, which, taking into account that $i_{l-1} = j_{l-1}$, give $i_l = i_{l-1} + 1$ and $j_l = i_{l-1} + 2$. Let $F_\gamma = (F_\beta \cup \{i_l\}) \setminus \max(F_\beta)$. It is clear that $\mathbf{x}_{F_\gamma} \in \mathcal{G}(I_t(L_n^2))$ and $\gamma < \beta$ since $\mathbf{x}_{F_\gamma} >_{lex} \mathbf{x}_{F_\beta}$.

If $i_1 \neq j_1$, then $i_1 < j_1$ and we consider $i_l = \max\{i_s \in F_\alpha \setminus F_\beta \ : i_s < \max(F_\beta)\}$. Then the set $F_\gamma = (F_\beta \cup \{i_l\}) \setminus \max(F_\beta)$ has the property that $\mathbf{x}_{F_\gamma} \in \mathcal{G}(I_t(L_n^2))$ and $\gamma < \beta$ since $\mathbf{x}_{F_\gamma} >_{lex} \mathbf{x}_{F_\beta}$. $\square$

The following corollary is now straightforward.

Corollary 2. *If $t \geq \left[\frac{n}{2}\right]$, then $I_t(L_n^2)$ has a linear resolution.*

In order to compute the projective dimension, one has to determine the generators of ideals L_k, (see Proposition 1 and the above notations).

Proposition 9. *Let $t \geq \left[\frac{n}{2}\right]$ and $\mathcal{G}(I_t(L_n^2)) = \{\mathbf{x}_{F_1}, \ldots, \mathbf{x}_{F_r}\}$, $r \geq 2$, where $\mathbf{x}_{F_1} >_{lex} \cdots >_{lex} \mathbf{x}_{F_r}$. Then*

$$\langle \mathbf{x}_{F_1}, \ldots, \mathbf{x}_{F_{k-1}} \rangle : \langle \mathbf{x}_{F_k} \rangle = \langle x_\alpha \ : \ \min(F_k) - 2 \leq \alpha < \max(F_k) \text{ and } x_\alpha \nmid \mathbf{x}_{F_k} \rangle.$$

Proof. We prove by double inclusion.

"$\supseteq$" Let us choose some α such that $\min(F_k) - 2 \leq \alpha < \max(F_k)$ and $x_\alpha \nmid \mathbf{x}_{F_k}$. Then

$$x_\alpha \mathbf{x}_{F_k} = \frac{x_\alpha \mathbf{x}_{F_k}}{x_{\max(F_k)}} x_{\max(F_k)} \in \langle \mathbf{x}_{F_1}, \ldots, \mathbf{x}_{F_{k-1}} \rangle$$

since $x_\alpha \mathbf{x}_{F_k} / x_{\max(F_k)} \in \mathcal{G}(I_t(L_n^2))$ and $x_\alpha \mathbf{x}_{F_k} / x_{\max(F_k)} >_{lex} \mathbf{x}_{F_k}$.

"$\subseteq$" Let $m \in \langle \mathbf{x}_{F_1}, \ldots, \mathbf{x}_{F_{k-1}} \rangle : \langle \mathbf{x}_{F_k} \rangle$, that is $m \mathbf{x}_{F_k} \in \langle \mathbf{x}_{F_1}, \ldots, \mathbf{x}_{F_{k-1}} \rangle$.

If $m \in \langle \mathbf{x}_{F_1}, \ldots, \mathbf{x}_{F_{k-1}} \rangle$, then there is some $\mathbf{x}_{F_i}$ which divides m. Since $I_t(L_n^2)$ has linear quotients, there is some $\gamma < k$ and some $l \in F_i$ such that $F_\gamma = (F_k \cup \{l\}) \setminus \{\max(F_k)\}$. One has that $l \geq \min(\mathbf{x}_{F_k}) - 2$ and $l < \max(F_k)$ according to the proof of Proposition 8. Moreover, $x_l \mid \mathbf{x}_{F_i} \mid m$ so $m \in \langle x_\alpha \ : \ \min(F_k) - 2 \leq \alpha < \max(F_k)$ and $x_\alpha \nmid \mathbf{x}_{F_k} \rangle$.

We assume now that $m \notin \langle \mathbf{x}_{F_1}, \ldots, \mathbf{x}_{F_{k-1}} \rangle$. Therefore, there is some $i < k$ such that $\mathbf{x}_{F_i} \mid m \mathbf{x}_{F_k}$, that is, $F_i \subseteq \operatorname{supp}(m) \cup F_k$. Since $I_t(L_n^2)$ is a squarefree monomial ideal which has linear quotients, there is some $l \in F_i \setminus F_k$ and some $\gamma < k$ such that $l = F_\gamma \setminus F_k$. By the proof of Proposition 8, $l \geq \min(\mathbf{x}_{F_k}) - 2$ and $l < \max(F_k)$. Moreover, $x_l \mid \mathbf{x}_{F_i} \mid m$ so $m \in \langle x_\alpha \ : \ \min(F_k) - 2 \leq \alpha < \max(F_k)$ and $x_\alpha \nmid \mathbf{x}_{F_k} \rangle$. $\square$

We can now compute the projective dimension of $I_t(L_n^2)$ for $t \geq \left[\frac{n}{2}\right]$.

Corollary 3. *If* $t \geq \left[\frac{n}{2}\right]$, *then* $\operatorname{pd}(S/I_t(L_n^2)) = n - t + 1$.

Proof. According to Proposition 1, $\operatorname{pd}(I_t(L_n^2)) = \max\{r_k : 1 \leq k \leq r\}$. One may note that there is a monomial m in $\mathcal{G}(I_t(L_n^2))$ such that $\min(m) = 1$ and $\max(m) = n$ since $t \geq \left[\frac{n}{2}\right]$. Therefore, $\operatorname{set}(m) = [n] \setminus \operatorname{supp}(m)$. Hence $\operatorname{pd}(I_t(L_n^2)) \geq n - t$ and this is the maximal possible. The statement follows. $\square$

We may also characterize in this case the property of path ideals of being Cohen-Macaulay.

Corollary 4. *If* $t \geq \left[\frac{n}{2}\right]$, *then* $S/I_t(L_n^2)$ *is Cohen-Macaulay if and only if* $t = n-1$.

Proof. One has that $S/I_t(L_n^2)$ is Cohen-Macaulay if and only if $\operatorname{depth}(S/I_t(L_n^2) = \dim(S/I_t(L_n^2))$, that is, $\operatorname{pd}(S/I_t(L_n^2)) = \operatorname{ht}(S/I_t(L_n^2))$, which is equivalent to $n-t+1 = 2$ and $t = n-1$. $\square$

We recall that if $I \subset S$ is a homogeneous ideal and $\sqrt{I}$ its radical, then the *arithmetical rank* of I is defined as

$$\operatorname{ara}(I) = \min\{r \in \mathbb{N} \colon \text{ there exist } a_1, \ldots, a_r \in I \text{ such that } \sqrt{I} = \sqrt{(a_1, \ldots, a_r)}\}.$$

For the squarefree monomial ideal case, an upper bound of the arithmetical rank is given by H.G. Gräbe [11].

Theorem 6 ([11, Theorem 1])**.** *Let* $I \subset S$ *be a squarefree monomial ideal. Then*

$$\operatorname{ara}(I) \leq n - \operatorname{indeg}(I) + 1,$$

where $\operatorname{indeg}(I)$ *is the initial degree of* I, *that is,* $\operatorname{indeg}(I) = \min\{q \colon I_q \neq 0\}$.

A lower bound for the arithmetical rank of a squarefree monomial ideal is given in [19].

Corollary 5 ([19, Theorem 1])**.** *Let* $I \subset S$ *be a squarefree monomial ideal. Then* $\operatorname{pd}_S(S/I) \leq \operatorname{ara}(I) \leq n - \operatorname{indeg}(I) + 1$.

Corollary 6. *If* $t \geq \left[\frac{n}{2}\right]$, *then* $\operatorname{pd}(S/I_t(L_n^2)) = \operatorname{ara}(S/I_t(L_n^2))$.

Proof. One has that $n - t + 1 \leq \operatorname{pd}(S/I_t(L_n^2)) \leq \operatorname{ara}(S/I_t(L_n^2)) \leq n - t + 1$, where the last inequality is given in [11]. $\square$

Next, we aim at characterizing all path ideals of L_n^2 which have a linear resolution. In order to do this, we need to recall the notion of edge diameter of a d-uniform properly connected hypergraph defined in [13].

Definition 8. *Let* $\mathcal{H}$ *be a* d*-uniform properly-connected hypergraph. The edge diameter of* $\mathcal{H}$ *is*

$$\operatorname{diam}(\mathcal{H}) = \max\{\operatorname{dist}_{\mathcal{H}}(E, H) : E, H \in \mathcal{H}\}.$$

One may characterize the property of having linear first syzygies in terms of edge diameter:

Theorem 7 ([13, Theorem 7.4])**.** *Suppose that $\mathcal{H}$ is a d-uniform properly-connected hypergraph. Then $I(\mathcal{H})$ has linear first syzygies if and only if* $\operatorname{diam}(\mathcal{H}) \leq d$*.*

Now we can characterize all path ideals of L_n^2 which have a linear resolution.

Theorem 8. *Let $\mathcal{L}_t(n)$ be the hypergraph whose edge ideal is $I_t(L_n^2)$. The following are equivalent:*

(*i*) $I_t(L_n^2)$ *has linear quotients.*

(*ii*) $I_t(L_n^2)$ *has a linear resolution.*

(*iii*) $I_t(L_n^2)$ *has linear first syzygies.*

(*iv*) $\operatorname{diam}(\mathcal{L}_t(n)) \leq t$.

(*v*) $t \geq \left[\frac{n}{2}\right]$.

Proof. The implications $(v) \Rightarrow (i) \Rightarrow (ii) \Rightarrow (iii) \Rightarrow (iv)$ are known to be true. Indeed, $(v) \Rightarrow (i)$ follows by Proposition 8, $(i) \Rightarrow (ii)$ is known by [7, Lemma 4.1] and $(iii) \Rightarrow (iv)$ is true by Theorem 7. We only have to prove that $(iv) \Rightarrow (v)$.

Let us assume that $\operatorname{diam}(\mathcal{H}) \leq t$, that is, $\operatorname{dist}_{\mathcal{H}}(E,H) \leq t$ for all E, H edges of $\mathcal{H}$. Since the sets $E = \{1,\ldots,t\}$ and $H = \{n-t+1,\ldots,n\}$ are edges in $\mathcal{H}$, we must have $\operatorname{dist}(E,H) \leq t$. If $t \geq n-t+1$, then $n \leq 2t-1$. If $t < n-t+1$, then $\operatorname{dist}(E,H) \leq t$ implies that there is an irredundant proper chain of length t which connects E and H and, by Construction 1, we must have that $n-t+1 \leq t+2$, that is, $n \leq 2t+1$. Thus $t \geq \left[\frac{n}{2}\right]$. □

5. Open questions and remarks

The study of path ideals of powers of the line graph is the next step in order to understand and compute invariants of path ideals of an arbitrary graph. There has been an intensive work in this direction and invariants of path ideals of trees, cycles and cycle posets have been studied [1, 2, 3, 9, 14, 20, 18].

The property of being sequentially Cohen-Macaulay has been characterized for trees, cycles and partial answers were given for cycle posets. In the case of path ideals if L_n^2, it is clear that they are sequentially Cohen-Macaulay for $t = 2$, that is, for edge ideals of the graph since L_n^2 is a chordal graph [10]. Moreover, the examples show that the following question has a positive answer for arbitrary t:

Question 1. *Is it true that the path ideal $I_t(L_n^2)$ is sequentially Cohen–Macaulay, for all $t \geq 2$?*

Moreover, the examples show that the bounds obtained for the projective dimension are sharp. For the Castelnuovo-Mumford regularity, one may see that the bounds from Corollary 1 are not sharp. Examples show that the following result could be true:

Question 2. *Let $n = c(t+1) + r$, where $c = \left[\frac{n}{t+1}\right]$ and $0 \leq r \leq t$. Is it true that*

$$\operatorname{reg}(S/I_t(L_n^2)) = \begin{cases} \left[\frac{n}{t+1}\right](t-1), & \text{if } 2 \leq r \leq t \\ \left(\left[\frac{n}{t+1}\right]-1\right)(t-1), & \text{if } 0 \leq r \leq 1? \end{cases}$$

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References

[1] A. Alilooee, S. Faridi, *Graded Betti numbers of path ideals of cycles and lines*, to appear in J. Algebra Appl., arXiv:1110.6653.

[2] A. Banerjee, *Regularity of path ideals of gap free graphs*, to appear in J. Pure Appl. Algebra, arXiv:1409.3583.

[3] R. R. Bouchat, H. T. Hà, A. O'Keefe, *Path ideals of rooted trees and their graded Betti numbers*, J. Combin. Theory Ser. A **118**(2011), 2411–2425.

[4] R. R. Bouchat, H. T. Hà, A. O'Keefe, *Corrigendum to "Path ideals of rooted trees and their graded Betti numbers" [J. Combin. Theory Ser. A* **118***(2011), 2411–2425]*, J. Combin. Theory Ser. A **119**(2012), 1610–1611.

[5] W. Bruns, J. Herzog, *Cohen–Macaulay Rings*, Cambridge University Press, Cambridge, 1993.

[6] A. Conca, E. De Negri, *M-sequences, graph ideals and ladder ideals of linear type*, J. Algebra **211**(1999), 599–624.

[7] A. Conca, J. Herzog, *Castelnuovo–Mumford regularity of products of ideals*, Collect. Math. **54**(2003), 137–152.

[8] S. Eliahou, M. Kervaire, *Minimal resolutions of some monomial ideals*, J. Algebra **129**(1990), 1–25.

[9] N. Erey, *Multigraded Betti numbers of some path ideals*, arXiv:1410.8242.

[10] C. A. Francisco, A. Van Tuyl, *Sequentially Cohen–Macaulay edge ideals*, Proc. Amer. Math. Soc. **135**(2007), 2327–2337.

[11] H.-G. Gräbe, *Über den arithmetischen Rang quadratfreier Potenzproduktideale*, Math. Nachr. **120**(1985), 217–227.

[12] G.-M. Greuel, G. Pfister, H. Schönemann, *Singular 2.0. A Computer Algebra System for Polynomial Computations*, Centre for Computer Algebra, University of Kaiserslautern, 2001, `http://www.singular.uni-kl.de`.

[13] H. T. Há, A. Van Tuyl, *Monomial ideals, edge ideals of hypergraphs, and their graded Betti numbers*, J. Algebraic Combin. **27**(2008), 215–245.

[14] J. He, A. Van Tuyl, *Algebraic properties of the path ideal of a tree*, Comm. Algebra **38**(2010), 1725–1742.

[15] J. Herzog, T. Hibi, *Monomial Ideals*, Graduate Texts in Mathematics 260, Springer, 2011.

[16] J. Herzog, Y. Takayama, *Resolutions by mapping cones*, Homology, Homotopy Appl. **4**(2002), 277–294.

[17] D. Kiani, S. Saeedi Madani, *Betti numbers of path ideals of trees*, Comm. Algebra **44**(2016), 5376–5394.

[18] M. Kubitzke, A. Olteanu, *Algebraic properties of classes of path ideals of posets*, J. Pure Appl. Algebra **218**(2014), 1012–1033.

[19] G. Lyubeznik, *On the local cohomology modules $H^i_{\mathfrak{a}}(R)$ for ideals $\mathfrak{a}$ generated by monomials in an R-sequence*, in: *Complete Intersections*, (S. Greco, R. Strano, Eds.), Lecture Notes in Mathematics 1092, Springer–Verlag, 1984, 214–220.

[20] S. S. Madani, D. Kiani, N. Terai, *Sequentially Cohen–Macaulay path ideals of cycles*, Bull. Math. Soc. Sci. Math. Roumanie **54**(2011), 353–363.

[21] B. Sturmfels, V. Welker, *Commutative algebra of statistical ranking*, J. Algebra **361**(2012), 264–286.