Sharp well-posedness of the Ostrovsky, Stepanyams and Tsimring equation

Amin Esfahani\textsuperscript{1,*}

\textsuperscript{1} School of Mathematics and Computer Science, Damghan University, Damghan 36 715-364, Iran

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Abstract. In this paper, we study the Ostrovsky, Stepanyams and Tsimring equation. We show that the associated initial value problem is locally well-posed in Sobolev spaces $H^s(\mathbb{R})$ for $s > -3/2$. We also prove that our result is sharp in the sense that the flow map of this equation fails to be $C^2$ in $H^s(\mathbb{R})$ for $s < -3/2$.

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1. Introduction

This paper is concerned with the well-posedness of the following initial value problem (IVP) for the Ostrovsky, Stepanyams and Tsimring (OST) equation:

\[
\begin{aligned}
&u_t + u_{xxx} - \eta(H u_x + H u_{xxx}) + uu_x = 0, \quad x \in \mathbb{R}, \; t \geq 0, \\
u(x,0) = u_0(x),
\end{aligned}
\]

where $u = u(x,t)$ is a real-valued function, $\eta > 0$ and $H$ denotes the usual Hilbert transformation given by

\[
H\varphi(x) = \frac{1}{\pi} \text{p.v.} \int_{\mathbb{R}} \frac{\varphi(x-y)}{y} \, dy,
\]

for $\varphi \in \mathcal{S}(\mathbb{R})$. Equation (1) was derived by Ostrovsky et al. in [18] to describe the radiational instability of long non-linear waves in a stratified flow caused by internal wave radiation from a shear layer.

We recall that the IVP for (1) is locally well-posed in Banach space $X$ if the solution uniquely exists in a certain time interval $[-T,T]$ (unique existence), the solution describes a continuous curve in $X$ in the interval $[-T,T]$ whenever initial data belong to $X$ (persistence), and the solution varies continuously depending upon the initial data (continuous dependence), i.e. continuity of application $u_0 \mapsto u(t)$ from $X$ to $C([-T,T];X)$.

Note that the OST equation is a modification of the well-known KdV equation

\[
u_t + u_{xxx} + uu_x = 0.
\]

*Corresponding author. Email address: esfahani@du.ac.ir (A. Esfahani)
It is known that the KdV equation arises in modeling of one-dimensional long wavelength surface waves propagating in weakly nonlinear dispersive media [1, 4, 12, 22], as well as the evolution of weakly nonlinear ion acoustic waves in plasmas [21]. Different from the KdV equation which is of purely dispersive type, the OST equation is of the dispersive–dissipative type.

A model similar to (1) is the Korteweg-de Vries-Kuramoto-Sivashinsky (KdV-KS) equation

\[
\begin{align*}
&u_t + u_{xxxx} + \eta(u_{xx} + u_{xxxx}) + u_x^2 = 0, \quad x \in \mathbb{R}, \quad t \geq 0, \\
&u(x,0) = u_0(x).
\end{align*}
\]  

This equation arises as a model for long waves on a viscous fluid flowing down an inclined plane and describing drift waves in plasma [8, 20]. The IVP for (2) was studied by Biagioni et al. [3]. They proved that (2) is well-posed in $H^s(\mathbb{R})$ for $s \geq 1$, by using the properties of the semi-group associated with the linear problem. They also obtained a global solution in $H^s(\mathbb{R})$ for $s \geq 1$, making use of the conserved quantities for the Korteweg-de Vries equation. Recently, Carvajal and Panthee in [7], considered the derivative equation of (2) and obtained the local well-posedness of (2) in $H^s(\mathbb{R})$ for $s > -3/4$ (see also [6]).

The first work on the well-posedness of the IVP for (1) was carried out by Alvarez in [2]. He proved that (1) is locally well-posed in $H^s(\mathbb{R})$ for $s > 1/2$ and globally well-posed in $H^s(\mathbb{R})$ for $s \geq 1$. In [5], Carvajal improved these results. He proved that (1) is locally well-posed in $H^s(\mathbb{R})$, for $s \geq 0$, and globally well-posed in $L^2(\mathbb{R})$. Zhao and Cui in [23] used the ideas of Molinet and Ribaud in [15, 16, 17], employed the method of bilinear estimate in the Bourgain-type spaces and proved that (1) is locally well-posed in $H^s(\mathbb{R})$ for $s > -3/4$; which coincides with the sharp local well-posedness result for the KdV equation established by Kenig et al. in [14]. The authors in [24] improved their previous results by showing that the IVP for (1) is locally well-posed in $H^s(\mathbb{R})$ for $s > -1$.

In this paper we shall prove that (1) is locally well-posed in $H^s(\mathbb{R})$ for $s > -3/2$. Indeed, we use purely dissipative methods as applied by Dix in [9] to study the IVP for the KdV-Burgers equation

\[
\begin{align*}
&u_t + u_{xxx} + uu_x = u_{xx}, \quad x \in \mathbb{R}, \quad t \geq 0 \\
&u(x,0) = u_0(x).
\end{align*}
\]  

The main ingredient consists of applying a fixed-point theorem to the integral equation associated to (1) in time-weighted spaces.

Regarding the sharpness of our result, we establish that the flow map of the OST equation fails to be $C^2$ in $H^s(\mathbb{R})$ for $s < -3/2$. This means that a Picard iteration cannot be used to obtain a solution of (1).

Before presenting the precise statement of our main result, let us first introduce some definitions and notations.

Without loss of generality, later on we assume that $\eta = 1$. We shall denote by $\hat{\varphi}$ the Fourier transform of $\varphi$, defined as

\[\hat{\varphi}(\xi) = \int_{\mathbb{R}} \varphi(x)e^{-ix\xi} \, dx.\]
For $s \in \mathbb{R}$, by $H^s(\mathbb{R})$ we denote the nonhomogeneous Sobolev space defined by

$$H^s(\mathbb{R}) = \{ \varphi \in \mathcal{S}'(\mathbb{R}) : \| \varphi \|_{H^s(\mathbb{R})} < \infty \},$$

where

$$\| \varphi \|_{H^s(\mathbb{R})} = \left\| (1 + \xi^2)^{s/2} \hat{\varphi}(\xi) \right\|_{L^2(\mathbb{R})},$$

and $\mathcal{S}'(\mathbb{R})$ is the space of tempered distributions.

For any positive numbers $a$ and $b$, the notation $a \lesssim b$ means that there exists a positive constant $c$ such that $a \leq cb$; and we denote $a \sim b$ when, $a \lesssim b$ and $b \lesssim a$.

For $s \in \mathbb{R}$ and $u_0 \in H^s(\mathbb{R})$, consider the following linear problem associated to (1):

$$\begin{cases}
  u_t + u_{xxxx} - \mathcal{H} u_x - \mathcal{H} u_{xxxx} = 0, & x \in \mathbb{R}, \ t \in \mathbb{R}, \\
  u(x,0) = u_0(x).
\end{cases}
$$

(4)

The unique solution of (4) is given by the semigroup $\{U(t)\}_{t \geq 0}$ defined as follows:

$$u(t) = U(t)u_0 = \int_{\mathbb{R}} e^{t(i|\xi|^3 - |\xi|^3 + |\xi|)} \hat{u}_0(\xi) \ d\xi.$$

The main results of this paper read as follows:

**Theorem 1.** Let $s > -3/2$. Then for all $u_0 \in H^s(\mathbb{R})$, there exist $T = T(\|u_0\|_{H^s(\mathbb{R})}) > 0$, a space

$$\mathcal{X}^s_T \hookrightarrow C([0,T]; H^s(\mathbb{R}))$$

and a unique solution $u(t)$ of (1) such that $u(0) = u_0$. Moreover, $u \in C((0,T); H^\infty(\mathbb{R}))$ and the map solution

$$F : H^s(\mathbb{R}) \rightarrow \mathcal{X}^s_T \cap C([0,T]; H^s(\mathbb{R})), \quad u_0 \mapsto u,$$

is smooth.

**Theorem 2.** Let $s < -3/2$, if there exists some $T > 0$ such that the Cauchy problem (1) is locally well-posed in $H^s(\mathbb{R})$, then the flow-map data solution

$$F : H^s(\mathbb{R}) \rightarrow C([0,T]; H^s(\mathbb{R})), \quad u_0 \mapsto u(t)$$

is not $C^2$ at zero.

The rest of this paper is as follows. In Section 2 we present the time-weighted space $\mathcal{X}^s_T$ and obtain some basic linear and bilinear estimates in this space. Section 3 is devoted to proving the local well-posedness in this space. We also establish that the flow map of the OST equation fails to be $C^2$ in $H^s(\mathbb{R})$ for $s < -3/2$. 

2. Linear and bilinear estimates

In this section, we introduce a suitable Banach space in order to derive appropriate linear and bilinear estimates.

To prove Theorem 1, we will make the assumption $-3/2 < s < 0$, since the case $0 \leq s$ follows by similar arguments. Our strategy is to use a contraction argument on the integral equation associated to (1):

$$u(t) = \Phi(u(t)) := U(t)u_0 + \frac{1}{2} \int_0^t U(t-t')\partial_x (u^2(t')) \, dt'.$$

For $0 < T \leq T^* = \min\{1, 9|s|/2\}$, we define the Banach space

$$\mathcal{X}_T^s = \{u \in C([0,T]; H^s(\mathbb{R})) : \|u\|_{\mathcal{X}_T^s} < \infty\},$$

where

$$\|u\|_{\mathcal{X}_T^s} = \sup_{t \in [0,T]} \left(\|u(t)\|_{H^s(\mathbb{R})} + t^{s/3}\|u(t)\|_{L^2(\mathbb{R})}\right).$$

We note that $T^* = 1$, if $s \leq -2/9$.

First we state the following lemma which is useful in establishing smoothness properties for the semigroup of (1). The proof is straightforward.

**Lemma 1.** For any $a > 0$ and $0 < t \leq 9a$, we have for all $\xi \in \mathbb{R}$,

$$\xi^{2a} e^{-t(|\xi|^3 - |\xi|)} \leq \rho^{2a} e^{-t(\rho^3 - \rho)} =: \psi(a,t),$$

where

$$\rho = \frac{(9a + \sqrt{81a^2 - t^2})^{1/3}}{3} t^{-1/3} + \frac{t^{1/3}}{3 (9a + \sqrt{81a^2 - t^2})^{1/3}}.$$

Moreover, if $a = 0$, then (6) holds for \(\psi(0,t) = \exp\left(\frac{2t}{3\sqrt{3}}\right)\).

Now, we will turn our attention to estimate the linear part in $\mathcal{X}_T^s$.

**Proposition 1.** Let $0 < T \leq T^*$, $s < 0$ and $u_0 \in H^s(\mathbb{R})$, then

$$\sup_{t \in [0,T]} \|U(t)u_0\|_{H^s(\mathbb{R})} \leq e^{\frac{2T}{3\sqrt{3}}} \|u_0\|_{H^s(\mathbb{R})},$$

and

$$\sup_{t \in [0,T]} t^{s/3}\|U(t)u_0\|_{L^2(\mathbb{R})} \lesssim \mathcal{Y}_s(T)\|u_0\|_{H^s(\mathbb{R})},$$

where

$$\mathcal{Y}_s(t) = e^{\frac{2t}{3\sqrt{3}}} + t^{s/3}\psi(|s|/2, t)$$

is a continuous nondecreasing function on $[0,T^*]$ and $\psi$ is defined as in Lemma 1.
Inequality (7) follows immediately from Lemma 1. To prove inequality (8), we first observe from 0 < T ≤ 1 that

\[ t^{s/3} \leq \frac{(1 + t^{2/3})^{s/2}}{(1 + \xi^2)^{s/2}}, \]

for all t ∈ [0, T]. Hence, by using the Plancherel theorem and the definition of \( U(t) \), we deduce that

\[
\int_0^T U(t) u_0 \|L^2(\mathbb{R}) \leq \left\| \left( 1 + t^{2/3} \right)^{s/2} e^{-tR(\xi^3 - |\xi|)} \left( 1 + \xi^2 \right)^{s/2} \bar{u}_0(\xi) \right\|_{L^2(\mathbb{R})} \\
\lesssim \left\| e^{-tR(\xi^3 - |\xi|)} \right\|_{L^\infty(\mathbb{R})} + \left\| \left( t^{2/3} \xi^2 \right)^{s/2} e^{-tR(\xi^3 - |\xi|)} \right\|_{L^\infty(\mathbb{R})} \|u_0\|_{H^s(\mathbb{R})}.
\]

Lemma 1 implies the desired inequality in (8).

The next step is to derive the bilinear estimate.

**Proposition 2.** Let 0 ≤ t ≤ T ≤ T* and s ∈ (-3/2, 0); then

\[
\int_0^T U(t - t') \partial_x(uv)(t') dt' \lesssim e^{2\sqrt{T/T'}} T(2s + 3)^{3/6} \|u\|_{X^s_T} \|v\|_{X^s_T},
\]

for all u, v ∈ \( X^s_T \), where the constant of the above inequality depends only on s.

**Proof.** Let 0 ≤ t ≤ T. We have \((1 + \xi^2)^{s/2} \leq |\xi|^s\), since s < 0. So by using the Minkowski inequality and the definition of \( U(t) \), we obtain that

\[
\int_0^T U(t - t') \partial_x(uv)(t') dt' \leq 0
\]

\[
\quad \int_0^T 0(1 + \xi^2)^{s/2} e^{(t-t')^3} (u(t')v(t')^\wedge)(\xi) \right\|_{L^2(\mathbb{R})} \|u(t') \ast v(t') (\xi) \right\|_{L^\infty(\mathbb{R})} dt'.
\]

The Young inequality implies that

\[
\left\| \bar{u}(t') \ast \bar{v}(t') \right\|_{L^\infty(\mathbb{R})} \leq \frac{\|u\|_{X^s_T} \|v\|_{X^s_T}}{|t'|^{2s+3/2}}.
\]

Therefore, by changing the variable, we obtain

\[
\int_0^T U(t - t') \partial_x(uv)(t') dt' \leq \left( \int_0^T \left\| \xi^{1+s} e^{-tR(\xi^3 - |\xi|)} \right\|_{L^2(\mathbb{R})} \frac{1}{|t - t'|^{s/3}} dt' \right) \|u\|_{X^s_T} \|v\|_{X^s_T}.
\]
To estimate the integral on the right-hand side of (12), we use a change of the variable to deduce that
\[
\left\| \xi^{1+s} \partial_x \left( \xi^{3-|\xi|^3} \right) \right\|_{L^2(\mathbb{R})} 
\leq \left| t' \right|^{-2(2s+3)/6} \left\| e^{\left( |\xi|^{s/3} - |\xi|^3 \right)/2} \right\|_{L^\infty(\mathbb{R})} \left\| \xi^{1+s} e^{-|\xi|^3/2} \right\|_{L^2(\mathbb{R})},
\]
where in the last inequality we used the following inequality
\[
e^{\left( |\xi|^{s/3} - |\xi|^3 \right)/2} \leq \frac{2}{\sqrt{2\pi}}, \quad \forall \xi \in \mathbb{R}.
\]
Therefore, we get from (12), (13) and a change of the variable that
\[
\left\| \int_0^t U(t - t') \partial_x (uv)(t') dt' \right\|_{H^s(\mathbb{R})} 
\leq 2^{2\sqrt{2\pi} - \sqrt{2\pi}} |t|^{-2(2s+3)/6} \left( \int_0^1 \left| t' \right|^{-2(2s+3)/6} |1 - t'|^{2s/3} dt' \right) \left\| u \right\|_{X_T} \left\| v \right\|_{X_T}
\]
for all \(0 \leq t \leq T\). On the other hand, a similar argument allows us to deduce for all \(0 \leq t \leq T\) that
\[
\left\| \int_0^t \left( t - t' \right) \partial_x (uv)(t') dt' \right\|_{L^2(\mathbb{R})} 
\leq t|^{s/3} \left( \int_0^t \left\| \xi e^{\left( t-t' \right) (|\xi| - |\xi|^3)} \right\|_{L^2(\mathbb{R})} \left\| u(t') \right\|_{L^\infty(\mathbb{R})} \left\| v(t') \right\|_{L^\infty(\mathbb{R})} dt' \right) \left\| u \right\|_{X_T} \left\| v \right\|_{X_T}
\]
\[
\leq 2^{2\sqrt{2\pi} - \sqrt{2\pi}} \left( \int_0^1 \left| t' \right|^{-2s/3} |1 - t'|^{-2s/3} dt' \right) \left\| u \right\|_{X_T} \left\| v \right\|_{X_T}
\]
and
\[
\int_0^t \left| \xi \right|^{1+s} \partial_x \left( \xi^{3-|\xi|^3} \right) dt' 
\leq 2^{2\sqrt{2\pi} - \sqrt{2\pi}} \left( \int_0^1 \left| t' \right|^{-2(2s+3)/6} |1 - t'|^{2s/3} dt' \right) \left\| u \right\|_{X_T} \left\| v \right\|_{X_T}.
\]
This completes the proof. \(\Box\)

Remark 1. If we consider \(s' > s > -3/2\), then after modifying the space \(X_T^{s'}\) by
\[
\tilde{X}_T^{s'} = \left\{ u \in X_T^{s'}; \left\| u \right\|_{X_T^{s'}} < \infty \right\}
\]
then
\[
\left\| u \right\|_{\tilde{X}_T^{s'}} = \left\| u \right\|_{X_T^{s'}} + \sup_{t \in [0,T]} \left\| t|^{s/3} \left( 1 - \partial_x^2 \right)^{(s'-s)/2} u(t) \right\|_{L^2(\mathbb{R})}
\]
and using
\[
(1 + \xi^2)^{s'/2} \lesssim (1 + \xi^2)^{s/2} (1 + \xi^2)^{(s'-s)/2} + (1 + \xi^2)^{s'/2} (1 + (\xi - \xi_1)^2)^{(s'-s)/2}
\]
and Proposition 2 we can deduce that for $s > s' > -3/2$, we have (see (10))
\[
\left\| \int_0^t U(t - t') \partial_x (uv)(t') \, dt' \right\|_{\mathcal{F}_x^{s'}} \lesssim e^{2\sqrt{T'} / \sqrt{27} T} e^{\theta(s)} \left( \|u\|_{\mathcal{F}_x^{s'}} \|v\|_{\mathcal{F}_x^{s'}} + \|v\|_{\mathcal{F}_x^{s'}} \|u\|_{\mathcal{F}_x^{s'}} \right).
\]

**Remark 2.** We should note that Proposition 2 holds for $s \geq 0$. Indeed since $H^s(\mathbb{R})$ is an algebra for $s > 1/2$, then bilinear estimate (9) holds easily. When $s \in [0,1/2]$, we have
\[
\left\| \int_0^t U(t - t') \partial_x (uv)(t') \, dt' \right\|_{H^s(\mathbb{R})} \lesssim \left\| \int_0^t \varphi(t - t') \ast \partial_x (uv)(t') \, dt' \right\|_{H^s(\mathbb{R})},
\]
where
\[
\varphi(t) = \int_{\mathbb{R}} e^{ix\xi} e^{(|\xi|^2 - |\xi|^3 + |\xi|) \frac{t}{t'}} \, d\xi.
\]
Observe that for any $1 \leq p \leq \infty$ and $\nu \geq 0$, we have for some $K > 0$ that
\[
\|D^\nu \varphi(t)\|_{L^p(\mathbb{R})} \lesssim e^K t^{-\frac{1}{2}(\nu + \frac{1}{p})} \lesssim t^{-\frac{1}{2}(\nu + \frac{1}{p})},
\]
for $0 \leq t \leq T \leq 1$, where $D^\nu \varphi = |\xi|^\nu \varphi$. Then by using the fractional Leibnitz rule, we get from (15), (16) and the Sobolev embedding that
\[
\left\| \int_0^t U(t - t') \partial_x (uv)(t') \, dt' \right\|_{H^s(\mathbb{R})} \lesssim \int_0^t \|\partial_x \varphi(t - t')\|_{L^2(\mathbb{R})} \left( \|D^s(uv)(t')\|_{L^1(\mathbb{R})} \right) \, dt' 
\]
\[
\lesssim \int_0^t (t - t')^{s/3-1/2} \|u(t')\|_{L^2(\mathbb{R})} \|v(t')\|_{H^s(\mathbb{R})} \lesssim T^{\theta(s)} \|u\|_{\mathcal{F}_x^{s'}} \|v\|_{\mathcal{F}_x^{s'}},
\]
where $\langle \cdot \rangle = 1 + |\cdot|$ and $\theta(s) > 0$ for any $s \geq 0$.

Next, we derive a regularity property which will be helpful in the regularity property in Theorem 1.

**Proposition 3.** Let $0 \leq t \leq T \leq T^*$, $s \in (-3/2, 0)$ and $\kappa \in [0, s + 3/2)$; then
\[
\forall: t \mapsto \int_0^t U(t - t') \partial_x u^2(t') \, dt',
\]
is in $C([0,T]; H^{s+\kappa}(\mathbb{R}))$, for all $u \in \mathcal{F}_x^s$.

**Proof.** Let $t_0, t_1 \in [0,T]$ be fixed such that $t_0 < t_1$. Then by the Minkowski inequality, we have
\[
\|\forall(t_1) - \forall(t_0)\|_{H^{s+\kappa}(\mathbb{R})} \leq \forall_1(t_0, t_1) + \forall_2(t_0, t_1),
\]
where
\[ V_1(t_0, t_1) = \int_{t_0}^{t_1} \left\| U(t_1 - t') \partial_x (u^2(t')) \right\|_{L^2(\mathbb{R})} \, dt', \]
and
\[ V_2(t_0, t_1) = \int_{0}^{t_0} \left\| (U(t_1 - t') - U(t_0 - t')) \partial_x (u^2(t')) \right\|_{L^2(\mathbb{R})} \, dt'. \]

By performing some straightforward computations, analogously to the proof of Proposition 2, we obtain that
\[ V_1(t_0, t_1) \leq \left( \int_{t_0}^{t_1} \left\| (1 + \xi^2)^{(1+s+\kappa)/2} e^{(t_1-t')(|\xi| - |\xi|)} \right\|_{L^2(\mathbb{R})} \left| t' - t_0 \right|^{-2|\xi|/3} \, dt' \right) \| u \|^2_{X^s_t}. \]
\[ \lesssim e^{2\sqrt{T}/\sqrt{2T} (t_1 - t_0) (2s + 2\kappa + 3)/6} \left[ \int_{t_0}^{t_1} \left| 1 - t' \right|^{-2|\xi|/3} \, dt' \right] \| u \|^2_{X^s_t}. \]

Now, by using the hypotheses, we get that
\[ \lim_{t_1 \to t_0} V_1(t_0, t_1) = 0. \]

On the other hand, we have
\[ V_2(t_0, t_1) \leq \left( \int_{0}^{t_0} \left\| g(t_0, t_1, t', \xi) \right\|_{L^2(\mathbb{R})} \left| t' - t_0 \right|^{-2|\xi|/3} \, dt' \right) \| u \|^2_{X^s_t}, \]
where
\[ g(t_0, t_1, t', \xi) = \xi^{s+\kappa+1} \left[ e^{(t_1-t')(|\xi| - |\xi|)} e^{i(t_1-t')\xi^2} - e^{(t_0-t')(|\xi| - |\xi|)} e^{i(t_0-t')\xi^2} \right]. \]

It is clear that \( g(t_0, t_1, t', \xi) \) tends to zero pointwise for almost every \( \xi \in \mathbb{R} \) and \( t' \in [0, t_0] \) when \( |t_1 - t_0| \to 0 \). Hence
\[ |g(t_0, t_1, t', \xi)| \lesssim \chi_{\{|\xi| \leq 1\}}(\xi) e^{2\sqrt{T}/\sqrt{2T}} + |\xi|^{s+\kappa+1} e^{(t_0-t')(|\xi| - |\xi|)}. \]
Thus, we deduce from the Lebesgue dominated convergence theorem that
\[ \left\| g(t_0, t_1, t', \xi) \right\|_{L^2(\mathbb{R})} \to 0, \]
as \( t_1 \to t_0 \). Using again the Lebesgue dominated convergence theorem, we conclude that
\[ \lim_{t_1 \to t_0} V_2(t_0, t_1) = 0. \]
This completes the proof.
3. Local existence and ill-posedness

All the elements are now in place to mount a proof of the local well-posedness result in Theorem 1.

**Proof of Theorem 1.** Let \( s > -3/2 \) and \( u_0 \in H^s(\mathbb{R}) \). We are going to show that the operator \( \Phi \) defined in (5) is a contraction in some closed ball of \( \mathcal{X}_T^s \). By Propositions 1 and 2, there exist two positive constants \( C = C(s) \) and \( \theta = \theta(s) \) such that

\[
\| \Phi(u) \|_{\mathcal{X}_T^s} \leq C \left( \| u_0 \|_{H^s(\mathbb{R})} + T^\theta \| u \|_{\mathcal{X}_T^s}^2 \right),
\]

(17)

and

\[
\| \Phi(u) - \Phi(v) \|_{\mathcal{X}_T^s} \leq CT^\theta \| u - v \|_{\mathcal{X}_T^s} \| u + v \|_{\mathcal{X}_T^s},
\]

(18)

for all \( u, v \in \mathcal{X}_T^s \) and \( 0 < T \leq T^* \). Now we define

\[
\mathcal{X}_T^s(b) = \{ u \in \mathcal{X}_T^s : \| u \|_{\mathcal{X}_T^s} \leq b \} \quad \text{with} \quad b = 2C\| u_0 \|_{H^s(\mathbb{R})}.
\]

and we choose

\[
0 < T < \min \left\{ 1, (2Cb)^{-1/\theta} \right\}.
\]

Estimates (17) and (18) imply that \( \Phi \) is a contraction on the Banach space \( \mathcal{X}_T^s(b) \); so that we deduce by the fixed point theorem, the existence of a unique solution \( u \) of the integral equation (5) in \( \mathcal{X}_T^s(b) \) with the initial data \( u(0) = u_0 \). Note that Proposition 3 assures that \( \Phi(u) \in C([0, T]; H^s(\mathbb{R})) \).

The uniqueness of the solution of (5) on the whole space \( \mathcal{X}_T^s \) and the smoothness of the flow map solution follow by standard arguments (see for example [13]).

Note that a similar contraction argument shows that the existence result holds for any \( s' > s > -3/2 \), in the time interval \([0, T]\) with \( T = T(\| u_0 \|_{H^s(\mathbb{R})}) \) (see Remark 1). Finally, we know that the map \( t \mapsto U(t)u_0 \) is continuous in the time interval \([0, T]\) with respect to the topology of \( H^\infty(\mathbb{R}) \). Since our solution \( u \) belongs to \( \mathcal{X}_T^s \), we deduce from Proposition 3 that there exists \( \kappa > 0 \) such that the map \( V \) belongs to \( C([0, T]; H^{s+\kappa}(\mathbb{R})) \), so that

\[
u \in C \left( [0, T]; H^{s+\kappa}(\mathbb{R}) \right).
\]

Therefore, by a standard bootstrapping argument, using the uniqueness result and the fact that the time interval of the existence of the solutions depends only on the \( H^s(\mathbb{R}) \)-norm of the initial data, we deduce that

\[
u \in C \left( (0, T]; H^\infty(\mathbb{R}) \right).
\]

\[\square\]

**Remark 3.** A standard argument similar to [3], one can observe that if \( u_0 \in H^s(\mathbb{R}) \), for \( s \geq 0 \), the corresponding local solution of (1) extends globally in time. More precisely, since the solution \( u \) of (1) is in \( C(\{(0, T]; H^\infty(\mathbb{R})) \), one only needs to prove an a priori estimate for \( u \). So \( u \) solves the Cauchy problem (1) in the classical sense. Recall that \( T = T(\| u_0 \|_{H^s(\mathbb{R})}) \). This allows us to take the \( L^2 \)-scalar product of (1) with \( u \), integrate by parts and use the properties of the Hilbert transform (see for
example \([10, 11]\), the Gagliardo-Nierenberg inequality and the Young inequality to obtain
\[
\frac{1}{2} \frac{d}{dt} \|u(t)\|_{L^2}(\mathbb{R}) = \|D^{1/2}u\|_{L^3}(\mathbb{R})^2 - \|D^{3/2}u\|_{L^2}(\mathbb{R})^2 \\
\leq C\|D^{3/2}u\|_{L^2(\mathbb{R})}^{2/3} \|u\|_{L^3(\mathbb{R})}^{4/3} - \|D^{3/2}u\|_{L^2(\mathbb{R})}^2 \leq C\|u\|_{L^2(\mathbb{R})}^2,
\]
where \(C > 0\) is independent of \(t\). Then by the Gronwall inequality, it yields
\[
\|u(t)\|_{L^2(\mathbb{R})} \leq \|u(0)\|_{L^2(\mathbb{R})} e^{CT}, \quad \text{for all } t \in [0, T].
\]

Next, we are going to show that our well-posedness result is sharp. We will first prove that we cannot solve the Cauchy problem (1) in \(H^s(\mathbb{R})\) using the fixed point theorem when \(s < -3/2\). Then we show that this fact implies Theorem 2.

**Remark 4.** With a slight modification, the proofs of Theorems 2 and 3 (below) are very similar to Pastrán’s results in his thesis [19]. The author should mention that he proved Theorems 2 and 3 independent of Pastrán’s thesis in [19], and for the sake of completeness of this paper, the author gives the proofs in details here.

**Theorem 3.** Let \(s < -3/2\) and \(T > 0\). Then, there does not exist any space \(\mathcal{X}_T^s\) such that \(\mathcal{X}_T^s\) is continuously embedded in \(C([0, T]; H^s(\mathbb{R}))\), i.e.
\[
\|u\|_{L^\infty(\mathbb{R})} \leq \|u\|_{\mathcal{X}_T^s}, \quad \forall u \in \mathcal{X}_T^s
\]
and such that
\[
\|U(t)u_0\|_{\mathcal{X}_T^s} \leq \|u_0\|_{H^s(\mathbb{R})}, \quad \forall u_0 \in H^s(\mathbb{R})
\]
and
\[
\left\| \int_0^t U(t-t')\partial_x \left( (U(t')u_0)^2 \right) \, dt' \right\|_{\mathcal{X}_T^s} \leq \|u_0\|_{H^s(\mathbb{R})}^2
\]
for all \(u, v \in \mathcal{X}_T^s\).

**Proof.** Suppose that there exists a space \(\mathcal{X}_T^s\) as in Theorem 3. Take \(u_0 \in H^s(\mathbb{R})\), \(u(t) = U(t)u_0\), and fix \(0 < t < T\). Then by using relations (19), (20) and (21), we see that
\[
\left\| \int_0^t U(t-t')\partial_x \left( (U(t')u_0)^2 \right) \, dt' \right\|_{H^s(\mathbb{R})} \leq \|u_0\|_{H^s(\mathbb{R})}^2.
\]
We will show that (22) fails for an appropriate choice of \(u_0\), which would lead to a contradiction. Define \(u_0\) by
\[
\tilde{u}_0(\xi) = N^{-s} \gamma^{-1/2} (\chi_{I_1}(\xi) + \chi_{I_2}(\xi)),
\]
where \(N \gg 1\), \(\gamma = N^{1-\epsilon_0} (0 < \epsilon_0 \ll 1\) fixed) and
\[
I_1 = [N, N + 2\gamma], \quad I_2 = [-N - 2\gamma, -N].
\]
It is easy to see that
\[
\|u_0\|_{H^s(\mathbb{R})} \sim 1.
\]
Then, we use the definition of \( U(t) \) and Fubini’s theorem to get

\[
\left| \tilde{h}(t, ξ) \right| = \left| \lambda \int_{0}^{t} U(t-t')\partial_{x} \left( (U(t')u_{0})^{2} \right) \, dt' \right| \bigg|_{\xi} \tag{1}
\]

\[
= \int_{0}^{t} i\xi e^{(t-t')(|ξ|^{3}-(|ξ|-|ξ_{1}|))} \hat{U}(t')\hat{u}_{0} \ast \hat{U}(t')\hat{u}_{0} (ξ) \, dt'
\]

\[
\leq e^{i\xi t} \int_{\mathbb{R}} i\xi \hat{u}_{0}(ξ_{1})\hat{u}_{0}(ξ_{2})f(t, ξ, ξ_{1}) \, dξ_{1}
\]

\[
\leq \frac{1}{\gamma N^{2s}} \int_{\mathcal{M}} ξ f(t, ξ, ξ_{1}) \, dξ_{1},
\]

where

\[
f(t, ξ, ξ_{1}) = \frac{e^{-t(|ξ|^{3}−|ξ_{2}|−|ξ_{1}|)|ξ|}e^{it(ξ_{1}^{2}+ξ_{2}^{2}−ξ^{2})} − e^{-t(|ξ|^{3}−|ξ|)}},
\]

\[
ξ_{2} = ξ − ξ_{1},
\]

\[
ω(ξ, ξ_{1}) = |ξ_{1}| − |ξ|^{3} − |ξ_{2}| + |ξ_{2}| + |ξ| + 3iξ_{1}ξ_{2}.
\]

and

\[
\mathcal{M} = \{ ξ_{1} : ξ_{1} ∈ I_{1}, ξ_{2} ∈ I_{2} \}.
\]

When \( ξ_{1} ∈ I_{1} \) and \( ξ_{2} ∈ I_{2} \), we deduce that \( ξ ∈ [2N, 2N + 4|ξ|] \) and \( ω(ξ, ξ_{1}) ≤ N^{3} \). Now we choose a sequence of times \( t_{N} = N^{−3}/γ \), so that \( e^{−t(|ξ|^{3}−|ξ|)|ξ|} \sim e^{−N^{3}t_{N}} \sim e^{−N^{−3}γ} > C > 0 \). Hence

\[
\left| \frac{e^{-t(|ξ|^{3}−|ξ_{2}|−|ξ_{1}|)|ξ|}e^{it(ξ_{1}^{2}+ξ_{2}^{2}−ξ^{2})} − 1}{ω(ξ, ξ_{1})} \right| \leq \frac{1}{N^{3+2γ}} + O \left( \frac{1}{N^{3+2γ}} \right).
\]

Therefore,

\[
\| h(\cdot, t) \|_{H^{s}(\mathbb{R})} ≥ N^{−s−3/2−3γ}/2.
\]

Hence, we obtain that

\[
N^{−s−3/2−3γ}/2 ≤ 1, \quad \forall \ N \gg 1;
\]

which contradicts the assumption \( s < −3/2 \).

A proof of Theorem 2 is now in sight.

**Proof of Theorem 2.** Let \( s < −3/2 \), suppose that there exists \( T > 0 \) such that the Cauchy problem (1) is locally well-posed in \( H^{s}(\mathbb{R}) \) in the time interval \([0, T]\) and that the flow map solution \( \mathcal{F} : H^{s}(\mathbb{R}) \to C([0, T]; H^{s}(\mathbb{R})) \) is \( C^{2} \) at the origin. When \( u_{0} \in H^{s}(\mathbb{R}) \), we will denote \( u_{0}(t) = \mathcal{F}(u_{0})(t) \) the solution of equation (1) with initial datum \( u_{0} \). This means that \( u_{0} \) is a solution of the integral equation

\[
u_{0}(t) = \mathcal{F}(u_{0})(t) = U(t)u_{0} − \frac{1}{2} \int_{0}^{t} U(t-t')\partial_{x} \left( u_{0}^{2} \right) (t') \, dt'.
\]
By computing the Fréchet derivative of $\mathcal{F}$ at $\varphi$ in the direction $u_0$, we obtain that
\[ d_{\varphi}\mathcal{F}(u_0)(t) = U(t)u_0 - \int_0^t U(t-t')\mathcal{B}[u_0(t'), d_{\varphi}\mathcal{F}(u_0)(t')] \, dt', \tag{23} \]
where $\mathcal{B}[\varphi, \psi] = (\varphi \psi)_x$. Since the Cauchy problem (1) is supposed to be well-posed, we know by using the uniqueness that $\mathcal{F}(0)(t) = u_0(t) = 0$ and then we deduce from (23) that
\[ d_0\mathcal{F}(u_0)(t) = U(t)u_0. \tag{24} \]
Using (23), we compute the second Fréchet derivative at the origin in the direction $(u_0, \psi)$ and using (24), we deduce that
\[ d_{u_0}^2\mathcal{F}(u_0, \psi)(t) = -\int_0^t U(t-t')\mathcal{B}[U(t')\psi, U(t')u_0] \, dt'. \]
The assumption of $C^2$ regularity of $\mathcal{F}$ at the origin would imply that
\[ d_{u_0}^2\mathcal{F} \in \mathcal{L} \left( H^s(\mathbb{R}) \times H^s(\mathbb{R}), H^s(\mathbb{R}) \right), \]
which would lead to the following inequality
\[ \|d_{u_0}^2\mathcal{F}(u_0, \psi)(t)\|_{H^s(\mathbb{R})} \lesssim \|u_0\|_{H^s(\mathbb{R})} \|\psi\|_{H^s(\mathbb{R})}, \tag{25} \]
for all $u_0, \psi \in H^s(\mathbb{R})$. But (25) is equivalent to (22) which has been shown to fail in the proof of Theorem 3.

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\section*{References}


