

## Isoptic curves of conic sections in constant curvature geometries

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**Abstract.** In this paper, we consider the isoptic curves in 2-dimensional geometries of constant curvature  $\mathbf{E}^2$ ,  $\mathbf{H}^2$ ,  $\mathcal{E}^2$ . The topic is widely investigated in the Euclidean plane  $\mathbf{E}^2$ , see for example [1] and [15] and the references given there. In the hyperbolic and elliptic plane (according to [18]), there are few results in this topic (see [3] and [4]). In this paper, we give a review of the known results on isoptics of Euclidean and hyperbolic curves and develop a procedure to study the isoptic curves in the hyperbolic and elliptic plane geometries and apply it to some geometric objects, e.g. proper conic sections. For the computations we use classical models based on the projective interpretation of hyperbolic and elliptic geometry and in this manner the isoptic curves can be visualized.

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**Key words:** isoptic curves, non-Euclidean geometry, conic sections, projective geometry

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### 1. Introduction

Let  $G$  be one of the constant curvature plane geometries, the Euclidean  $\mathbf{E}^2$ , the hyperbolic  $\mathbf{H}^2$ , and the elliptic  $\mathcal{E}^2$ . The isoptic curve of a given plane curve  $\mathcal{C}$  is the locus of points  $P \in G$ , where  $\mathcal{C}$  is seen under a given fixed angle  $\alpha$  ( $0 < \alpha < \pi$ ). An isoptic curve formed by the locus of tangents meeting at right angles is called orthoptic curve. The name isoptic curve was suggested by Taylor in [14].

First, we consider the Euclidean plane geometry ( $G = \mathbf{E}^2$ ). The easiest case is where  $\mathcal{C}$  is a line segment, then the set of all points for which a line segment can be seen at angle  $\alpha$  contains of two circular arcs with central angle  $2\alpha$  symmetric with respect to the segment. In the special case of  $\alpha = \frac{\pi}{2}$ , we get exactly one circle called Thales circle (without the endpoints of the given segment) with the center in the middle of the line segment.

In [1] and [2], the isoptic curves of the closed, strictly convex curves are studied, using their support function. Papers [16] and [17] deal with curves having a circle or an ellipse for an isoptic curve. Further, curves appearing as isoptic curves are well studied in Euclidean plane geometry  $\mathbf{E}^2$ , see e.g. [6, 15]. Isoptic curves of conic sections have been studied in [5] and [12]. A lot of papers concentrate on the properties of the isoptics, e.g. [8, 7], and the reference given there.

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There are a lot of possibilities to give the equations of the isoptics of conic sections (see e.g [6]), for instance, they can be determined by the *construction method of the tangent lines from an outer point*. We have illustrated this procedure (see [13]) in the following figure:

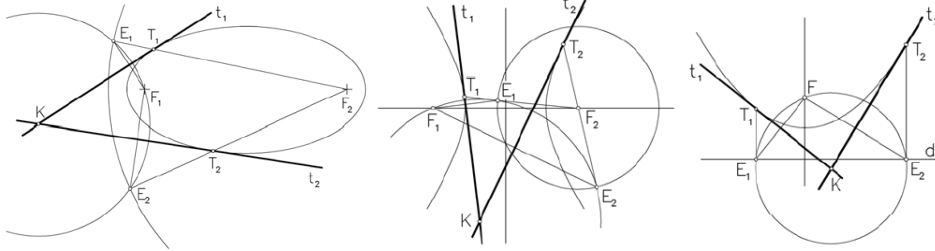


Figure 1: Tangent lines from outer point  $K$

To get the isoptics, we have to solve the system of equations generated by two circle equations (ellipse, hyperbola) or a circle and a line equation (parabola) and using the scalar product we have to fix the angle of the tangent lines. In the case of the hyperbola, there is no proper touching point, if the outer point is contained by one of its asymptotes, but the asymptotes are the tangent lines at the curve's ideal points. From this method, we get the following equations for the isoptic curves (see [6]):

$$\text{Ellipse : } \cos \alpha = -\frac{a^2 + b^2 - x^2 - y^2}{\sqrt{(-a^2 + b^2 + x^2)^2 + 2y^2(a^2 - b^2 + x^2) + y^4}},$$

where the ellipse is given by its equation  $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ ,

$$\text{Hyperbola : } \cos^2 \alpha = \frac{(-a^2 + b^2 + x^2 + y^2)^2}{(a^2 + b^2 - x^2)^2 + 2y^2(a^2 + b^2 + x^2) + y^4},$$

where the hyperbola is given by its equation  $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ ,

$$\text{Parabola : } \cos \alpha = -\frac{y}{\sqrt{(p-y)^2 + x^2}},$$

where the axis  $x$  is the directrix, and the focus equals  $(0, p)$ .

**Remark 1.**

1. In the case of a hyperbola, the two asymptotes split the space into four domains, two of them contain a hyperbola branch (focal domains), the other ones are empty. Let  $P$  be an outer point of the hyperbola. If  $P$  is in a focal domain, then the tangent lines touch the same branch of the hyperbola, else they touch both of the branches. In these cases, the isoptic angles are complementary to each other, i.e. they sum up to  $\pi$ . Therefore, we take the square of the equation and thus we obtain both types of isoptic curves.

2. The numerator is greater than zero for every  $(x, y)$  if  $b > a$ . Therefore, the isoptic curves do not exist in the interval

$$\left( \arccos\left(\frac{b^2 - a^2}{b^2 + a^2}\right), \arccos\left(\frac{a^2 - b^2}{a^2 + b^2}\right) \right)$$

if the condition  $b > a$  holds. Otherwise, the isoptic curve exist for every  $\alpha \in (0, \pi)$ .

We have illustrated the isoptic curves of some conic sections in Euclidean plane  $\mathbf{E}^2$  in Figures 2-3:

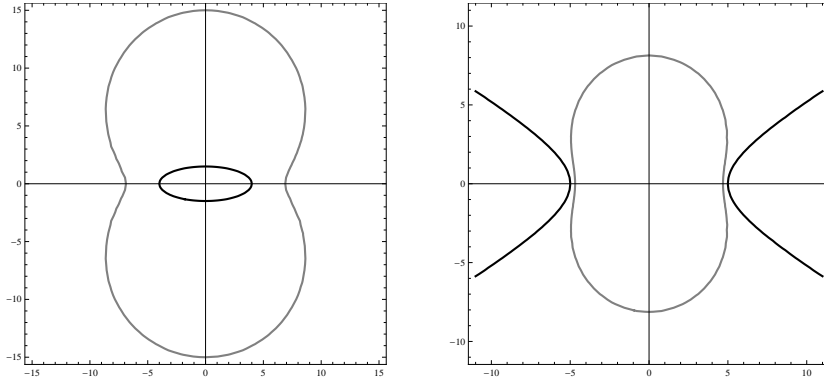


Figure 2: Isoptic curve to the Euclidean ellipse (left) and hyperbola (right) with parameters:  $a = 4$ ,  $b = 1, 5$ ,  $\alpha = \pi/6$ , and  $a = 5$ ,  $b = 3$ ,  $\alpha = \pi/3$

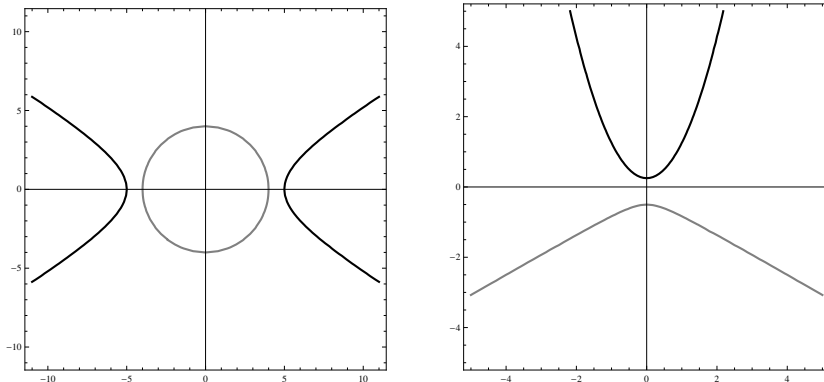


Figure 3: Isoptic curve to the Euclidean hyperbola (left) and parabola (right) with parameters:  $a = 5$ ,  $b = 3$ ,  $\alpha = \pi/2$ , and  $p = 1/2$ ,  $\alpha = \pi/3$

In the case of hyperbolic planar geometry there are only a few results. The isoptic curves of the hyperbolic line segment, ellipses and parabolas are determined in [3] and [4].

As far as we know, there are no results in elliptic geometry  $\mathcal{E}^2$ , but we conjecture that there might exist a few in spherical geometry.

In this paper, we develop a method based on the projective interpretation of hyperbolic and elliptic geometry to determine the isoptic curve of a given plane

curve  $\mathcal{C}$  and we apply our procedure to the hyperbolic hyperbola with proper foci, elliptic line segments and elliptic conic sections. Moreover, we visualize them for some angles.

## 2. The projective model

We use homogeneous coordinates  $\mathbf{x} = (x^0 : x^1 : x^2)$  and  $\mathbf{u} = (u_0 : u_1 : u_2)$  in order to represent points  $X$  as well as straight lines  $u$  in projective space  $\mathbb{P}^3$ . Sometimes we write  $X = \mathbf{x}\mathbb{R}$  in order to express that the point  $X$  is determined by a certain vector  $\mathbf{x} \in \mathbb{R}^3$  and its non-trivial scalar multiples, similarly  $u = \mathbf{u}\mathbb{R}$  for lines. A point  $X = \mathbf{x}\mathbb{R}$  and a straight line  $u = \mathbf{u}\mathbb{R}$  are incident if  $\mathbf{x} \cdot \mathbf{u}^T = 0$  with  $\cdot$  denoting usual matrix multiplication.

Constant curvature plane geometries can be represented in projective space  $\mathbb{P}^3$  using the bilinear form

$$\langle \mathbf{x}, \mathbf{y} \rangle = \epsilon x^0 y^0 + x^1 y^1 + x^2 y^2, \quad (1)$$

where  $\epsilon = 0, -1, +1$ , respectively, Euclidean, hyperbolic and elliptic geometries. Now we consider hyperbolic and elliptic planes, therefore  $\epsilon = \pm 1$ .

Hyperbolic and elliptic distances and angles can be computed with the help of the bilinear form (1), see [10]. The distance  $d(X, Y)$  of two points  $X = \mathbf{x}\mathbb{R}$  and  $Y = \mathbf{y}\mathbb{R}$  is given by

$$C(s) = \frac{\epsilon \langle \mathbf{x}, \mathbf{y} \rangle}{\sqrt{\langle \mathbf{x}, \mathbf{x} \rangle \langle \mathbf{y}, \mathbf{y} \rangle}}, \quad (2)$$

where  $C(s)$  is cosine in elliptic geometry, and hyperbolic cosine in hyperbolic geometry.

Further, we find the angle  $\alpha(u, v)$  enclosed by two straight lines  $u = \mathbf{u}\mathbb{R}$  and  $v = \mathbf{v}\mathbb{R}$ , respectively, with

$$\cos \alpha = \frac{\epsilon \langle \mathbf{u}, \mathbf{v} \rangle}{\sqrt{\langle \mathbf{u}, \mathbf{u} \rangle \langle \mathbf{v}, \mathbf{v} \rangle}}. \quad (3)$$

## 3. Isoptic curve of the line segment on the hyperbolic and elliptic plane

In this section, we examine the hyperbolic and elliptic cases together.

Let two points  $A$  and  $B$  be given in the plane. Without loss of generality, we can assume that their homogeneous coordinates are  $A = (1 : a : 0)$  and  $B = (1 : -a : 0)$ , where  $(a \in ]0, 1])$ . We consider two straight lines  $u$  and  $v$ , where  $u = (1 : u_1 : u_2)^T$  passes through points  $A$  and  $P$ , and  $v = (1 : v_1 : v_2)^T$  passes through points  $B$  and

$P$ . By the incidence formula we get the following equations:

$$\begin{aligned} A \in u &\Leftrightarrow (1, a, 0) \begin{pmatrix} 1 \\ u_1 \\ u_2 \end{pmatrix} = 0 \Leftrightarrow u_1 = -\frac{1}{a}, \\ B \in v &\Leftrightarrow (1, -a, 0) \begin{pmatrix} 1 \\ v_1 \\ v_2 \end{pmatrix} = 0 \Leftrightarrow v_1 = \frac{1}{a}. \end{aligned} \quad (4)$$

$$\begin{aligned} P \in u &\Leftrightarrow (1, x, y) \begin{pmatrix} 1 \\ u_1 \\ u_2 \end{pmatrix} = 0 \Leftrightarrow u_2 = -\frac{a-x}{ya}, \quad y \neq 0, \\ P \in v &\Leftrightarrow (1, x, y) \begin{pmatrix} 1 \\ v_1 \\ v_2 \end{pmatrix} = 0 \Leftrightarrow v_2 = -\frac{a+x}{ya}, \quad y \neq 0. \end{aligned} \quad (5)$$

The angle  $\alpha$  between the above straight lines can be determined by the formula (3):

$$\cos(\alpha) = \frac{\epsilon(\epsilon + u_1v_1 + u_2v_2)}{\sqrt{(\epsilon + u_1^2 + u_2^2)(\epsilon + v_1^2 + v_2^2)}}$$

with  $\epsilon = \pm 1$  if  $G$  is elliptic or hyperbolic. Substituting coordinates from (4) and (5) into the above equation, we obtain:

**Theorem 1.** *Let us suppose that a line segment is given by  $A = (1 : a : 0)$  and  $B = (1 : -a : 0)$ . Then for a given  $\alpha$  ( $0 < \alpha < \pi$ ), the  $\alpha$ -isoptic curve of  $AB$  in the hyperbolic and elliptic plane has an equation of the form:*

$$\cos(\alpha) = \frac{\epsilon(\epsilon - \frac{1}{a^2} + \frac{a^2 - x^2}{y^2 a^2})}{\sqrt{(\epsilon + \frac{1}{a^2} + (\frac{a-x}{ya})^2)(\epsilon + \frac{1}{a^2} + (\frac{a+x}{ya})^2)}}, \quad (6)$$

where  $\epsilon = \pm 1$  if  $G$  is either the elliptic or the hyperbolic plane.

**Remark 2.**

1. We obtain the orthoptic curve  $\mathcal{G}_{\pi/2}$  if  $\alpha = \pi/2$  with the equation:

$$\frac{x^2}{a^2} + \frac{y^2}{\frac{a^2}{1-\epsilon a^2}} = 1. \quad (7)$$

*This is an ellipse (without endpoints of the given segment) in the Euclidean sense, and it can be called the Thales curve.*

*In the hyperbolic plane, if we increase the parameter  $a$ , then the Thales curve tends to a hypercycle (or an equidistant curve). That means the hypercycle is a special type of orthoptic curves*

$$x^2 + 2y^2 = 1.$$

2. In the hyperbolic plane, if  $a \rightarrow 1$ , then equation (6) converges to the following equation

$$x^2 + \left( \frac{y}{\cos\left(\frac{\alpha}{2}\right)} \right)^2 = 1.$$

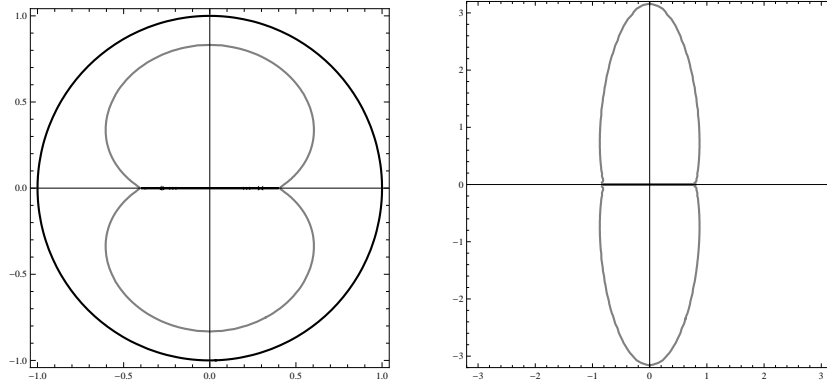


Figure 4: The isoptic curve to hyperbolic (left) and elliptic (right) line segment with parameters: hyperbolic:  $a = 0.4$ ,  $\alpha = \pi/6$ , elliptic:  $a = 0.8$ ,  $\alpha = 8\pi/18$

## 4. The general method

The procedure above can be used to develop a more general method to determine the isoptic curves. Let a conic section  $C$  and one of its point  $P$  be given. Using the implicit function theorem and the equation of  $C$ , we can determine the equation of the tangent line in this point. After that, a system of equations to the coordinates of the tangent point from an exterior point  $K$  can be given. This point has to satisfy the equation of the given curve and the tangent lines to this point have to contain  $K$ . This system can be solved for every  $K = (1 : x^0 : y^0)$  outer point with respect to the parameters  $x^0, y^0$ . In the general case, the formulas of the solutions are complicated. Now, we have to follow the upper method using the coordinates of the tangent points. The equation of the tangent lines from  $K$  can be determined by solving a system of equations for its coordinates. Finally, we have to fix the angle of the straight lines and we get the equation of the isoptic curve.

### 4.1. On the hyperbolic plane

#### 4.1.1. Equation of the hyperbolic ellipse and hyperbola

Now, we define the *proper central conic sections* and give their equations.

**Definition 1.** The *proper hyperbolic ellipse* is the locus of all points of the hyperbolic plane whose distances to two proper fixed points add to the same constant  $2a$ .

**Definition 2.** The *proper hyperbolic hyperbola* is the locus of points where the absolute value of the difference of the distances to the two proper foci is a constant  $2a$ .

We discuss the ellipse and the hyperbola together. We can suppose that the two foci are equidistant from the origin  $O$ , both located on the axis  $x$  with coordinates  $\mathbf{f}_1\mathbb{R} = F_1 = (1 : f : 0)$  and  $\mathbf{f}_2\mathbb{R} = F_2 = (1 : -f : 0)$  where  $0 < f < 1$ . Let  $\mathbf{p}\mathbb{R} = P = (1 : x : y) \in \mathbf{H}^2$  a point of the conic section. Using (2) we obtain

$$\begin{aligned} & \epsilon_1 \cosh^{-1} \left( \frac{-\langle \mathbf{p}, \mathbf{f}_1 \rangle}{\sqrt{\langle \mathbf{p}, \mathbf{p} \rangle \langle \mathbf{f}_1, \mathbf{f}_1 \rangle}} \right) + \epsilon_2 \cosh^{-1} \left( \frac{-\langle \mathbf{p}, \mathbf{f}_2 \rangle}{\sqrt{\langle \mathbf{p}, \mathbf{p} \rangle \langle \mathbf{f}_2, \mathbf{f}_2 \rangle}} \right) \\ &= 2a \Leftrightarrow \epsilon_2 \cosh^{-1} \left( \frac{-(-1 - xf)}{\sqrt{(-1 + x^2 + y^2)(-1 + f^2)}} \right) \\ &= 2a - \epsilon_1 \cosh^{-1} \left( \frac{-(-1 + xf)}{\sqrt{(-1 + x^2 + y^2)(-1 + f^2)}} \right), \end{aligned}$$

where  $\epsilon_{1,2} = \pm 1$  and  $\epsilon_1 + \epsilon_2 \geq 0$ . Applying the hyperbolic cosine to both sides, we get the following equations:

$$\begin{aligned} \frac{1 + xf}{\sqrt{(-1 + x^2 + y^2)(-1 + f^2)}} &= \cosh(2a) \frac{1 - xf}{\sqrt{(-1 + x^2 + y^2)(-1 + f^2)}} \\ &\quad - \epsilon_1 \sinh(2a) \sinh \left( \cosh^{-1} \left( \frac{1 - xf}{\sqrt{(-1 + x^2 + y^2)(-1 + f^2)}} \right) \right). \end{aligned}$$

The next equation is obtained by applying the formula  $\sinh(\cosh^{-1}(t)) = \sqrt{t^2 - 1}$  and by multiplying both sides by  $\sqrt{(-1 + x^2 + y^2)(-1 + f^2)}$ .

$$1 + xf = \cosh(2a)(1 - xf) - \epsilon_1 \sinh(2a) \sqrt{(1 - xf)^2 - (1 - x^2 - y^2)(1 - f^2)}.$$

Now, if we simplify this equation and take its square, there is no  $\epsilon$  therein. Finally, we get the following equation:

$$\left( \frac{x}{\tanh(a)} \right)^2 + \frac{y^2}{1 + \frac{1}{(f^2 - 1) \cosh^2(a)}} = 1. \quad (8)$$

If the distance between the two foci is less than  $2a$ , it is an ellipse; if it is greater, then it is a hyperbola since we have

$$\begin{aligned} 2a <> d(F_1, F_2) &\Leftrightarrow \cosh(2a) <> \cosh(d(F_1, F_2)) = \frac{-\langle F_1, F_2 \rangle}{\sqrt{\langle F_1, F_1 \rangle \langle F_2, F_2 \rangle}} = \frac{1 + f^2}{1 - f^2} \\ &\Leftrightarrow 2 \cosh^2(a) - 1 <> \frac{2}{1 - f^2} - 1 \Leftrightarrow 1 <> \frac{1}{\cosh^2(a)(1 - f^2)} \\ &\Leftrightarrow 1 + \frac{1}{\cosh^2(a)(f^2 - 1)} <> 0. \end{aligned}$$

Therefore, the hyperbolic ellipse and hyperbola are also an ellipse and a hyperbola in the model.

#### 4.1.2. Isoptic curve of hyperbolic ellipse and hyperbola

Now, we will use the above described method to determine the isoptic curves to hyperbolic ellipses and hyperbolas.

The first step is to determine the equation of the tangent lines ( $y \neq 0$ ):

$$y' = -\frac{x}{y} \left( 1 + \frac{f^2}{\sinh^2(a)(f^2 - 1)} \right). \quad (9)$$

The equation above converges continuously to  $x^2 = \tanh^2(a)$ , if  $y \rightarrow 0$ . After that, we have to solve the following system of equations for  $\tilde{X}_i = (1 : \tilde{x}_i : \tilde{y}_i)$ , ( $i = 1, 2$ ), where  $X = (1 : x : y)$  is a point in the Cayley-Klein model:

$$\begin{aligned} -\frac{\tilde{x}}{\tilde{y}} \left( 1 + \frac{f^2}{\sinh^2(a)(f^2 - 1)} \right) (x - \tilde{x}) + \tilde{y} &= y \\ \left( \frac{\tilde{x}}{\tanh(a)} \right)^2 + \frac{\tilde{y}^2}{1 + \frac{1}{(f^2 - 1) \cosh^2(a)}} &= 1. \end{aligned} \quad (10)$$

It is not so hard to determine the roots, but because of the complexity of the result, we ignore it. Now we need  $u = (1 : u_1 : u_2)^T$  and  $v = (1 : v_1 : v_2)^T$  straight lines, fits on  $P$ ,  $\tilde{X}_1$  and  $P$ ,  $\tilde{X}_2$ , respectively.

Using that  $(\tilde{x}_i, \tilde{y}_i)$  is known, we get the following system of equations:

$$\begin{aligned} 1 + u_1 x + u_2 y &= 0 \\ 1 + u_1 \tilde{x}_1 + u_2 \tilde{y}_1 &= 0, \end{aligned} \quad (11)$$

$$\begin{aligned} 1 + v_1 x + v_2 y &= 0 \\ 1 + v_1 \tilde{x}_2 + v_2 \tilde{y}_2 &= 0. \end{aligned} \quad (12)$$

Solving these systems, we can determine the tangent lines for all  $X[\mathbf{x}]$  outside of the conic section. The last step in the method of the previous section is to fix the angle. We summarize our results in the following theorem:

**Theorem 2.** *Let a hyperbolic ellipse or hyperbola be centered at the origin in the projective model given by its semimajor axis  $a$  and foci  $F_1 = (1 : f : 0)$ ,  $F_2 = (1 : -f : 0)$ , ( $0 < f < 1$ ) such that  $2a > d(F_1, F_2)$  or  $2a < d(F_1, F_2)$  holds. The  $\alpha$  and  $(\pi - \alpha)$ -isoptic curves ( $0 < \alpha < \pi$ ) of the considered ellipse or hyperbola in the hyperbolic plane has the equation*

$$\cos^2(\alpha) = \frac{((f^2 - 1) \cosh(2a) (x^2 + y^2 - 1) + f^2 x^2 - 1)^2}{-2(f^2 - 1) y^2 (f^2 + x^2) + (f^2 - x^2)^2 + (f^2 - 1)^2 y^4}, \quad (13)$$

where  $x^2 + y^2 \leq 1$  holds.

**Remark 3.**

1. The orthoptic curves of hyperbolic hyperbola exist if

$$f \leq \sqrt{1 - \frac{1}{\cosh(2a)}}$$



and it is an ellipse, similarly to the hyperbolic ellipse, with the following equation

$$(1 - f^2) \cosh(2a) (-1 + x^2 + y^2) + f^2 x^2 = 1.$$

2. We remark here, without any calculations, that the isoptic curve of the hyperbolic ellipse does not exist in the following interval:

$$\alpha \in \left( \arccos \left( \frac{(f^2 - 1) \cosh(2a) + 1}{f^2} \right), \arccos \left( \frac{(1 - f^2) \cosh(2a) - 1}{f^2} \right) \right),$$

$$\text{if } \left( \frac{1}{1-f^2} + 1 \right) \operatorname{sech}^2(a) > 2.$$

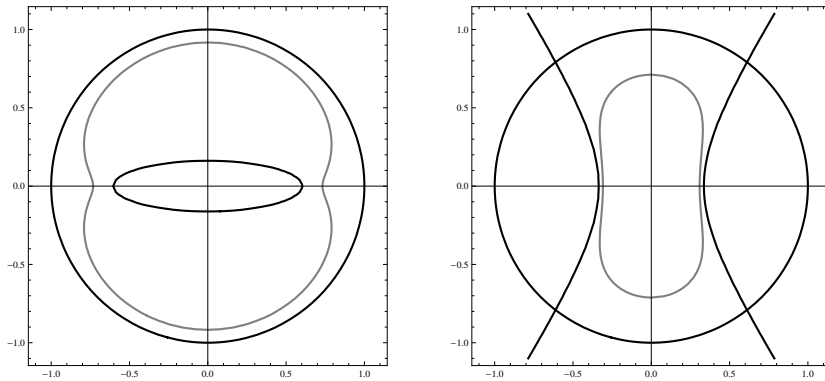


Figure 5: The isoptic curve to a hyperbolic ellipse (with  $a = 0.7$ ,  $f = 0.59$ ,  $\alpha = \pi/6$ ; left) and a hyperbola (with  $a = 0.35$ ,  $f = 0.55$ ,  $\alpha = \pi/6$ ; right)

#### 4.1.3. The equation and the isoptic curve of the hyperbolic parabola

In this section, we define the *proper hyperbolic parabolas* and give their equations.

**Definition 3.** The hyperbolic proper parabola is the set of points  $(X = \mathbf{x}\mathbb{R} = (1 : x : y))$  in the hyperbolic plane that are equidistant to a proper point (the focus  $F$ ) and a proper line (the directrix  $e$ ) ( $s = d(X; F) = d(X; e)$ ).

Without loss of generality, we can assume that the directrix ( $e$ ) is the axis  $x$  and the coordinates of the focus point are  $F = (1 : 0 : p)$ .

We remark that the coordinates of the foot point  $X'$  of the perpendicular dropped from  $X$  to the  $x$ -axis are  $X' = (1 : x : 0)$ . Using formula (2) we find

$$\cosh(s) = \frac{1 - py}{\sqrt{1 - x^2 - y^2} \sqrt{1 - p^2}} = \frac{1 - x^2}{\sqrt{1 - x^2 - y^2} \sqrt{1 - x^2}}. \quad (14)$$

The equation of the proper hyperbolic parabola is obtained by (14):

$$x^2 + \frac{(1 - py)^2}{1 - p^2} = 1. \quad (15)$$

Using the above method, we have to solve the following system of equations for coordinates  $\tilde{x}, \tilde{y}$ :

$$\begin{aligned} \frac{\tilde{x}(1-p^2)}{p-\tilde{y}p^2}(x-\tilde{x})+\tilde{y} &= y \\ x^2+\frac{(1-py)^2}{1-p^2} &= 1. \end{aligned} \quad (16)$$

In accordance with the method, we have to solve the system of equations (11), (12). Now, the following theorem holds:

**Theorem 3.** *Let a proper hyperbolic parabola be given in the projective model by its focus  $\mathbf{f}\mathbb{R} = F = (1 : 0 : p)$  and its directrix  $e$  which coincides with the  $x$ -axis. The  $\alpha$ -isoptic curves of this parabola ( $0 < \alpha < \pi$ ) in the hyperbolic plane have the equation:*

$$\cos(\alpha) = \frac{y(py-1)}{\sqrt{(x^2-1)((p^2(x^2-1)+2py+y^2-x^2))}}. \quad (17)$$

**Remark 4.** *The orthoptic curve of the hyperbolic parabola is a straight line;  $y = 0$ .*

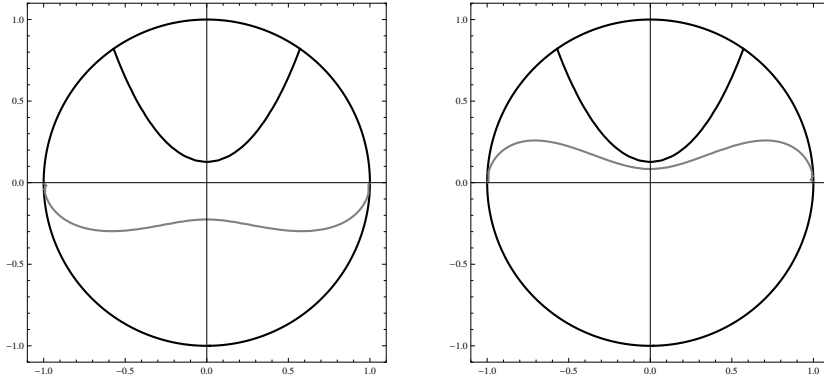


Figure 6: The isoptic curve to a hyperbolic parabola with parameters  $p = 0.25$ ,  $\alpha = \pi/3$  (left),  $p = 0.25$ ,  $\alpha = 2\pi/3$  (right)

## 4.2. On the elliptic plane

In this section, we will discuss the equations of the conic sections and their isoptics in the elliptic plane  $\mathcal{E}^2$ .

We remark that in the elliptic plane the maximum distance between two points is less than  $\frac{\pi}{2}$ ; therefore, in some cases the given curve cannot be seen under an arbitrarily small angle.

### 4.2.1. Equation of the elliptic ellipse and hyperbola

We will follow the deduction process detailed in the previous section to determine the equation of the elliptic ellipse and hyperbola, having the following definitions:

**Definition 4.** *The elliptic ellipse is the locus of all points of the elliptic plane whose distances to two fixed points add to the same constant  $2a$ .*

**Definition 5.** *The elliptic hyperbola is the locus of points where the absolute value of the difference of the distances to the two foci is a constant  $2a$ .*

Let us suppose that the two foci are symmetric with respect to origin  $O$ , both fits on the axis  $x$ . Then their coordinates are  $F_1 = (1 : f : 0)$  and  $F_2 = (1 : -f : 0)$ , where  $(0 < f < \pi/2)$ . Let  $P = (1 : x : y) \in \mathcal{E}^2$  a point of the conic section.

Using (2) we obtain the following equation:

$$\begin{aligned} & \epsilon_1 \cos^{-1} \left( \frac{\langle \mathbf{p}, \mathbf{f}_1 \rangle}{\sqrt{\langle \mathbf{p}, \mathbf{p} \rangle \langle \mathbf{f}_1, \mathbf{f}_1 \rangle}} \right) + \epsilon_2 \cos^{-1} \left( \frac{\langle \mathbf{p}, \mathbf{f}_2 \rangle}{\sqrt{\langle \mathbf{p}, \mathbf{p} \rangle \langle \mathbf{f}_2, \mathbf{f}_2 \rangle}} \right) \\ &= 2a \Leftrightarrow \epsilon_2 \cos^{-1} \left( \frac{(1 - xf)}{\sqrt{(1 + x^2 + y^2)(1 + f^2)}} \right) \\ &= 2a - \epsilon_1 \cos^{-1} \left( \frac{(1 + xf)}{\sqrt{(1 + x^2 + y^2)(1 + f^2)}} \right), \end{aligned}$$

where  $\epsilon_{1,2} = \pm 1$  and  $\epsilon_1 + \epsilon_2 \geq 0$ .

Repeating the procedure described in the previous section, we get the equation

$$\left( \frac{x}{\tan(a)} \right)^2 + \frac{y^2}{\frac{1}{(1+f^2)\cos^2(a)} - 1} = 1. \quad (18)$$

If the distance between the two foci is less than  $2a$ , it is an ellipse, else it is a hyperbola.

$$2a <> d(F_1, F_2) \Leftrightarrow \dots \Leftrightarrow 1 - \frac{1}{\cos^2(a)(1 + f^2)} <> 0$$

We have to take the implicit derivative of (18) and solve the system of equations for the tangent points  $\tilde{X}_i = (1 : \tilde{x}_i : \tilde{y}_i)$ ,  $(i = 1, 2)$ ,

$$\begin{aligned} & -\frac{\tilde{x}}{\tilde{y}} \left( 1 - \frac{f^2}{\sin^2(a)(1 + f^2)} \right) (x - \tilde{x}) + \tilde{y} = y \\ & \left( \frac{\tilde{x}}{\tan(a)} \right)^2 + \frac{\tilde{y}^2}{\frac{1}{(1+f^2)\cos^2(a)} - 1} = 1. \end{aligned} \quad (19)$$

By solving the system of equations (11), (12) to these roots  $\tilde{X}_{1,2}$  we get the following.

**Theorem 4.** *Let an elliptic ellipse or hyperbola be centered at the origin of the projective model given by its semimajor axis  $a$  and its foci  $F_1 = (1 : f : 0)$ ,  $F_2 = (1 : -f : 0)$ ,  $(0 < f < 1)$  such that  $2a > d(F_1, F_2)$  or  $2a < d(F_1, F_2)$  holds. The  $\alpha$ -isoptic and  $(\pi - \alpha)$ -isoptic curves  $(0 < \alpha < \pi)$  of the considered ellipse or hyperbola in the elliptic plane have the equation*

$$\cos^2(\alpha) = \frac{((1 + f^2) \cos(2a) (x^2 + y^2 + 1) + f^2 x^2 - 1)^2}{2(1 + f^2) y^2 (f^2 + x^2) + (f^2 - x^2)^2 + (1 + f^2)^2 y^4}. \quad (20)$$

**Remark 5.**

1. The orthoptic curve of the elliptic ellipse and hyperbola is an ellipse with the following equation:

$$(1 + f^2) \cos(2a) (x^2 + y^2 + 1) + f^2 x^2 = 1.$$

2. The isoptic curve of the elliptic hyperbola exists if the following formula holds true

$$\left( \cos \alpha \leq \max \left( \frac{1 - (1 + f^2) \cos(2a)}{f^2}, f^2 + (1 + f^2) \cos(2a) \right) \right) \\ \wedge \left( a \geq \frac{\pi}{6} \vee \left( f \leq \sqrt{\frac{1}{\cos(2a)} - 1} \right) \vee (\alpha \notin I) \right),$$

where

$$I = \left( \arccos \left( \frac{(1 + f^2) \cos(2a) - 1}{f^2} \right), \arccos \left( \frac{1 - (1 + f^2) \cos(2a)}{f^2} \right) \right).$$

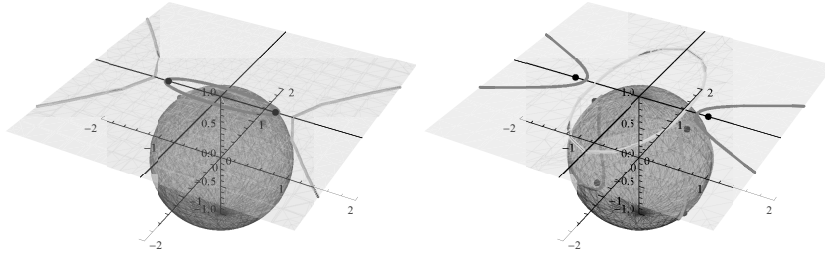


Figure 7: The isoptic curve to an elliptic ellipse (with  $a = 0.7$ ,  $f = 0.8$ ;  $\alpha = \pi/3$ ; left) and a hyperbola (with  $a = 0.7$ ,  $f = 1$ ;  $\alpha = \pi/2$ ; right)

#### 4.2.2. The equation and the isoptic curve of elliptic parabola

Similarly to the hyperbolic case, we define the *elliptic parabolas* and give their equations.

**Definition 6.** An elliptic parabola is the set of points  $(X = (1 : x : y) \in \mathcal{E}^2)$  in the elliptic plane that are equidistant to a proper point (the focus  $F$ ) and a proper line (the directrix  $e$ ) ( $s = d(X; F) = d(X; e)$ ).

Without loss of generality, we can assume that the directrix ( $e$ ) is the axis  $x$  and the coordinates of the focus point are  $F = (1 : 0 : p)$ . The distances  $d(X; F)$  and  $d(X; e)$  can be computed by the formula (3):

$$\cos(s) = \frac{1 + py}{\sqrt{1 + x^2 + y^2} \sqrt{1 + p^2}} = \frac{1 + x^2}{\sqrt{1 + x^2 + y^2} \sqrt{1 + x^2}}. \quad (21)$$

From (21) we obtain the equation of the elliptic parabola:

$$-x^2 + \frac{(1+py)^2}{1+p^2} = 1. \quad (22)$$

Using the same method, we can solve the following system of equations for  $(\tilde{x}, \tilde{y})$ :

$$\begin{aligned} \frac{\tilde{x}(1+p^2)}{p+\tilde{y}p^2}(x-\tilde{x}) + \tilde{y} &= y \\ -x^2 + \frac{(1+py)^2}{1+p^2} &= 1. \end{aligned} \quad (23)$$

Finally, we have to solve the system of equations (11), (12). Now, the following theorem holds

**Theorem 5.** *Let an elliptic parabola be given in the projective model by its focus  $F = (1 : 0 : p)$  and its directrix  $e$  which coincides with an  $x$ -axis. The  $\alpha$ -isoptic curve of this parabola ( $0 < \alpha < \pi$ ) in the elliptic plane has the equation*

$$\cos(\alpha) = \frac{y(py+1)}{\sqrt{(x^2+1)((p^2(x^2+1)-2py+y^2+x^2))}}. \quad (24)$$

**Remark 6.** *The orthoptic curve of the elliptic parabola contains two straight lines  $y = 0$  and  $y = -\frac{1}{p}$ .*

**Remark 7.** *The figures of the isoptic curves also confirm the fact that in elliptic geometry there is only one class of conic sections. However, in the affine model of the projective plane used, these can be considered separately.*

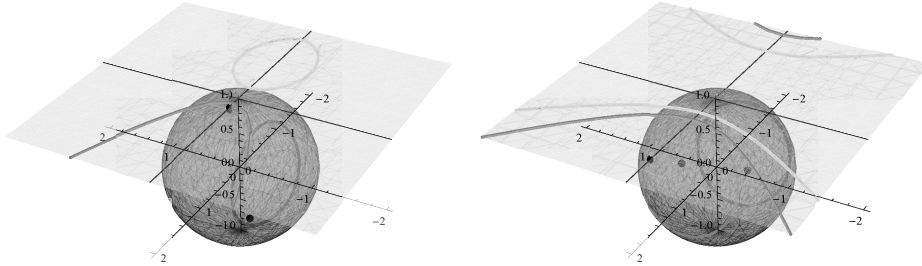


Figure 8: The isoptic curve to an elliptic parabola with parameters  $p = 0.25$ ;  $\alpha = \pi/3$  (left),  $p = 1.5$ ,  $\alpha = 2\pi/3$  (right)

In this paper, we consider the hyperbolic conic sections with proper foci, the problem is actual for the general types of conic section types in the hyperbolic plane (see [9]). Moreover, similar questions are interesting for other plane geometries, e.g. in the Minkowski plane.

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