Compactly generated rectifiable spaces or paratopological groups

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Abstract. A rectifiable space (or a paratopological group) $G$ is compactly generated if $G = \langle K \rangle$ for some compact subset $K$ of $G$. In this paper, we mainly discuss compactly generated rectifiable spaces or paratopological groups. The main results are that: (1) each $\sigma$-compact metrizable rectifiable space containing a dense compactly generated rectifiable subspace is compactly generated; (2) a metrizable rectifiable space is compactly generated if and only if it is $\sigma$-compact and finitely generated modulo open sets; (3) any $\sigma$-compact paratopological group can be embedded as a closed paratopological subgroup in some compactly generated paratopological group. Finally, we consider generalized metric properties of compactly generated rectifiable spaces.

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1. Introduction

Recall that a topological group $G$ is a group $G$ with a (Hausdorff) topology such that the product map from $G \times G$ onto $G$ is jointly continuous and the inverse map of $G$ onto itself associating $x^{-1}$ with arbitrary $x \in G$ is continuous. A paratopological group $G$ is a group $G$ with a topology such that the product maps of $G \times G$ into $G$ is jointly continuous. A topological space $G$ is said to be a rectifiable space [4] provided that there are a surjective homeomorphism $\varphi : G \times G \to G \times G$ and an element $e \in G$ such that $\pi_1 \circ \varphi = \pi_1$, and for every $x \in G$ we have $\varphi(x, x) = (x, e)$, where $\pi_1 : G \times G \to G$ is the projection to the first coordinate. If $G$ is a rectifiable space, then $\varphi$ is called a rectification on $G$. It is well known that rectifiable spaces and paratopological groups are all good generalizations of topological groups. It is easy to see that a topological group $G$ with the neutral element $e$ has a rectification $\varphi(x, y) = (x, x^{-1}y)$. However, there exists a paratopological group which is not a rectifiable space; Sorgenfrey line ([8, Example 1.2.2]) is such an example. Also, the 7-dimensional sphere $S_7$ is rectifiable but not a topological group [21, §3]. In fact, it

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is even not a semitopological group, because each (locally) compact semitopological group is a topological group [7]. Further, it is easy to see that both paratopological groups and rectifiable spaces are homogeneous.

Recently, the study of rectifiable spaces has become an interesting topic in topological algebra, see [1, 11, 13, 14, 15, 16, 20, 21].

2. Preliminaries

The following theorem was announced for the first time in [4], and the readers can see the proof in [5, 11, 20].

**Theorem 1** (see [4]). A topological space $G$ is rectifiable if and only if there exist an element $e \in G$ and two continuous maps $p : G^2 \to G$, $q : G^2 \to G$ such that for any $x \in G, y \in G$ the next identities hold:

$$p(x, q(x, y)) = q(x, p(x, y)) = y, q(x, x) = e.$$  

In fact, we can assume that $p = \pi_2 \circ \varphi^{-1}$ and $q = \pi_2 \circ \varphi$ in Theorem 1. If we fix a point $x \in G$, then $f_x, g_x : G \to G$ defined with $f_x(y) = p(x, y)$ and $g_x(y) = q(x, y)$, for each $y \in G$, are homeomorphisms. We denote $f_x, g_x$ by $p(x, G), q(x, G)$, respectively.

If $G$ is a rectifiable space, then we shall call the map $p$ the multiplication on $G$. Moreover, sometimes we shall write $x \cdot y$ instead of $p(x, y)$ and $A \cdot B$ instead of $p(A, B)$ for any $A, B \subset G$. Therefore, $q(x, y)$ is an element such that $x \cdot q(x, y) = y$; since $x \cdot e = x \cdot q(x, x) = x$ and $x \cdot q(x, e) = e$, it follows that $e$ is a right neutral element for $G$ and $q(x, e)$ is a right inverse for $x$. Hence a rectifiable space $G$ is a topological algebraic system with binary operations $p, q$, 0-ary operation $e$ and identities as above. It is easy to see that this algebraic system need not satisfy the associative law about the multiplication operation $p$. Clearly, every topological loop is rectifiable.

If $G$ is a rectifiable space (or a paratopological group) and $X \subset G$, then we use $\langle X \rangle$ to denote the smallest rectifiable subspace of $G$ which contains $X$. A set $X$ algebraically generates $G$ if $G = \langle X \rangle$.

Recall that a rectifiable space $G$ (a paratopological group) is:

1. $\sigma$-compact if $G = \bigcup \{K_n : n \in \mathbb{N}\}$, where each $K_n$ is compact, and
2. compactly generated if $G = \langle K \rangle$ for some compact subset $K$ of $G$.

**Note 1.** (a): Obviously, each compactly generated rectifiable space is $\sigma$-compact. However, there exists a compactly generated paratopological group which is not $\sigma$-compact. Indeed, let $X$ be an uncountable compact space, and let $AP(X)$ be a free Abelian paratopological group. Then $-X$ is closed discrete in $AP(X)$ [17], which implies that $AP(X)$ is not $\sigma$-compact. Moreover, $AP(X)$ is not a topological group.

(b): There exists a countable, metrizable, and compactly generated paratopological group which is not a topological group. Indeed, let the rational number $\mathbb{Q}$ with the subspace topology of Sorgenfrey line. Then $\mathbb{Q}$ is a countable, metrizable paratopological group which is not a topological group. Put $S = \{0\} \cup \{\frac{1}{n} : n \in \mathbb{N}\}$; then $\mathbb{Q} = \langle S \rangle$. Therefore, $\mathbb{Q}$ is compactly generated.
(c): Sorgenfrey line is not a compactly generated paratopological group since each compact subset of Sorgenfrey line is countable [2, 3.3.b].

All spaces considered in this paper are supposed to be $T_1$ and regular unless stated otherwise. The notation $\mathbb{N}$ denotes the set of all positive integer numbers. The letter $e$ denotes the neutral element of a group or the right neutral element of a rectifiable space. Readers may refer to [2, 8, 10] for notations and terminology not explicitly given here.

3. Compactly generated rectifiable spaces

In this section, we mainly discuss compactly generated rectifiable spaces. Firstly, we give some technical lemmas.

Lemma 1 (see [9]). Let $\{U_n : n \in \mathbb{N}\}$ be a local base at point $e$ of a topological space $G$ such that $U_{n+1} \subset U_n$ for all $n \in \mathbb{N}$. Assume that $\{F_n : n \in \mathbb{N}\}$ is a sequence of subsets of $G$ such that

1. each $F_n$ is compact, and
2. $F_n \subset \overline{U_n}$.

Then $K = \bigcup\{F_n : n \in \mathbb{N}\} \cup \{e\}$ is compact. Moreover, if each $F_n$ is finite, then for each enumeration $i : \mathbb{N} \to K$ a sequence $\{i(n) : n \in \mathbb{N}\}$ converges to $e$.

Let $A$ be a subspace of a rectifiable space $G$. Then $A$ is called a rectifiable subspace [14] of $G$ if we have $p(A, A) \subset A$ and $q(A, A) \subset A$.

Lemma 2 (see [14]). Let $G$ be a rectifiable space. If $V$ is an open rectifiable subspace of $G$, then $V$ is closed in $G$.

Lemma 3. Let $H$ be a dense rectifiable subspace of a rectifiable space $G$. Then for each open rectifiable subspace $E$ of $H$ there exists an open rectifiable subspace $E'$ of $G$ such that $E' \cap H = E$.

Proof. Let

$$E' = \bigcup\{V : V \text{ is open in } G \text{ and } \operatorname{cl}_G(V) \cap H \subset E\}.$$ 

Obviously, $E'$ is an open subset of $G$ and $E' \cap H = E$. Now, we shall prove that $E'$ is a rectifiable subspace of $G$.

Indeed, suppose that $a, b \in E'$. It follows from the definition of $E'$ that there exist open sets $U$ and $V$ in $G$ such that $a \in U, b \in V, \operatorname{cl}_G(U) \cap H \subset E$ and $\operatorname{cl}_G(V) \cap H \subset E$. By the density of $H$ in $G$, we have $\operatorname{cl}_G(U \cap H) = \operatorname{cl}_G(U)$ and $\operatorname{cl}_G(V \cap H) = \operatorname{cl}_G(V)$. Therefore, it follows from the continuity of $p$ in $G$ that

$$p(U, V) \subset p(\operatorname{cl}_G(U), \operatorname{cl}_G(V)) = p(\operatorname{cl}_G(U \cap H), \operatorname{cl}_G(V \cap H)) \subset \operatorname{cl}_G(p(U \cap H, V \cap H)).$$

Then we have $\operatorname{cl}_G(p(U, V)) = \operatorname{cl}_G(p(U \cap H, V \cap H))$, and

$$\operatorname{cl}_G(p(U, V)) \cap H = \operatorname{cl}_G(p(U \cap H, V \cap H)) \cap H \subset \operatorname{cl}_G(p(E, E)) \cap H = \operatorname{cl}_G(E) \cap H = E.$$
A dense rectifiable subspace of a connected rectifiable space has no proper open rectifiable subspaces.

Let $c, d \in E'$. Then there exist open sets $O, W$ in $G$ such that $c \in O, d \in W$, $c \in G(O) \cap H \subset E$ and $cl_G(W) \cap H \subset E$. Obviously, $q(O, W)$ is open in $G$. Moreover, it is also easy to see that $cl_G(q(O, W)) = cl_G(q(O \cap H, W \cap H))$. Since $cl_G(q(O, W)) \cap H = cl_G(q(O \cap H, W \cap H)) \cap H \subset cl_G(q(E, E)) \cap H \subset cl_G(E) \cap H = E$, it follows that $q(c, d) \in q(O, W) \subset E'$.

\textbf{Corollary 1.} A dense rectifiable subspace of a connected rectifiable space has no proper open rectifiable subspaces.

\textbf{Proof.} By Lemma 2, each open rectifiable subspace of a rectifiable space is closed, and therefore, a connected rectifiable space cannot have proper open rectifiable subspaces. Now the result follows from Lemma 3. \hfill \Box

\textbf{Lemma 4 (see [14]).} Let $G$ be a rectifiable space. If $Y$ is a dense subset of $G$ and $U$ is an open neighborhood of the right neutral element $e$ of $G$, then $G = Y \cdot U$.

\textbf{Theorem 2.} If a $\sigma$-compact metrizable rectifiable space $G$ contains a dense compactly generated rectifiable subspace, then $G$ is also compactly generated.

\textbf{Proof.} Let $H$ be a dense rectifiable subspace of $G$ such that $H$ is generated by some compact set $E$, and let $G = \bigcup \{K_n : n \in \mathbb{N}\}$, where each $K_n$ is compact. Since $G$ is metrizable, the point $e$ has a countable local base $\{U_n : n \in \mathbb{N}\}$, where $U_{n+1} \subset U_n$ for each $n \in \mathbb{N}$. By the density of $H$ in $G$, it follows from Lemma 4 that $p(H, U_n) = G$ for each $n \in \mathbb{N}$. For each $n \in \mathbb{N}$, there exists a finite subset $F_n$ of $H$ such that $K_n \subset p(F_n, U_n)$, and put $L_n = U_n \cap q(F_n, K_n)$, then each $K_n \subset p(F_n, L_n)$ since $K_n \subset p(F_n, q(F_n, K_n))$. Obviously, each $L_n$ is compact and, by Lemma 1, $L = \bigcup \{L_n : n \in \mathbb{N}\}$ is also compact. Therefore,

$$G = \bigcup \{K_n : n \in \mathbb{N}\} \subset \bigcup \{p(F_n, L_n) : n \in \mathbb{N}\} \subset \bigcup \{p(H, L_n) : n \in \mathbb{N}\} \subset p(H, L).$$

Since $H$ is generated by $E$, $G$ is generated by the compact set $E \cup L$. Therefore, $G$ is compactly generated. \hfill \Box

\textbf{Corollary 2.} If a $\sigma$-compact metrizable rectifiable space $G$ contains a dense finitely generated rectifiable subspace, then $G$ is also compactly generated.

Next, we define the notion of finitely generated modulo open sets in rectifiable spaces which contains all compactly generated rectifiable spaces.

\textbf{Definition 1.} We will say that a rectifiable space (or a paratopological group) $G$ is finitely generated modulo open sets if for each non-empty open rectifiable subspace $H$ of $G$ there exists a finite subset $F$ of $G$ such that $G = (F \cup H)$. 

\textbf{Proposition 1.} Let $G$ be a rectifiable space. Then the following conditions are equivalent:

1. $G$ is finitely generated modulo open sets;
2. for each non-empty open subset $V$ of $G$ there exists a finite subset $F$ of $G$ such that $G = \langle F \cup V \rangle$.

**Proof.** Obviously, (2) $\Rightarrow$ (1).

(1) $\Rightarrow$ (2). Let $V$ be a non-empty open subset $V$ of $G$. Let $H$ be the rectifiable subspace generated by $V$, that is, $H = \langle V \rangle$. Obviously, $H$ is open in $G$, and so by (2) there is a finite set $F \subset G$ such that $G = \langle F \cup H \rangle = \langle F \cup \langle V \rangle \rangle = \langle F \cup V \rangle$.

Theorem 3. If a rectifiable space $G$ is compactly generated, then it is finitely generated modulo open sets.

**Proof.** Assume that $G = \langle K \rangle$, where $K$ is a compact set. Let $H$ be an open rectifiable subspace of $G$. Then $\mathcal{H} = \{g \cdot H : g \in G\}$ is an open covering of $G$. Since $K$ is compact, there exist finitely many elements of $\mathcal{H}$, say $g_1 \cdot H, \ldots, g_n \cdot H$, which cover $K$. Put $F = \{g_1, \ldots, g_n\}$. Then $G = \langle F \cup H \rangle$.

Theorem 4. Let $G$ be a metrizable rectifiable space $G$ and $A$ a countable subset of $G$. Suppose that $G$ is finitely generated modulo open sets. Then $G$ contains a sequence $S$ converging to $e$ of $G$ such that $A \subset \langle S \rangle$.

**Proof.** Let $A = \{a_n : n \in \omega\}$. Since $G$ is metrizable, let $\{U_n : n \in \omega\}$ be a local base at $e$ such that $G = U_0 \supseteq U_1 \supseteq \cdots \supseteq U_n \supseteq \cdots$.

Since $G$ is finitely generated modulo open sets, for each $n \in \omega$ we can fix a finite set $F_n$ such that $G = \langle F_n \cup U_{n+1} \rangle$.

By induction on $n$, we will define a sequence $\{B_n : n \in \omega\}$ of finite subsets of $G$ with the following properties:

(a) $B_n \subset U_n$;
(b) $G = \langle B_0 \cup B_1 \cup \cdots \cup B_n \cup U_{n+1} \rangle$,
(c) $a_n \in \langle B_0 \cup B_1 \cup \cdots \cup B_n \rangle$.

To begin with, let $B_0 = F_0 \cup \{a_0\}$; then $B_0$ satisfies all three conditions (a)-(c). Assume that we have already defined finite sets $B_0, B_1, \ldots, B_{n-1}$ satisfying all three conditions (a)-(c). By (b), $F_n \cup \{a_n\} \subset \langle B_0 \cup B_1 \cup \cdots \cup B_{n-1} \cup U_n \rangle$. Since $F_n$ is finite, we can find a finite set $B_n \subset H_n$ such that $F_n \cup \{a_n\} \subset \langle B_0 \cup B_1 \cup \cdots \cup B_{n-1} \cup B_n \rangle$.

Clearly, (a)-(c) are satisfied.

Put $S = \cup \{B_n : n \in \omega\}$. By (c), $A \subset \langle S \rangle$. By Lemma 1 and (a), $S$ can be enumerated as a sequence converging to $e$.

Theorem 5. Let $G$ be a $\sigma$-compact metrizable rectifiable space $G$. Then $G$ is compactly generated if and only if $G$ is finitely generated modulo open sets.
Proof. By Theorem 3, we only need to prove the sufficiency. Suppose that for each open rectifiable subspace $H$ of $G$ there exists a finite set $F$ such that $G = \langle F \cup H \rangle$. Obviously, $G$ is separable, and let $D$ be a countable dense subset of $G$. By Theorem 4, $G$ has a dense compactly generated rectifiable subspace, and by Theorem 2, $G$ is compactly generated.

Corollary 3. A metrizable rectifiable space $G$ is compactly generated if and only if $G$ is $\sigma$-compact and finitely generated modulo open sets.

A rectifiable space without proper open rectifiable subspaces trivially satisfies condition (2) of Proposition 1. Therefore, we have the following corollary.

Corollary 4. A $\sigma$-compact metrizable rectifiable space $G$ without proper open rectifiable subspaces is compactly generated.

By Corollaries 1 and 4, we also have the following corollary.

Corollary 5. A $\sigma$-compact dense rectifiable subspace of a connected metrizable rectifiable space $G$ is compactly generated.

Corollary 6. A $\sigma$-compact connected metrizable rectifiable space $G$ is compactly generated.

By Theorems 3 and 4, it is easy to prove the following theorem.

Theorem 6. A countable metrizable rectifiable space is compactly generated if and only if it is compactly generated by a sequence converging to the right neutral element $e$.

4. Compactly generated paratopological groups

In this section, we mainly discuss compactly generated paratopological groups.

Lemma 5. Let $G$ be a paratopological group. If $Y$ is a dense subset of $G$ and $U$ is an open neighborhood of the neutral element $e$ of $G$, then $G = Y^{-1} \cdot U$.

Proof. For arbitrary $g \in G$, since $Y$ is a dense subset of $G$, we have $Ug^{-1} \cap Y \neq \emptyset$. Take $x \in Ug^{-1} \cap Y$. Then $g \in x^{-1}U \subset Y^{-1} \cdot U$.

The proof of the following theorem is similar to that of Theorem 2.

Theorem 7. If a $\sigma$-compact first-countable paratopological group $G$ contains a dense compactly generated subgroup, then $G$ is also compactly generated.

Proof. Let $H$ be a dense subgroup of $G$ such that $H$ is generated by some compact set $E$, and let $G = \bigcup \{K_n : n \in \mathbb{N}\}$, where each $K_n$ is compact. Since $G$ is first-countable, the point $e$ has a countable local base $\{U_n : n \in \mathbb{N}\}$, where $U_{n+1} \subset U_n$ for each $n \in \mathbb{N}$. By the density of $H$ in $G$, it follows from Lemma 5 that $HU_n = G$ for each $n \in \mathbb{N}$. For each $n \in \mathbb{N}$, there exists a finite subset $F_n$ of $H$ such that $K_n \subset F_nU_n$, and put $L_n = \overline{U_n} \cap (F_n)^{-1}K_n$, then each $K_n \subset F_nL_n$, since $K_n \subset$
Under the class of paratopological groups, we can obtain all results from
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Theorem 9.
Clearly, Z is a compact subspace of G
K let i : G → G be a topological isomorphism, and we can identify Gn with its image in σΠ under the natural embedding. Suppose that
G = \bigcup \{K_n : n ∈ \mathbb{N}\}
= \bigcup \{F_n L_n : n ∈ \mathbb{N}\} \subset \bigcup \{HL_n : n ∈ \mathbb{N}\} \subset HL.
Since H is generated by E, G is generated by the compact set E ∪ L. Therefore, G is compactly generated.

Note 2. Under the class of paratopological groups, we can obtain all results from Proposition 1 to Theorem 5 and Corollary 4 to Theorem 6 in Section 3 by similar proofs. In fact, the respective counterparts also hold for first-countable paratopological groups and this condition is weaker than the metrizability.

Since a compactly generated rectifiable space G is σ-compact, G has Souslin property, see [18] or [19]. Moreover, E.A. Reznichenko showed that every σ-compact Hausdorff paratopological group has Souslin property, see [2, Theorem 5.7.12]. However, the following question is still open.

Question 1. Let G be a compactly generated paratopological group. Does G have Souslin property?

Theorem 8. Any σ-compact paratopological group G can be embedded as a closed paratopological subgroup in some compactly generated paratopological group.

Proof. Let σΠ = σΠ\{G_n : n ∈ \mathbb{Z}\} be the σ-product of copies of G with the topology induced from Tikhonov power \( G^2 \), where σΠ is a σ-product with the neutral element e as a distinguished point. Then σΠ is also a paratopological group. For each n ∈ \mathbb{Z}, let \( i_n : G \to G_n \) be a topological isomorphism, and we can identify \( G_n \) with its image in \( \sigma\Pi \) under the natural embedding. Suppose that \( G = \bigcup \{K_n : n ∈ \mathbb{Z}\} \), where each \( K_n \) is compact. Let K denote the subspace \( \bigcup_{n∈\mathbb{Z}} i_n(K_n) \) of the paratopological group \( σ\Pi \). Since \( K \) is closed in the compact subspace \( \Pi\{K_n : n ∈ \mathbb{Z}\} \) of the paratopological group \( G^2 \) under the natural embedding \( \sigma\Pi \to G^2 \). \( K \) is compact in \( σ\Pi \).

The group \( \mathbb{Z} \) of integers with the discrete topology acts on the paratopological group \( σ\Pi \) by shifting coordinates: for \( x = (x_n)_{n∈\mathbb{Z}} ∈ σ\Pi \) and \( k ∈ \mathbb{Z}, k \cdot x \) is the element of \( σ\Pi \) whose n-th coordinate is \( x_{n+k} \). Let \( G' \) denote the semidirect product \( σ\Pi \rtimes \mathbb{Z} \). Assume 1 is the smallest positive element of \( \mathbb{Z} \). Then the space \( K \cup \{1\} \) is a compact subspace of \( G' \) and \( G' = (K \cup \{1\}) \) in \( G' \).

Clearly, \( Z ⊆ H \). Next, we shall prove that, for each \( m ∈ \mathbb{Z}, G_m ⊆ H \). Take arbitrary \( x ∈ G_m \). Then \( i_m^{-1}(x) ∈ K_n \) for some \( n ∈ \mathbb{Z} \). Let \( a \) be the element \( (i_n i_m^{-1}(x), 0) \) and \( b \) the element \( (e, m − n) \) of the semidirect product \( G' \). Clearly, \( a ∈ K ⊆ H \) and \( b ∈ H \), and hence \( ba \) belongs to \( H \). However, it is easy to see that \( ba = x \).

If \( G \) be countable, then each of the sets \( K_n \) can be assumed finite. A simple analysis of the topological structure of the space \( K \cup \{1\} \) enables us to obtain

Theorem 9. Any countable paratopological group G can be embedded as a closed paratopological subgroup in some paratopological group algebraically generated by a subspace homeomorphic to the one-point compactification \( \partial\mathbb{N} \) of a countable discrete space.

Question 2. Can any σ-compact rectifiable space G be embedded as a closed rectifiable subspace in some compactly generated rectifiable space?
5. Generalized metrizability properties of compactly generated rectifiable spaces

A closed mapping $f$ is called perfect if each fiber is compact.

**Proposition 2.** Suppose that $F$ is a compact subspace of a rectifiable space $G$. Then the restriction $p$ and $q$ to the subspace $F \times G$ is a perfect and open mapping of $F \times G$ onto $G$.

**Proof.** We firstly prove that the restriction $p$ to the subspace $F \times G$ is a perfect and open mapping of $F \times G$ onto $G$.

Let $f : F \times G \to F \times G$ be defined by $f(x, y) = (x, p(x, y))$ for each $(x, y) \in F \times G$. Obviously, $f$ is continuous, one-to-one, and $f(F \times G) = F \times G$. Moreover, $f^{-1}(x, y) = (x, q(x, y))$. Therefore, $f^{-1}$ is also continuous. Thus $f$ is a homeomorphism. For $i = 1, 2$, denote by $\pi_i$ the projection of $F \times G$ onto the $i$-th factor. Since $p(x, y) = \pi_2(x, p(x, y)) = \pi_2 f(x, y)$ for all $x \in F$ and $y \in G$, $p$ is the composition of $f$ and $\pi_2$, that is, $p = \pi_2 \circ f$. Since $F$ is compact, it follows from [8, Theorem 3.1.16] that $\pi_2$ is closed. Then $p$ is closed since $f$ is a homeomorphism and $\pi_2$ is closed.

For each $y \in G$, $p^{-1}(y) = f^{-1}(F \times \{y\}) = \bigcup \{(x, q(x, y)) : x \in F\}$ is closed in the compact subspace $F \times q(F, y)$. Indeed, let $(x, q(z, y)) \in (F \times q(F, y)) \setminus p^{-1}(y)$, where $x, z \in F$. Then $q(x, y) \neq q(z, y)$, and thus there exist two open sets $U$ and $V$ in $G$ such that $q(x, y) \in U$, $q(z, y) \in V$ and $U \cap V = \emptyset$. Since $q$ is continuous, there exists an open neighborhood $W$ of $e$ such that $q(x \cdot W, y \cdot W) \subset U$ and $q(z \cdot W, y \cdot W) \subset V$. Then $(x \cdot W, V)$ is an open neighborhood of $(x, q(z, y))$. However, since $q(x \cdot w, y) \subset U$ for each $w \in W$, it follows that $(x \cdot W, V) \cap p^{-1}(y) = \emptyset$. Therefore, $p^{-1}(y)$ is closed in $F \times q(F, y)$, and thus it is compact. Then $p$ is perfect.

Let $O$ be an open subset of $F \times G$. Put $O' = \pi_1(O)$. For each $x \in O'$, let $U_x = \{y \in G : (x, y) \in O\}$; then $O_x$ is open in $G$ as the projection of the open subset $O \cap \pi_1^{-1}(x)$ of $\{x\} \times G$ onto the second factor. Therefore, $p(O) = \bigcup_{x \in O'} p(x, O_x)$ is open in $G$, which implies that $p$ is an open mapping.

As for the mapping $q$, we only redefine the mapping $f$ by $(x, y) = (x, q(x, y))$ for each $(x, y) \in F \times G$, and the rest of the proof is immediate.  

**Corollary 7.** Suppose that $F$ is a compact subspace of a rectifiable space $G$, and that $M$ is a closed subspace of $G$. Then $p(F, M)$ and $q(F, M)$ are all closed in $G$.

**Note 3.** Corollary 7 gives an affirmative answer to the following question. Recently, L.X. Peng and S.J. Guo [16] have also obtained Corollary 7. However, we prove Corollary 7 by a different method.

**Question 3 (see [15]).** Let $G$ be a rectifiable. If $F, P$ are compact and closed subsets of $G$, respectively, is $P \cdot F$ or $F \cdot P$ closed in $G$?

Since the restriction of a perfect mapping to a closed subspace is again a perfect mapping, it follows from Corollary 7 and Proposition 2 that we have the following corollary.

**Corollary 8.** Suppose that $F$ is a compact subspace of a rectifiable space $G$, and that $M$ is a closed subspace of $G$. Then the restriction $p$ and $q$ to the subspace $F \times M$ is a perfect mapping of $F \times M$ onto a closed subspace of $G$. 


A space $G$ is of countable tightness if for each subset $A$ of $G$ and each point $x \in \text{cl}(A)$ there exists a countable subset $D$ of $A$ such that $x \in \text{cl}(D)$.

**Theorem 10.** Suppose that $F$ is a compact subspace of a rectifiable space $G$ and that $M$ is a closed subspace of $G$. Suppose also that both $F$ and $M$ have countable tightness. Then both spaces $p(F, M)$ and $q(F, M)$ have countable tightness, too.

**Proof.** Since perfect mappings do not increase the tightness and the tightness of the product $F \times M$ is countable by [8, 3.12.8(a)], it follows from Corollary 8 that both spaces $p(F, M)$ and $q(F, M)$ have countable tightness, too.

**Theorem 11.** Suppose that $F$ is a compact metrizable subspace of a rectifiable space $G$, and that $M$ is a closed metrizable subspace of $G$. Then both spaces $p(F, M)$ and $q(F, M)$ are metrizable, too.

**Proof.** By Corollary 7, $p(F, M)$ and $q(F, M)$ are closed in $G$. Since perfect mappings preserve the metrizability [8, Theorem 4.4.15], it follows from Corollary 8 that $p(F, M)$ and $q(F, M)$ are metrizable.

A network for a space $X$ is a collection $\mathcal{F}$ of subsets of $X$ such that whenever $x \in U$ with $U$ open, there exists $F \in \mathcal{F}$ with $x \in F \subset U$.

**Theorem 12.** Let $G$ be a rectifiable space, and let $H$ be a rectifiable subspace of $G$ compactly generated by a compact metrizable space $F$. Suppose further that $G = p(H, M)$, where $M$ is a closed metrizable subspace of $G$. Then $G$ is the union of a countable family of closed metrizable subspaces.

**Proof.** By induction on $n$, we can define a sequence $\{A_n : n \in \omega\}$ of subsets of $G$ such that:

1. $A_0 = F \cup p(F, F) \cup q(F, F)$;
2. $A_1 = p(A_0, A_0) \cup q(A_0, A_0)$;
3. $A_n = p(A_{n-1}, A_{n-1}) \cup q(A_{n-1}, A_{n-1})$.

Obviously, each $p(A_n, A_n), q(A_n, A_n), A_n$ are compact. Since compact space with a countable network is metrizable [10], it follows from Theorem 11 that each $A_n$ is also metrizable. Since $H = \langle F \rangle$, $H = \bigcup_{n \in \omega} A_n$. Since $G = p(H, M)$, it follows from Theorem 11 again that $G$ is the union of a countable family of closed metrizable subspaces.

A neighborhood assignment for a space $X$ is a function $\varphi$ from $X$ to the topology of $X$ such that $x \in \varphi(x)$ for each point $x \in X$. A space $X$ is a $D$-space[6], if for any neighborhood assignment $\varphi$ for $X$ there is a closed discrete subset $D$ of $X$ such that $X = \bigcup_{d \in D} \varphi(d)$.

**Corollary 9.** Let $G$ be a rectifiable space, and let $H$ be a rectifiable subspace of $G$ compactly generated by a compact metrizable space $F$. Suppose further that $G = p(H, M)$, where $M$ is a closed metrizable subspace of $G$. Then $G$ is a $D$-space.
Proof. It is well known that each metrizable space is a $D$-space. Hence $M$ is a $D$-space, and then each $p(h, M)$ is a $D$-space, too. Since a countable infinite union of closed $D$-subspaces is $D$ [3], it follows that $G = p(H, M) \cup_{h \in H} p(h, M)$ is a $D$-space.

Recall that a space $X$ has a quasi-$G_\delta$-diagonal provided there is a sequence $\{G(n) : n \in \mathbb{N}\}$ of collections of open subsets of $X$ such that for any distinct points $x, y \in X$ there is a number $n$ with $x \in \text{st}(x, G(n)) \subset X \setminus \{y\}$.

**Theorem 13.** Let $G$ be a compactly generated Tychonoff rectifiable space, and $Y = bG \setminus G$ have locally quasi-$G_\delta$-diagonal, where $bG$ is a Hausdorff compactification of $G$. Then $G$ satisfies one of the following conditions:

1. $G$ is locally compact;
2. $G$ is separable and metrizable.

**Proof.** Suppose that $G$ is nowhere locally compact. Since $G$ is $\sigma$-compact, it follows from [14, Theorem 7.3] that $G$ is separable and metrizable.

A space $X$ is said to have a regular $G_\delta$-diagonal if the diagonal $\Delta = \{(x, x) : x \in X\}$ can be represented as the intersection of the closures of a countable family of open neighborhoods of $\Delta$ in $X \times X$.

Since a rectifiable space with a countable pseudocharacter has a regular $G_\delta$-diagonal [14] and a paracompact space with a $G_\delta$-diagonal is submetrizable [10], we have the following proposition.

**Proposition 3.** If $G$ is a compactly generated rectifiable space with a countable pseudocharacter, then $G$ is submetrizable.

**Proposition 4.** Let $G$ be a compactly generated Tychonoff rectifiable space with a countable pseudocharacter, and let $Y = bG \setminus G$ be Lindelöf, where $bG$ is a Hausdorff compactification of $G$. Then $G$ is separable and metrizable.

**Proof.** Since $Y = bG \setminus G$ is Lindelöf, $G$ is countable type [12], and thus $G$ is a $p$-space [1]. Then $G$ is a $\sigma$-compact $p$-space with a $G_\delta$-diagonal, hence it is separable and metrizable [10, Corollaries 3.8 and 3.20].

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**References**