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# On an extension of the Stone-Weierstrass theorem*

Kuppum V. Srikanth[1,†] and Raj Bhawan Yadav[1]

[1] *Department of Mathematics, Indian Institute of Technology Guwahati, Guwahati 781 039, Assam, India*

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**Abstract.** The classical Stone-Weierstrass Theorem has been generalized and extended in different directions. Theorem 1 of [2] (D. Hill, E. Passow, L. Raymon, *Approximation with interpolatory constraints*, Illinois J. Math. **20**(1976), 65–71) may be viewed as one such extension involving finitely many interpolatory constraints. This article generalizes the latter theorem to the case where the constraints are on an arbitrary closed subset of the compact metric space under consideration. We also present an alternative proof of the cited Theorem.

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## 1. Introduction

Let $X$ be a compact metric space. $\mathbb{F}$ denotes either the field of real numbers $\mathbb{R}$ or the field of complex numbers $\mathbb{C}$. $C(X, \mathbb{F})$ denotes the collection of $\mathbb{F}$-valued continuous functions on $X$ with the sup-norm. Let $\mathcal{A}$ denote a subset of $C(X, \mathbb{F})$.

Let $k$ be any natural number. Further, let

$$S = \{x_1, x_2, \dots, x_k\} \subset X \quad (\text{with } x_i \neq x_j \text{ for distinct } i \text{ and } j)$$

and let

$$V = \{v_1, v_2, \dots, v_k\} \subset \mathbb{F}.$$

Define

$$C_S^V(X, \mathbb{F}) = \{f \in C(X, \mathbb{F}) | f(x_1) = v_1, f(x_2) = v_2, \dots, f(x_k) = v_k\}$$

and

$$\mathcal{A}_S^V(X, \mathbb{F}) = \{f \in \mathcal{A} | f(x_1) = v_1, f(x_2) = v_2, \dots, f(x_k) = v_k\}.$$

When $\mathcal{A}$ is a separating unital self-adjoint sub-algebra of $C(X, \mathbb{C})$, the Stone-Weierstrass theorem ([4]) asserts that $\mathcal{A}$ is dense in $C(X, \mathbb{C})$. We recommend [3] as a survey on density results like the Weierstrass Approximation Theorem, the Stone-Weierstrass Theorem and other related results. Theorem 1 of [2] asserts that $\mathcal{A}_S^V(X,\mathbb{R})$

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†Corresponding author. *Email addresses:* `kvsrikanth@iitg.ernet.in` (K. V. Srikanth), `r.yadav@iitg.ernet.in` (R. B. Yadav)

 

is dense in $C_S^V(X, \mathbb{R})$ when $\mathcal{A}$ is a unital sub-algebra of $C(X, \mathbb{R})$ which separates points of $X$. An independent proof of this theorem is provided in Section 2 of this article. A natural generalization of this theorem would be to allow for an arbitrary closed $S \subset X$, as our interpolating set. To proceed, we examine the notion of a separating unital algebra in Section 3. A version of the Stone-Weierstrass Theorem with an arbitrary closed subset of the metric space as the interpolating set is stated and proved in Section 4 as Theorem 3. Theorem 4 is the complex version of the latter theorem.

## 2. Finitely many interpolatory constraints

**Theorem 1.** *For a natural number $k$, let $S = \{x_1, x_2, \cdots, x_k\} \subset X$ (with $x_i \neq x_j$ for distinct $i$ and $j$) and let $V = \{v_1, v_2, \cdots, v_k\} \subset \mathbb{R}$. If $\mathcal{A}$ is a unital sub-algebra of $C(X, \mathbb{R})$ which separates points of $X$, then $\mathcal{A}_S^V(X, \mathbb{R})$ is dense in $C_S^V(X, \mathbb{R})$.*

**Proof**. Case (a): Assume that the $\{v_i\}_1^k$ are all distinct, that is, $v_i \neq v_j$ for every $i \neq j$, where $i, j \in \{1, \cdots, k\}$.

Let $f \in C_S^V(X, \mathbb{R})$. By the Stone-Weierstrass theorem, there is a sequence $\{p_n\}$ in $\mathcal{A}$ such that $p_n \to f$ uniformly. Define a new sequence of functions $f_n : X \to \mathbb{R}$ via

$$f_n(x) = \sum_{1 \leq i \leq k} v_i \, \frac{\prod_{1 \leq j \leq k, j \neq i} \big(p_n(x) - p_n(x_j)\big)}{\prod_{1 \leq j \leq k, j \neq i} \big(p_n(x_i) - p_n(x_j)\big)}.$$

Dropping finitely many functions $f_n$, if necessary, ensures well-definedness of the sequence $\{f_n\}$. Further,

$$f_n \to \sum_{1 \leq i \leq k} v_i \, \frac{\prod_{1 \leq j \leq k, j \neq i} \big(f(x) - f(x_j)\big)}{\prod_{1 \leq j \leq k, j \neq i} \big(f(x_i) - f(x_j)\big)} \text{ uniformly.} \tag{1}$$

However, we note that

$$f(x) - \sum_{1 \leq i \leq k} v_i \, \frac{\prod_{1 \leq j \leq k, j \neq i} \big(f(x) - f(x_j)\big)}{\prod_{1 \leq j \leq k, j \neq i} \big(f(x_i) - f(x_j)\big)}$$

is a polynomial in $f(x)$ of degree $k-1$ which has at least $k$ roots viz., $v_1, v_2, \ldots, v_k$. Hence for each $x \in X$,

$$f(x) = \sum_{1 \leq i \leq k} v_i \, \frac{\prod_{1 \leq j \leq k, j \neq i} \big(f(x) - f(x_j)\big)}{\prod_{1 \leq j \leq k, j \neq i} \big(f(x_i) - f(x_j)\big)}. \tag{2}$$

From 1 and 2 we conclude that $\{f_n\}$ converges uniformly to the given $f$. Clearly, for each $n$, $f_n(x_i) = v_i$ for every $i \in \{1, 2, \ldots, k\}$ and $f_n \in \mathcal{A}_S^V(X, \mathbb{R})$.

Case (b): When two or more of $v_i$ become equal, we assume without loss of generality that the prescribed values $\{v_i\}_1^k$ are ordered as $v_1 \leq v_2 \leq \cdots \leq v_k$. For some positive real number $\alpha$, let $\{w_j\}_1^k$ be the strictly increasing sequence of real numbers given by $w_j = v_j + j\alpha$ for each $j \in \{1, 2, \ldots, k\}$. By Urysohn's Lemma, there exists an $h \in C(X, \mathbb{R})$ such that $h(x_i) = w_i$ for every $i \in \{1, 2, \ldots, k\}$. Now, define $f^{\pm} = \frac{1}{2} f \pm h$. Clearly, $f^{\pm}(x_i) \neq f^{\pm}(x_j)$ for every $i, j \in \{1, 2, \ldots, k\}$ with $i \neq j$.

Applying our conclusion in Case (a) to $f^{\pm}$, we get two sequences $f_n^{\pm} : X \to \mathbb{R}$ such that

$$f_n^{\pm} \to f^{\pm} \text{ uniformly on } X$$

satisfying

$$f_n^{\pm}(x_i) = f^{\pm}(x_i) \text{ for each } i \in \{1, 2, \ldots, k\}.$$

Let $f_n = f_n^{+} + f_n^{-}$. Clearly $f_n \to f$ uniformly on $X$ and $f_n \in \mathcal{A}_S^V(X, \mathbb{R})$. $\square$

**Theorem 2.** *For a natural number $k$, let $S = \{x_1, x_2, \ldots, x_k\} \subset X$ (with $x_i \neq x_j$ for distinct $i$ and $j$) and let $V = \{a_1 + ib_1, a_2 + ib_2, \ldots, a_k + ib_k\} \subset \mathbb{C}$. Let $\mathcal{A}$ be a unital sub-algebra of $C(X, \mathbb{C})$ such that if $f \in \mathcal{A}$, then $\bar{f} \in \mathcal{A}$. Further, assume that $\mathcal{A}$ separates points of $X$. Then $\mathcal{A}_S^V(X, \mathbb{C})$ is dense in $C_S^V(X, \mathbb{C})$.*

**Proof**. Let $\mathcal{A}_{\mathbb{R}} = \{f \in \mathcal{A} \mid f(x) \in \mathbb{R} \text{ for all } x \in X\}$. Clearly, $\mathcal{A}_{\mathbb{R}}$ is a unital sub-algebra of $\mathcal{A}$ over $\mathbb{R}$. Further, if $f \in \mathcal{A}$ separates points $x, y \in X$, we note that either the real part of $f$ satisfies $\Re(f)(x) \neq \Re(f)(y)$ or the imaginary part of $f$ satisfies $\Im(f)(x) \neq \Im(f)(y)$. Since both $\Re(f), \Im(f) \in \mathcal{A}_{\mathbb{R}}$, we conclude that $\mathcal{A}_{\mathbb{R}}$ separates points.

Applying Theorem 1, we get two sequences $\{g_n\}_1^{\infty}, \{h_n\}_1^{\infty}$ in $\mathcal{A}_{\mathbb{R}}$ such that $g_n \to \Re(f)$ uniformly and $h_n \to \Im(f)$ uniformly with $g_n(x_j) = a_j$ and $h_n(x_j) = b_j$ for $j \in \{1, 2, \ldots, k\}$. Let

$$f_n = g_n + ih_n$$

and we have

$$f_n \to f \text{ uniformly on } X \text{ with } f_n \in \mathcal{A}_S^V(X, \mathbb{C}).$$

$\square$

In [1], one can find an alternative proof of the latter theorem. However, these methods are inadequate to handle the general case of constraints on a closed subset of $X$.

## 3. Generalizing the notion of a separating algebra

The following definitions are immediate when we attempt to generalize the notion of a separating subset of $C(X, \mathbb{R})$ or $C(X, \mathbb{C})$.

**Definition 1.** *Let $k \geq 2$ be a fixed natural number. For $\mathcal{A} \subset C(X, \mathbb{F})$, $\mathcal{A}$ is $k$-separating, if, given any $k$ distinct $x_1, x_2, \ldots, x_k \in X$, there exists an $f \in \mathcal{A}$ such that $f(x_i) \neq f(x_j)$ for every $i, j \in \{1, 2, \ldots, k\}$ with $i \neq j$. Further, $\mathcal{A}$ is $k$-interpolating if given any $k$ distinct $x_1, x_2, \ldots, x_k \in X$, and arbitrary $v_1, v_2, \ldots, v_k \in \mathbb{F}$, there exists an $f \in \mathcal{A}$ such that $f(x_i) = v_i$ for every $i \in \{1, 2, \ldots, k\}$.*

Although a 2-separating subset $\mathcal{A}$ is equivalent to a separating $\mathcal{A}$, it is of interest to know whether requiring $\mathcal{A}$ to be $k$-separating or $k$-interpolating is more stringent than requiring $\mathcal{A}$ to be separating. The next proposition answers this question.

**Proposition 1.** *If $\mathcal{A}$ is a separating unital sub-algebra of $C(X,\mathbb{F})$ and $k \geq 2$ is any natural number, then $\mathcal{A}$ is $k$-separating and $k$-interpolating.*

**Proof**. We proceed by induction on $k$. For $k = 2$, let $x_1, x_2 \in X$ with $x_1 \neq x_2$. Further, let $v_1, v_2 \in \mathbb{F}$ be given. Since $\mathcal{A}$ is separating, there exists $g \in \mathcal{A}$ such that $g(x_1) \neq g(x_2)$. Now,

$$f(x) = v_1 + (v_2 - v_1)\frac{g(x) - g(x_1)}{g(x_2) - g(x_1)}$$

is in $\mathcal{A}$ and takes values $v_1$ and $v_2$ at $x_1$ and $x_2$, respectively. Thus $\mathcal{A}$ is 2-interpolating.

Assume that $\mathcal{A}$ is $k$-interpolating for $k = m$, a natural number $\geq 2$. And suppose we are given distinct $x_1, x_2, \ldots, x_{m+1} \in X$ and $v_1, v_2, \ldots, v_{m+1} \in \mathbb{F}$. Then, by induction hypothesis, there exist functions $f_1, f_2$ and $f_3 \in \mathcal{A}$ satisfying

$$\begin{aligned}
&f_1(x_1) = 0, f_1(x_2) = 0, \ldots, f_1(x_{m-1}) = 0, f_1(x_{m+1}) = 1;\\
&f_2(x_2) = 0, f_2(x_3) = 0, \ldots, f_2(x_m) = 0, f_2(x_{m+1}) = 1; \text{ and}\\
&f_3(x_1) = v_1, f_3(x_2) = v_2, \ldots, f_3(x_{m-1}) = v_{m-1}, f_3(x_m) = v_m.
\end{aligned}$$

Now, the function

$$f(x) = f_3(x) + f_1(x)f_2(x)\left(v_{m+1} - f_3(x_{m+1})\right) \in \mathcal{A}$$

and has the desired interpolating properties. This completes the induction.

We have proved that $\mathcal{A}$ is $k$-interpolating for each natural number $k$ and hence it is $k$-separating. ☐

The conclusion of the above Proposition 1 is implicit in the conclusion of Theorem 1.

## 4. Arbitrary interpolatory constraints

**Definition 2.** *Let $S$ be a closed subset of $X$ and $\mathcal{A} \subset C(X,\mathbb{F})$. We say that $\mathcal{A}$ is separating mod $S$, if, for any $T$ such that $S \subset T \subset X$, where $T \setminus S$ is finite and any continuous $g : T \to \mathbb{F}$, there exists an $f \in \mathcal{A}$ such that $f_{|T} = g$.*

By Proposition 1, a unital sub-algebra $\mathcal{A} \subset C(X,\mathbb{F})$ is separating if and only if it is separating mod $\varnothing$, the empty set. Thus, the notion of a separating subset of $C(X,\mathbb{F})$ is subsumed by Definition 2, at least for unital sub-algebras.

Our main Theorem 3 necessitates the following definition.

**Definition 3.** *Let $S$ be a closed subset of $X$ and $\mathcal{A} \subset C(X,\mathbb{F})$. We say that $\mathcal{A}$ is interpolating mod $S$, if, for any $f \in \bar{\mathcal{A}}$ and for any $T$ such that $S \subset T \subset X$, where $T \setminus S$ is finite, there is a sequence $\{f_n\}_1^\infty$ in $\mathcal{A}$ such that $f_n \to f$ uniformly on $X$ and $f_n(x) = f(x)$ for all $x \in T$.*

**Theorem 3.** *Let $X$ be a compact metric space and $S$ a closed subset of $X$ with $\mathcal{A} \subset C(X,\mathbb{R})$. Further, assume that $\mathcal{A}$ is a unital sub-algebra of $X$ which is separating mod $S$ and interpolating mod $S$. Then for any $f \in C(X,\mathbb{R})$, there exists a sequence $\{f_n\}_1^\infty$ in $\mathcal{A}$ such that $f_n \to f$ uniformly on $X$ with $f_n(x) = f(x)$ for every $x \in S$.*

**Proof**. For each natural number $n$, it suffices to produce an $f_n \in \mathcal{A}$ satisfying $f_n(x) = f(x)$ for every $x \in S$ and $||f_n - f|| < \frac{1}{n}$.

Pick any $u, v \in X \setminus S$. Since $\mathcal{A}$ is separating mod $S$, there exists $f_{S;u,v} \in \mathcal{A}$ such that $f_{S;u,v}(x) = f(x)$ for every $x \in S \cup \{u, v\}$.

Fix $u \in X \setminus S$. For a natural number $n$ and $v \in X \setminus S$, let

$$U_v = \left\{w \in X | f_{S;u,v}(w) < f(w) + \tfrac{1}{2n}\right\}.$$

$U_v$ is an open set containing $v$. Hence, $\{U_v\}_{v \in X}$ is an open cover of $X$. By compactness of $X$, there exist finitely many $v_1, v_2, \ldots, v_k$ such that $X = \bigcup_{i=1}^{k} U_{v_i}$. Define

$$h_{S;u} = \min\nolimits_{1 \le i \le k} f_{S;u,v_i}.$$

Now, $\bar{\mathcal{A}}$ being a closed unital subalgebra of $C(X, \mathbb{R})$ is a lattice and hence $h_{S;u} \in \bar{\mathcal{A}}$. Further,

$$h_{S;u}(x) = f(x) \text{ for every } x \in S \cup \{u\}$$

and

$$h_{S;u}(x) < f(x) + \tfrac{1}{2n} \text{ for every } x \in X.$$

Since $\mathcal{A}$ is interpolating mod $S$, there exists $g_{S;u} \in \mathcal{A}$ such that

$$g_{S;u}(x) < h_{S;u}(x) + \tfrac{1}{2n} < f(x) + \frac{1}{n} \text{ for every } x \in X$$

and

$$g_{S;u}(x) = h_{S;u}(x) = f(x) \text{ for every } x \in S \cup \{u\}.$$

Next, let

$$V_u = \left\{w \in X | g_{S;u}(w) > f(w) - \tfrac{1}{2n}\right\}.$$

Then, $\{V_u\}_{u \in X}$ is an open cover of $X$ which admits a finite subcover, say, $X = \bigcup_{i=1}^{l} V_{u_i}$. Define

$$q_n = \max_{1 \le i \le l} g_{S;u_i}.$$

Since $\bar{\mathcal{A}}$ is a lattice, $q_n \in \bar{\mathcal{A}}$ with $q_n(x) = f(x)$ for every $x \in S$ and

$$||q_n - f|| < \frac{1}{2n}.$$

Again, since $\mathcal{A}$ is interpolating mod $S$, there exists an $\{f_n\}$ in $\mathcal{A}$ such that

$$||f_n - q_n|| < \frac{1}{2n}$$

and $f_n(x) = q_n(x) = f(x)$ for every $x \in S$. By triangle inequality, $||f_n - f|| < \frac{1}{n}$. This completes the proof. ☐

**Theorem 4.** *Let $X$ be a compact metric space and $S$ a closed subset of $X$ with $\mathcal{A}$ a unital subalgebra of $C(X, \mathbb{C})$ such that if $f \in \mathcal{A}$, then $\bar{f} \in \mathcal{A}$. Further, assume that $\mathcal{A}$ is separating mod $S$ and interpolating mod $S$. Then for any $f \in C(X, \mathbb{C})$, there exists a sequence $\{f_n\}_1^\infty$ in $\mathcal{A}$ such that $f_n \to f$ uniformly on $X$ with $f_n(x) = f(x)$ for every $x \in S$.*

**Proof**. Let $\mathcal{A}_{\mathbb{R}} = \{f \in \mathcal{A} | f(x) \in \mathbb{R} \text{ for all } x \in X\}$. Clearly, $\mathcal{A}_{\mathbb{R}}$ is a unital subalgebra of $\mathcal{A}$ over $\mathbb{R}$. To prove that $\mathcal{A}_{\mathbb{R}}$ is separating mod $S$, take a continuous $g : T \to \mathbb{R}$ for some $T$ such that $S \subset T \subset X$ such that $T \setminus S$ is finite. Since $\mathcal{A}$ is separating mod $S$, we get an $f \in \mathcal{A}$ such that $f_{|T} = g$. Now, $\Re(f) \in \mathcal{A}_{\mathbb{R}}$ and $\Re(f)_{|T} = g$, which proves that $\mathcal{A}_{\mathbb{R}}$ is separating mod $S$.

Next, suppose $f \in \bar{\mathcal{A}}_{\mathbb{R}}$ and $S \subset T \subset X$ such that $T \setminus S$ is finite. Since $\bar{\mathcal{A}}_{\mathbb{R}} \subset \bar{\mathcal{A}}$ and $\mathcal{A}$ is interpolating mod $S$, we get a sequence $\{f_n\}_1^\infty$ in $\mathcal{A}$ such that

$$f_n \to f \text{ uniformly on } X \text{ and } f_n(x) = f(x) \text{ for all } x \in T.$$

Consequently, we get the sequence $\{\Re(f_n)\}_1^\infty$ in $\mathcal{A}_{\mathbb{R}}$ which converges to $\Re(f) = f$ and satisfies

$$\Re(f_n(x)) = f(x) \text{ for all } x \in T.$$

This proves that $\mathcal{A}_{\mathbb{R}}$ is interpolating mod $S$.

Applying Theorem 3, we get two sequences $\{g_n\}_1^\infty, \{h_n\}_1^\infty$ in $\mathcal{A}_{\mathbb{R}}$ such that $g_n \to \Re(f)$ uniformly and $h_n \to \Im(f)$ uniformly with $g_n(x) = \Re(f)$ and $h_n(x) = \Im(f)$ for all $x$ in $S$. Let

$$f_n = g_n + ih_n$$

and we have

$$f_n \to f \text{ uniformly with } f_n(x) = f(x) \text{ for all } x \in S.$$

□

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