Multiplicity of solutions for *p*-Laplacian equation in \mathbb{R}^N with indefinite weight^{*}

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Abstract. In this article, we study the existence of infinitely many nontrivial solutions for a class of superlinear p-Laplacian equations

$$-\Delta_p u + V(x)|u|^{p-2}u = f(x,u),$$

where the primitive of the nonlinearity f is of subcritical growth near ∞ in u and the weight function V is allowed to be sign-changing. Our results extend the recent results of Zhang and Xu [Q. Y. Zhang, B. Xu, *Multiplicity of solutions for a class of semilinear Schrödinger equations with sign-changing potential*, J. Math. Anal. Appl **377**(2011), 834–840].

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1. Introduction

In this paper, we are concerned with the study of the p-Laplacian equation

$$\begin{cases} -\Delta_p u + V(x)|u|^{p-2}u = f(x,u), \\ u \in W^{1,p}(\mathbb{R}^N), \end{cases}$$
(P)

where $\Delta_p u := \operatorname{div}(|\nabla u|^{p-2}\nabla u)$ is the *p*-Laplacian operator with p > 1. With the aid of variational methods, the existence and multiplicity of nontrivial solutions for problem (*P*) have been extensively investigated in the literature over the past several decades. (*P*) with a constant sign potential V(x) was considered in [7]. More precisely, if the potential is periodic or bounded and of constant sign, the author proved the existence of ground states of (*P*). In [13], the authors considered (*P*) with a potential V(x) that may change the sign.

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For p = 2, (P) turns into a kind of Schrödinger equation of the form

$$\begin{cases} -\Delta u + V(x)u = f(x, u) \\ u \in H^1(\mathbb{R}^N) \end{cases}, \qquad (P_1)$$

which has been studied extensively. For more results, we refer the reader to [2, 9, 10, 11, 12, 14] (for constant sign potential) and [3, 15, 16, 17] (for sign-changing potential).

The quasilinear case 1 appears in a variety of applications, such as non-Newtonian fluids, image processing, nonlinear elasticity and reaction -diffusion; for more details see [4].

In the present paper, we will study the existence of infinitely many nontrivial solutions of (P) under suitable conditions. Precisely, we assume that the potential V(x) and the nonlinearity f(x, u) satisfy the following conditions:

$\mathbf{H}(\mathbf{V}):$

 (V_1) $V \in C(\mathbb{R}^N, \mathbb{R})$ is bounded from below.

 (V_2) There exists r > 0 such that for any b > 0

$$\lim_{|y| \to \infty} \mu(\{x \in \mathbb{R}^N : V(x) \le b\} \cap B_r(y))) = 0,$$

where μ is the Lebesgue measure on \mathbb{R}^N .

H(f) :

 (f_1) $f \in C(\mathbb{R}^N, \mathbb{R})$ and there exist constants $c_1 > 0$ and $p < \alpha < p^*$ such that

$$|f(x,u)| \le c_1(|u|^{p-1} + |u|^{\alpha-1}), \ \forall (x,u) \in \mathbb{R}^N \times \mathbb{R},$$

where p^* denotes the critical Sobolev exponent, i.e., $p^* = \frac{Np}{N-p}$ for p < N and $p^* = +\infty$ for $p \ge N$.

 (f_2) $F(x,0) = 0, F(x,u) \ge 0$ for all $(x,u) \in \mathbb{R}^N \times \mathbb{R}$, and

$$\lim_{|u| \to \infty} \frac{F(x, u)}{|u|^p} = +\infty$$

uniformly in \mathbb{R}^N , where $F(x,t) := \int_0^t f(x,s) ds$.

 (f_3) There exists a constant $\theta \ge 1$ such that

$$\theta \mathcal{F}(x,u) \ge \mathcal{F}(x,su), \ \forall (x,u) \in \mathbb{R}^N \times \mathbb{R} \ s \in [0,1],$$

where
$$\mathcal{F}(x,t) := uf(x,u) - pF(x,u).$$

(f4) $f(x,-u) = -f(x,u), \forall (x,u) \in \mathbb{R}^N \times \mathbb{R}.$

Our main result reads as follows.

Theorem 1. Suppose that $\mathbf{H}(\mathbf{V})$ and $\mathbf{H}(\mathbf{f})$ are satisfied. Then problem (P) possesses infinitely many nontrivial solutions.

Remark 1. Condition $\mathbf{H}(\mathbf{V})$ is due to [17]. The condition (f_3) was introduced in [8], and then was used by many authors, for example, [7]. In addition, we note that the usual condition $f(x, u) = o(|u|^{p-1})$ as $|u| \to 0$ is not needed in our Theorem 1.1. Let V be a zig-zag function with respect to |x| defined by

$$V(x) = n \sin\left[(|x| - (n-1))\pi\right] - 1, \ n-1 \le |x| \le n, n \in \mathbb{N}.$$

It is easy to check that V satisfy $\mathbf{H}(\mathbf{V})$ in our Theorem 1.1.

Remark 2. The following condition used in [1] is even stronger than (V_2) : (V_3) $\mu(\{x \in \mathbb{R}^N : V(x) \leq M\}) < +\infty, \forall M > 0$. Note that for any M > 0, (V_3) implies $\{x \in \mathbb{R}^N : V(x) \leq M\}$ is a bounded set. Thus for |y| large, $\{x \in \mathbb{R}^N : V(x) \leq M\} \cap B_r(y) = \emptyset$. Hence, Theorem 1 generalizes the main results of [13]. On the other hand, our main result extends a result for (P) when p = 2 given by Zhang and Xu in [17] to the general problem (P) where p > 1. The approach of this paper is similar to that of [17], but our proof is much more complex and more delicate than that of the main result in [17].

2. Preliminaries

Firstly, by (V_1) , there exists a constant $V_0 > 0$ such that $\overline{V}(x) := V(x) + V_0 \ge 1$ for all $x \in \mathbb{R}^N$. Let $\overline{f}(x, u) := f(x, u) + V_0 |u|^{p-2} u$. Then it is easy to verify the following lemma.

Lemma 1. Problem (P) is equivalent to the following problem

$$\begin{cases} -\operatorname{div}(|\nabla u|^{p-2}\nabla u) + \overline{V}(x)|u|^{p-2}u = \overline{f}(x,u), \\ u \in W^{1,p}(\mathbb{R}^N). \end{cases}$$
(P₂)

Proof. Obviously, \overline{V} satisfies $\mathbf{H}(\mathbf{V})$, and $\overline{f}(x, -u) = -\overline{f}(x, u)$. Furthermore, we choose $c_1 = c_0 + V_0$. It results in

$$\begin{aligned} |\overline{f}(x,u)| &\leq |f(x,u)| + V_0 |u|^{p-1} \\ &\leq c_0 (|u|^{p-1} + |u|^{\alpha-1}) + V_0 |u|^{p-1} \\ &\leq c_1 (|u|^{p-1} + |u|^{\alpha-1}), \end{aligned}$$

so condition (f_1) holds.

We clearly have $\overline{F}(x, u) = F(x, u) + \frac{V_0}{p} |u|^p$, $\overline{F}(x, 0) = 0$, $\overline{F}(x, u) \ge 0$ for all $(x, u) \in \mathbb{R}^N \times \mathbb{R}$, and

$$\lim_{|u|\to\infty}\frac{\overline{F}(x,u)}{|u|^p} = \lim_{|u|\to\infty}\frac{F(x,u)}{|u|^p} + \frac{V_0}{p} = +\infty.$$

So, condition (f_2) holds.

Meanwhile, we have

$$\begin{aligned} \overline{\mathcal{F}}(x,u) &= u\overline{f}(x,u) - p\overline{F}(x,u) \\ &= u(f(x,u) + V_0|u|^{p-1}u) - p(F(x,u) + \frac{V_0}{p}|u|^p) \\ &= uf(x,u) - pF(x,u) \\ &= \mathcal{F}(x,u). \end{aligned}$$

Thus, we have

$$\theta \overline{\mathcal{F}}(x,u) = \theta \mathcal{F}(x,u) \ge \mathcal{F}(x,su) = \overline{\mathcal{F}}(x,su).$$

So, condition (f_3) holds.

Therefore, all the assumptions of Theorem 1 have been verified. Hence, we can assume without loss of generality that $V(x) \ge 1$ for all $x \in \mathbb{R}^N$. Hence, we consider the following function space

$$E = \left\{ u \in W^{1,p}(\mathbb{R}^N) : \int_{\mathbb{R}} (|\nabla u|^p + V(x)|u|^p) dx < +\infty \right\}$$

endowed with the norm

$$\|u\| = \left[\int_{\mathbb{R}^N} (|\nabla u|^p + V(x)|u|^p) dx\right]^{\frac{1}{p}}.$$

Clearly, E is a reflexive, separable Banach space. Evidently, E is continuously embedded into $W^{1,p}(\mathbb{R}^N)$ and hence continuously embedded into $L^q(\mathbb{R}^N)$ for $p \leq q \leq p^*$, i.e., there exists c > 0 such that

$$|u|_p \le c ||u||, \ \forall u \in E,\tag{1}$$

where $|u|_q$ denotes the usual norm in $L^q(\mathbb{R}^N)$ for $p \leq q < p^*$. In fact, we further have the following lemma due to [6].

Lemma 2 (see [6], Theorem 2.1). The embedding from E into $L^q(\mathbb{R}^N)$ is compact for $p \leq q < p^*$.

Lemma 3 (see [15], Lemma 2.2). Assume that $p_1, p_2 > 1$, $r \ge 1$ and $\Omega \subseteq \mathbb{R}^N$. Let g(x,t) be a Carathéodory function on $\Omega \times \mathbb{R}$ and satisfy

$$|g(x,t)| \le a_1 |t|^{\frac{p_1-1}{r}} + a_2 |t|^{\frac{p_2-1}{r}}, \ \forall (x,t) \in \Omega \times \mathbb{R},$$

where $a_1, a_2 \geq 0$. If $u_n \to u$ in $L^{p_1}(\Omega) \cap L^{p_2}(\Omega)$, and $u_n \to u$ a.e. $x \in \Omega$, then

$$\lim_{n \to \infty} \int_{\Omega} |g(x, u_n) - g(x, u)|^r |u_n - u| dx.$$

To prove Theorem 1, we consider the C^1 functional $\varphi: E \to \mathbb{R}$ defined by

$$\varphi(u) = \frac{1}{p} \int_{\mathbb{R}^N} (|\nabla u|^p + V(x)|u|^p) dx - \lambda \int_{\mathbb{R}^N} F(x, u) dx.$$
(2)

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It is clear that critical points of φ are weak solutions of (P). In order to find a critical point of this functional, we will use the following variant fountain theorem.

Let X be a Banach space with the norm $\|\cdot\|$ and $X = \bigoplus_{j \in \mathbb{N}} X_j$ with $\dim X_j < \infty$ for any $j \in \mathbb{N}$. Set $Y_k = \bigoplus_{j=1}^k X_j$ and $Z_k = \bigoplus_{j=k}^\infty X_j$. Consider the following C^1 -functional $\varphi_{\lambda}(u) := A(u) - \lambda B(u), \lambda \in [1, 2]$. Then we have

Theorem 2 (see [18], Theorem 2.1). Assume that the above functional φ_{λ} satisfies:

- (1) φ_{λ} maps bounded sets to bounded sets for $\lambda \in [1, 2]$, and $\varphi_{\lambda}(-u) = \varphi_{\lambda}(u)$ for all $(\lambda, u) \in [1, 2] \times E$.
- (2) $B(u) \ge 0$ for all $u \in E$, and $A(u) \to \infty$ as $||u|| \to \infty$.
- (3) There exist $r_k > \rho_k > 0$ such that

$$a_k(\lambda) := \inf_{u \in Z_k, \|u\| = \rho_k} \varphi_\lambda(u) > b_k(\lambda) := \inf_{u \in Y_k, \|u\| = r_k} \varphi_\lambda(u), \ \forall \lambda \in [1, 2].$$

Then

$$a_k(\lambda) \leq \zeta_k(\lambda) := \inf_{\gamma \in \Gamma_k} \max_{u \in B_k} \varphi_\lambda(\gamma(u)), \ \forall \lambda \in [1, 2],$$

where $B_k := \{u \in Y_k : ||u|| \le r_k\}$ and $\Gamma_k := \{\gamma \in C(B_k, X) : \gamma \text{ is odd}, \gamma|_{\partial B_k} = \text{id}\}.$ Moreover, for a.e. $\lambda \in [1, 2]$, there exists a sequence $\{u_m^k(\lambda)\}_k^\infty$ such that

$$\sup_{m} \|u_m^k(\lambda)\| < \infty, \ \varphi_{\lambda}'(u_m^k(\lambda)) \to 0 \text{ and } \varphi_{\lambda}(u_m^k(\lambda)) \to \zeta_k(\lambda) \text{ as } m \to \infty.$$

3. Proof of Theorem 1.1

In order to apply Theorem 2 to prove our main result, we define the functionals A, B and φ_{λ} on our working space E by

$$A(u) = \frac{1}{p} \int_{\mathbb{R}^N} (|\nabla u|^p + V(x)|u|^p) dx,$$

$$B(u) = \int_{\mathbb{R}^N} F(x, u) dx,$$

$$\varphi_{\lambda}(u) = A(u) - \lambda B(u),$$

(3)

for all $u \in E$ and $\lambda \in [1, 2]$. By condition (f_1) , we have

$$|F(x,u)| \le \frac{c_1}{p} |u|^p + \frac{c_1}{\alpha} |u|^\alpha \le c_1(|u|^p + |u|^\alpha), \ \forall (x,u) \in \mathbb{R}^N \times \mathbb{R}.$$
 (4)

Consequently, from $\mathbf{H}(\mathbf{V})$, (f_1) and proposition 2.3 in [13], we know that $\varphi_{\lambda} \in C^1(E, \mathbb{R})$ for all $\lambda \in [1, 2]$. Moreover, we have

$$\langle A'(u), v \rangle = \int_{\mathbb{R}^N} (|\nabla u|^{p-2} \nabla u \nabla v + V(x)|u|^{p-2} uv) dx$$

$$\langle B'(u), v \rangle = \int_{\mathbb{R}^N} f(x, u) v dx,$$

(5)

for all $u, v \in E$. Furthermore, we know that (see [5]), $A' : E \to E^*$ is a mapping of (S_+) -type, i.e., if $u_n \rightharpoonup u$ in E and $\lim \sup \langle A'(u_n), u_n - u \rangle \leq 0$, then $u_n \rightarrow u$ in E. $n \rightarrow \infty$

As E is a separable and reflexive Banach space, there exist $\{e_n\}_{n=1}^{\infty} \subset E$ and ${f_n}_{n=1}^{\infty} \subset E^*$ such that

$$f_n(e_m) = \begin{cases} 1, \text{ if } n = m, \\ 0, \text{ if } n \neq m, \end{cases}$$

 $E = \overline{\operatorname{span}\{e_n : n = 1, 2, \cdots, \}} \text{ and } E^* = \overline{\operatorname{span}\{f_n : n = 1, 2, \cdots, \}}. \text{ For } k = 1, 2, \cdots,$ denote $X_k = \operatorname{span}\{e_k\}, Y_k = \bigoplus_{j=1}^k X_j \text{ and } Z_k = \overline{\bigoplus_{j=k}^\infty X_j}.$ Claim 3.1. Let $l_p(k) := \sup_{\substack{u \in Z_k, ||u|| = 1 \\ u \in Z_k, ||u|| = 1}} |u|_p \to 0 \text{ and } l_\alpha(k) := \sup_{\substack{u \in Z_k, ||u|| = 1 \\ u \in Z_k, ||u|| = 1}} |u|_\alpha \to 0.$

Then $l_p(k) \to 0$ and $l_{\alpha}(k) \to 0$, as $k \to \infty$.

It is obvious that $l_p(k) \ge l_p(k+1) \ge 0$ and $l_\alpha(k) \ge l_\alpha(k+1) \ge 0$, so $l_p(k) \to l_p \ge 0$ and $l_\alpha(k) \to l_\alpha \ge 0$ as $k \to \infty$. For each $k = 1, 2, \cdots$, we take $u_k \in Z_k$, $||u_k|| = 1$ such that

$$0 \le l_p(k) - |u_k|_p < \frac{1}{k} \text{ and } 0 \le l_\alpha(k) - |u_k|_\alpha < \frac{1}{k}$$

As E is reflexive, $\{u_k\}$ has a weakly convergent subsequence, and without loss of generality, we can suppose that $u_k \rightharpoonup u$. We claim that u = 0. In fact, for any f_m $\in \{f_n : n = 1, 2, \cdots\}$, we have $f_m(u_k) = 0$ when k > m, so $f_m(u_k) \to 0$ as $k \to \infty$, this concludes $f_m(u) = 0$ for any $f_m \in \{f_n : n = 1, 2, \cdots\}$, therefore u = 0. Since E is compactly embedded to both $L^p(\mathbb{R}^N)$ and $L^{\alpha}(\mathbb{R}^N)$, we have that $u_k \to 0$ in $L^p(\mathbb{R}^N)$ and $u_k \to 0$ in $L^{\alpha}(\mathbb{R}^N)$. This implies $l_p(k) \to 0$ and $l_{\alpha}(k) \to 0$ as $k \to \infty$. This proves Claim 3.1.

Claim 3.2. There exists a positive integer k_1 and a sequence $\rho_k \to \infty$ as $k \to \infty$ such that

$$a_k(\lambda) := \inf_{u \in \mathbb{Z}_k, \|u\| = \rho_k} \varphi_\lambda(u) > 0, \ \forall k \ge k_1.$$
(6)

By (3) and (4), we have

$$\begin{aligned} \varphi_{\lambda}(u) &= A(u) - \lambda B(u) \\ &\geq \frac{1}{p} \|u\|^{p} - 2 \int_{\mathbb{R}^{N}} F(x, u) dx \\ &\geq \frac{1}{p} \|u\|^{p} - 2c_{1}(|u|_{p}^{p} + |u|_{\alpha}^{\alpha}) \\ &\geq \frac{1}{p} \|u\|^{p} - 2c_{1}(l_{p}^{p}(k)\|u\|^{p} + l_{\alpha}^{\alpha}(k)\|u\|^{\alpha}), \ \forall (\lambda, u) \in [1, 2] \times Z_{k}. \end{aligned}$$

$$(7)$$

By Claim 3.1, there exists a positive integer k_1 such that

$$2c_1 l_p^p(k) \le \frac{1}{2p}, \ \forall k \ge k_1.$$
(8)

For each $k \geq k_1$, choose

$$\rho_k := [8pc_1 l^{\alpha}_{\alpha}(k)]^{\frac{1}{p-\alpha}}.$$
(9)

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Then $\rho_k \to +\infty$ as $k \to \infty$, since $\alpha > p$. Using (8) and (9) in (7), for each $k \ge k_1$, we have

$$a_k(\lambda) = \inf_{u \in Z_k, \|u\| = \rho_k} \varphi_{\lambda}(u) \ge \frac{1}{4p} \rho_k^p > 0,$$

which concludes the proof of Claim 3.2.

Claim 3.3. For each $k \ge k_1$, there exist $r_k > \rho_k$ such that

$$b_k(\lambda) := \inf_{u \in Y_k, \|u\| = r_k} \varphi_\lambda(u) < 0, \ \forall k \ge k_1.$$

$$(10)$$

We first prove that for any finite dimensional subspace $F\subset E$ there exists a constant $\varepsilon>0$ such that

$$\operatorname{meas}(\{x \in \mathbb{R}^N : |u(x)| \ge \varepsilon ||u||\}) \ge \varepsilon, \ \forall u \in F \setminus \{0\}.$$
(11)

Suppose that this is not true. Then for any $n \in \mathbb{N}$ there exists $u_n \in F \setminus \{0\}$ such that

$$\max(\{x \in \mathbb{R}^N : |u_n(x)| \ge \frac{\|u_n\|}{n}\}) < \frac{1}{n}.$$

Set $v_n = \frac{u_n}{\|u_n\|} \in F$ for all $n \in \mathbb{N}$. Then $\|v_n\| = 1$ for all $n \in \mathbb{N}$, and

$$\operatorname{meas}\left(\left\{x \in \mathbb{R}^{N} : |v_{n}(x)| \geq \frac{1}{n}\right\}\right) < \frac{1}{n}, \, \forall n \in \mathbb{N}.$$
(12)

Passing to a subsequence if necessary, we may assume $v_n \to v_0$ in E for some $v_0 \in F$, since F is of finite dimension. Evidently, $||v_0|| = 1$. In view of Lemma 2 and the equivalence of any two norms on F, we have

$$\int_{\mathbb{R}^N} |v_n - v_0|^p dx \to 0, \text{ as } n \to \infty.$$
(13)

Since $v_0 \neq 0$, there exists a constant $\delta_0 > 0$ such that

$$\operatorname{meas}(\Omega_0) := \operatorname{meas}(\{x \in \mathbb{R}^N : |v_0(x)| \ge \delta_0\}) \ge \delta_0.$$
(14)

For $n \in \mathbb{N}$, let

$$\Omega_n := \Big\{ x \in \mathbb{R}^N : |v_n(x)| < \frac{1}{n} \Big\} \text{ and } \Omega_n^c := \mathbb{R}^N \backslash \Omega_n = \Big\{ x \in \mathbb{R}^N : |v_n(x)| \ge \frac{1}{n} \Big\}.$$

Then for n large enough, by (12) and (14), we have

$$\operatorname{meas}(\Omega_0 \cap \Omega_n) \ge \operatorname{meas}(\Omega_0) - \operatorname{meas}(\Omega_n^c) \ge \delta_0 - \frac{1}{n} \ge \frac{\delta_0}{2}.$$

Consequently, for n large enough, there holds

$$\int_{\mathbb{R}^N} |v_n - v_0|^p dx \ge \int_{\Omega_0 \cap \Omega_n} |v_n - v_0|^p dx$$
$$\ge \int_{\Omega_0 \cap \Omega_n} (|v_0| - |v_n|)^p dx$$
$$\ge \left(\delta_0 - \frac{1}{n}\right)^p \operatorname{meas}(\Omega_0 \cap \Omega_n)$$
$$\ge \left[\frac{\delta_0}{2}\right]^{p+1}.$$

This contradicts (13). Therefore, (11) holds. Note that Y_k is finite dimensional for each $k \in \mathbb{N}$. Then by (11), for each $k \in \mathbb{N}$, there exists a constant $\varepsilon_k > 0$ such that

$$\operatorname{meas}(\{x \in \mathbb{R}^N : |u(x)| \ge \varepsilon_k ||u||\}) \ge \varepsilon_k, \ \forall u \in Y_k \setminus \{0\}.$$
(15)

By (f_2) , for each $k \in \mathbb{N}$, there exists a constant $\eta_k > 0$ such that

$$F(x,u) \ge \frac{|u|^p}{\varepsilon_k^{p+1}}, \ \forall x \in \mathbb{R}^N \text{ and } |u| \ge \eta_k.$$
 (16)

Combining (3), (15), (16) and (f_2) , for any $k \in \mathbb{N}$ and $\lambda \in [1, 2]$, we have

$$\begin{split} \varphi_{\lambda}(u) &= A(u) - \lambda B(u) \\ &\leq \frac{1}{p} \|u\|^{p} - \int_{\mathbb{R}^{N}} F(x, u) dx \\ &\leq \frac{1}{p} \|u\|^{p} - \int_{\{x \in \mathbb{R}^{N} : |u(x)| \ge \varepsilon_{k} \|u\|\}} F(x, u) dx \\ &\leq \frac{1}{p} \|u\|^{p} - \int_{\{x \in \mathbb{R}^{N} : |u(x)| \ge \varepsilon_{k} \|u\|\}} \frac{|u|^{p}}{\varepsilon_{k}^{p+1}} dx \\ &\leq \frac{1}{p} \|u\|^{p} - \varepsilon_{k}^{p} \|u\|^{p} \frac{\operatorname{meas}(\{x \in \mathbb{R}^{N} : |u(x)| \ge \varepsilon_{k} \|u\|\})}{\varepsilon_{k}^{p+1}} \\ &\leq \frac{1}{p} \|u\|^{p} - \|u\|^{p} \\ &= -\frac{p-1}{p} \|u\|^{p}, \end{split}$$
(17)

for all $u \in Y_k$ with $||u|| \ge \frac{\eta_k}{\varepsilon_k}$. Choose $r_k > \max\{\rho_k, \frac{\eta_k}{\varepsilon_k}\}$ for all $k \ge k_1$. Then (17) implies

$$b_k(\lambda) := \inf_{u \in Y_k, \|u\| = r_k} \varphi_\lambda(u) \le -\frac{p-1}{p} \|r_k\|^p < 0, \ \forall k \ge k_1.$$

This proves Claim 3.3.

From (1), (3) and (4), it follows that φ_{λ} maps bounded sets to bounded sets uniformly for $\lambda \in [1, 2]$. By (f_4) , $\varphi_{\lambda}(-u) = \varphi_{\lambda}(u)$ for all $(\lambda, u) \in [1, 2] \times E$. Therefore, condition (1) of Theorem 2 holds. On the other hand, by (3) and (f_2) , condition (2) of Theorem 2 holds. In addition, Claim 3.2 and Claim 3.3 imply that condition (3) of Theorem 2 holds. Therefore, by Theorem 2, we have that for each $k \geq k_1$ and $\lambda \in [1, 2]$, there exists a sequence $\{u_m^k(\lambda)\}_{m=1}^{\infty} \subset E$ such that

$$\sup_{m} \|u_m^k(\lambda)\| < \infty, \ \varphi_{\lambda}'(u_m^k(\lambda)) \to 0 \text{ and } \varphi_{\lambda}(u_m^k(\lambda)) \to \zeta_k(\lambda), \text{ as } m \to \infty,$$
(18)

where

$$\zeta_k(\lambda) = \inf_{\gamma \in \Gamma_k} \max_{u \in B_k} \varphi_\lambda(\gamma(u)), \ \forall \lambda \in [1, 2],$$

with $B_k := \{u \in Y_k : ||u|| \le r_k\}$ and $\Gamma_k := \{\gamma \in C(B_k, X) : \gamma \text{ is odd}, \gamma|_{\partial B_k} = \text{id}\}.$

Furthermore, it follows from the proof of Claim 3.2 that $\zeta_k(\lambda) \in [c_k, d_k]$ for all $k \ge k_1$ and $\lambda \in [1, 2]$, where $d_k := \max_{u \in B_k} \varphi_1(u)$ and $c_k := \frac{1}{4p} \rho_k^p$.

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In view of (18), for each $k \geq k_1$, we can choose $\lambda_n \to 1$ (depending on k) and get the corresponding sequences satisfying

$$\sup_{m} \|u_m^k(\lambda_n)\| < \infty \text{ and } \varphi_{\lambda_n}'(u_m^k(\lambda_n)) \to 0, \text{ as } m \to \infty.$$
(19)

Claim 3.4. For each λ_n given above, the sequence $\{u_m^k(\lambda_n)\}_{m=1}^{\infty}$ has a strongly convergent subsequence. Furthermore, if we assume that $\lim_{m\to\infty} u_m^k(\lambda_n) = u_n^k \in E$

for all $n \in \mathbb{N}$ and $k \ge k_1$, then the sequence $\{u_n^k\}_{n=1}^{\infty}$ is bounded. In fact, by (19), without loss of generality, we may assume

$$u_m^k(\lambda_n) \rightharpoonup u_n^k \text{ in } E, \text{ as } m \to \infty$$
 (20)

for some $u_n^k \in E$. By Lemma 2, we have

$$u_m^k(\lambda_n) \to u_n^k \text{ in } L^p(\mathbb{R}^N), \text{ as } m \to \infty;$$

$$u_m^k(\lambda_n) \to u_n^k \text{ in } L^\alpha(\mathbb{R}^N), \text{ as } m \to \infty.$$
(21)

From the choice of the sequence $\{u_m^k(\lambda_n)\}_{m=1}^{\infty}$, we have

$$|\langle \varphi_{\lambda_n}'(u_m^k(\lambda_n)), u_m^k(\lambda_n) \rangle| \le \varepsilon_m, \ \varepsilon_m \downarrow 0,$$
(22)

which implies that

$$|\langle A'(u_m^k(\lambda_n)), u_m^k(\lambda_n) - u_n^k \rangle| - \lambda_n \int_{\mathbb{R}^N} f(x, u_m^k(\lambda_n))(u_m^k(\lambda_n) - u_n^k) dx \le \varepsilon_m, \quad (23)$$

for all $m \geq 1$.

From Lemma 3, hypothesis $\mathbf{H}(\mathbf{f})(f_1)$ and (21), we obtain

$$\lim_{m \to \infty} \int_{\mathbb{R}^N} f(x, u_m^k(\lambda_n)) (u_m^k(\lambda_n) - u_n^k) dx = 0.$$
(24)

Using (22) (23) and (24), we obtain

$$\lim \sup_{m \to \infty} \langle A'(u_m^k(\lambda_n)), u_m^k(\lambda_n) - u_n^k \rangle \le 0.$$
(25)

Because A' is a mapping of (S_+) -type, from (25), it follows that

$$u_m^k(\lambda_n) \to u_n^k \text{ in } E, \text{ as } m \to \infty.$$
 (26)

It remains to prove that the sequence $\{u_n^k\}_{n=1}^{\infty}$ is bounded. Suppose that this is not true. Then passing to a subsequence is necessary, we can assume that $||u_n^k|| \to \infty$ as $n \to \infty$. Set $w_n^k = \frac{u_n^k}{||u_n^k||}$, $n \ge 1$. We may assume that

$$\begin{aligned}
& w_n^k \to w_0^k \text{ in } E; \\
& w_n^k \to w_0^k \text{ in } L^p(\mathbb{R}^N); \\
& w_n^k \to w_0^k \text{ in } L^\alpha(\mathbb{R}^N); \\
& w_n^k(x) \to w_0^k(x) \text{ a.e. on } \mathbb{R}^N.
\end{aligned}$$
(27)

If $w_0^k = 0$, we choose a sequence $\{t_n\}_{n=1}^{\infty} \subset [0, 1]$ such that

$$\varphi_{\lambda_n}(t_n w_n^k) = \max_{t \in [0,1]} \varphi_{\lambda_n}(t w_n^k).$$
(28)

For any R > 0, we set $\overline{w}_n^k = \sqrt[p]{pR} w_n^k$. Combining (3), (4), (27) and (f₂), we have

$$0 \le \int_{\mathbb{R}^N} F(x, \overline{w}_n^k) dx \le c_1 \int_{\mathbb{R}^N} (|\overline{w}_n^k|^p + |\overline{w}_n^k|^\alpha) dx \to 0 \text{ as } n \to \infty.$$
(29)

Thus, for sufficiently large n, we have

$$\begin{split} \varphi_{\lambda_n}(t_n u_n^k) &\geq \varphi_{\lambda_n}(\overline{w}_n^k) \\ &= \frac{1}{p} \|\overline{w}_n^k\|^p - \lambda_n \int_{\mathbb{R}^N} F(x, \overline{w}_n^k) dx \\ &= R - \lambda_n \int_{\mathbb{R}^N} F(x, \overline{w}_n^k) dx \\ &\geq \frac{R}{2}, \end{split}$$

which implies that

$$\lim_{n \to \infty} \varphi_{\lambda_n}(t_n u_n^k) = +\infty.$$

On the one hand, it is clear from (18) (19) and (26) that

$$\varphi_{\lambda_n}'(u_n^k) = 0, \ \varphi_{\lambda_n}(u_n^k) \in [c_k, d_k], \ \forall n \in \mathbb{N} \text{ and } k \ge k_1.$$
(30)

On the other hand, from (28) it follows that

$$0 = t_n \frac{d}{dt} \bigg|_{t=t_n} \varphi_{\lambda_n}(t u_n^k) = \langle \varphi'_{\lambda_n}(t_n u_n^k), t_n u_n^k \rangle.$$
(31)

So from (3), (5), (28) and condition (f_3) , we conclude that

$$\begin{split} \varphi_{\lambda_n}(u_n^k) = &\varphi_{\lambda_n}(u_n^k) - \frac{1}{p} \langle \varphi_{\lambda_n}'(u_n^k), u_n^k \rangle \\ = &\frac{\lambda_n}{p} \int_{\mathbb{R}^N} \mathcal{F}(x, u_n^k) dx \\ \geq &\frac{\lambda_n}{\theta p} \int_{\mathbb{R}^N} \mathcal{F}(x, t_n u_n^k) dx \\ = &\frac{1}{\theta} \varphi_{\lambda_n}(t_n u_n^k) - \frac{1}{\theta p} \langle \varphi_{\lambda_n}'(t_n u_n^k), t_n u_n^k \rangle \\ = &\frac{1}{\theta} \varphi_{\lambda_n}(t_n u_n^k) \to +\infty \text{ as } n \to \infty. \end{split}$$

This provides a contradiction to (30).

If $w_0^k \neq 0$, then $\Theta := \{x \in \mathbb{R}^N : w_0^k(x) \neq 0\}$ has a positive Lebesgue measure. Thus for $x \in \Theta$, we have $u_n^k(x) \to \infty$. This, together with (3), (27) and (f₂), implies

$$\frac{1}{p} - \frac{\varphi_{\lambda_n}(u_n^k)}{\|u_n^k\|^p} = \lambda_n \int_{\mathbb{R}^N} \frac{F(x, u_n^k)}{\|u_n^k\|^p} dx$$
$$\geq \lambda_n \int_{\Theta} |w_n^k|^p \frac{F(x, u_n^k)}{|u_n^k|^p} dx \to \infty, \text{ as } n \to \infty,$$

which is a contradiction to (30) again. Therefore, the sequence $\{u_n^k\}_{n=1}^{\infty}$ is bounded. This proves that Claim 3.4 is true.

In view of Claim 3.4 and (30), for each $k \geq k_1$, using similar arguments in the proof of Claim 3.4, we can also prove that the sequence $\{u_n^k\}_{n=1}^{\infty}$ has a strong convergent subsequence with the limit u^k being just a critical point of $\varphi_1 = \varphi$. We already know that $\varphi(u^k) \in [c_k, d_k]$ for all $k \geq k_1$. Recalling that $c_k \to +\infty$ as $k \to \infty$, we deduce the existence of infinitely many nontrivial critical points of φ . Therefore, problem (P) possesses infinitely many nontrivial solutions. The proof of Theorem 1 is complete. \Box

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