# Multiplicity of solutions for $p$-Laplacian equation in $\mathbb{R}^{N}$ with indefinite weight* 

Qing-Mei Zhou ${ }^{1}$ and Ke-Qi Wang ${ }^{2, ~}{ }^{\dagger}$<br>${ }^{1}$ Library, Northeast Forestry University, Harbin, 150 040, P. R. China<br>${ }^{2}$ College of Mechanical and Electrical Engineering, Northeast Forestry University, Harbin, 150040 , P. R. China

Received June 18, 2014; accepted October 8, 2015


#### Abstract

In this article, we study the existence of infinitely many nontrivial solutions for a class of superlinear $p$-Laplacian equations $$
-\Delta_{p} u+V(x)|u|^{p-2} u=f(x, u),
$$ where the primitive of the nonlinearity $f$ is of subcritical growth near $\infty$ in $u$ and the weight function $V$ is allowed to be sign-changing. Our results extend the recent results of Zhang and Xu [Q. Y. Zhang, B. Xu, Multiplicity of solutions for a class of semilinear Schrödinger equations with sign-changing potential, J. Math. Anal. Appl 377(2011), 834-840].


AMS subject classifications: 35J60, 35J20, 47J30
Key words: $p$-Laplacian, sign-changing potential, superlinear problems, variational method, critical points

## 1. Introduction

In this paper, we are concerned with the study of the $p$-Laplacian equation

$$
\left\{\begin{array}{l}
-\Delta_{p} u+V(x)|u|^{p-2} u=f(x, u)  \tag{P}\\
u \in W^{1, p}\left(\mathbb{R}^{N}\right)
\end{array}\right.
$$

where $\Delta_{p} u:=\operatorname{div}\left(|\nabla u|^{p-2} \nabla u\right)$ is the $p$-Laplacian operator with $p>1$. With the aid of variational methods, the existence and multiplicity of nontrivial solutions for problem $(P)$ have been extensively investigated in the literature over the past several decades. $(P)$ with a constant sign potential $V(x)$ was considered in [7]. More precisely, if the potential is periodic or bounded and of constant sign, the author proved the existence of ground states of $(P)$. In [13], the authors considered $(P)$ with a potential $V(x)$ that may change the sign.

[^0]For $p=2,(P)$ turns into a kind of Schrödinger equation of the form

$$
\left\{\begin{array}{l}
-\Delta u+V(x) u=f(x, u)  \tag{1}\\
u \in H^{1}\left(\mathbb{R}^{N}\right)
\end{array}\right.
$$

which has been studied extensively. For more results, we refer the reader to $[2,9$, $10,11,12,14]$ (for constant sign potential) and [3, 15, 16, 17] (for sign-changing potential).

The quasilinear case $1<p<N$ appears in a variety of applications, such as non-Newtonian fluids, image processing, nonlinear elasticity and reaction-diffusion; for more details see [4].

In the present paper, we will study the existence of infinitely many nontrivial solutions of $(P)$ under suitable conditions. Precisely, we assume that the potential $V(x)$ and the nonlinearity $f(x, u)$ satisfy the following conditions:
$\mathbf{H}(\mathbf{V})$ :
$\left(V_{1}\right) V \in C\left(\mathbb{R}^{N}, \mathbb{R}\right)$ is bounded from below.
$\left(V_{2}\right)$ There exists $r>0$ such that for any $b>0$

$$
\left.\lim _{|y| \rightarrow \infty} \mu\left(\left\{x \in \mathbb{R}^{N}: V(x) \leq b\right\} \cap B_{r}(y)\right)\right)=0
$$

where $\mu$ is the Lebesgue measure on $\mathbb{R}^{N}$.
$\mathbf{H}(f)$ :
$\left(f_{1}\right) f \in C\left(\mathbb{R}^{N}, \mathbb{R}\right)$ and there exist constants $c_{1}>0$ and $p<\alpha<p^{*}$ such that

$$
|f(x, u)| \leq c_{1}\left(|u|^{p-1}+|u|^{\alpha-1}\right), \forall(x, u) \in \mathbb{R}^{N} \times \mathbb{R}
$$

where $p^{*}$ denotes the critical Sobolev exponent, i.e., $p^{*}=\frac{N p}{N-p}$ for $p<N$ and $p^{*}=+\infty$ for $p \geq N$.
$\left(f_{2}\right) F(x, 0)=0, F(x, u) \geq 0$ for all $(x, u) \in \mathbb{R}^{N} \times \mathbb{R}$, and

$$
\lim _{|u| \rightarrow \infty} \frac{F(x, u)}{|u|^{p}}=+\infty
$$

uniformly in $\mathbb{R}^{N}$, where $F(x, t):=\int_{0}^{t} f(x, s) d s$.
$\left(f_{3}\right)$ There exists a constant $\theta \geq 1$ such that

$$
\theta \mathcal{F}(x, u) \geq \mathcal{F}(x, s u), \forall(x, u) \in \mathbb{R}^{N} \times \mathbb{R} s \in[0,1]
$$

where $\mathcal{F}(x, t):=u f(x, u)-p F(x, u)$.
$\left(f_{4}\right) f(x,-u)=-f(x, u), \forall(x, u) \in \mathbb{R}^{N} \times \mathbb{R}$.
Our main result reads as follows.
Theorem 1. Suppose that $\mathbf{H}(\mathbf{V})$ and $\mathbf{H}(\mathbf{f})$ are satisfied. Then problem $(P)$ possesses infinitely many nontrivial solutions.

Remark 1. Condition $\mathbf{H}(\mathbf{V})$ is due to [17]. The condition $\left(f_{3}\right)$ was introduced in [8], and then was used by many authors, for example, [7]. In addition, we note that the usual condition $f(x, u)=o\left(|u|^{p-1}\right)$ as $|u| \rightarrow 0$ is not needed in our Theorem 1.1. Let $V$ be a zig-zag function with respect to $|x|$ defined by

$$
V(x)=n \sin [(|x|-(n-1)) \pi]-1, n-1 \leq|x| \leq n, n \in \mathbb{N} .
$$

It is easy to check that $V$ satisfy $\mathbf{H}(\mathbf{V})$ in our Theorem 1.1.
Remark 2. The following condition used in [1] is even stronger than $\left(V_{2}\right):\left(V_{3}\right)$ $\mu\left(\left\{x \in \mathbb{R}^{N}: V(x) \leq M\right\}\right)<+\infty, \forall M>0$. Note that for any $M>0,\left(V_{3}\right)$ implies $\left\{x \in \mathbb{R}^{N}: V(x) \leq M\right\}$ is a bounded set. Thus for $|y|$ large, $\left\{x \in \mathbb{R}^{N}: V(x) \leq\right.$ $M\} \cap B_{r}(y)=\emptyset$. Hence, Theorem 1 generalizes the main results of [13]. On the other hand, our main result extends a result for $(P)$ when $p=2$ given by Zhang and $X u$ in [17] to the general problem $(P)$ where $p>1$. The approach of this paper is similar to that of [17], but our proof is much more complex and more delicate than that of the main result in [17].

## 2. Preliminaries

Firstly, by $\left(V_{1}\right)$, there exists a constant $V_{0}>0$ such that $\bar{V}(x):=V(x)+V_{0} \geq 1$ for all $x \in \mathbb{R}^{N}$. Let $\bar{f}(x, u):=f(x, u)+V_{0}|u|^{p-2} u$. Then it is easy to verify the following lemma.

Lemma 1. Problem $(P)$ is equivalent to the following problem

$$
\left\{\begin{array}{l}
-\operatorname{div}\left(|\nabla u|^{p-2} \nabla u\right)+\bar{V}(x)|u|^{p-2} u=\bar{f}(x, u)  \tag{2}\\
u \in W^{1, p}\left(\mathbb{R}^{N}\right)
\end{array}\right.
$$

Proof. Obviously, $\bar{V}$ satisfies $\mathbf{H}(\mathbf{V})$, and $\bar{f}(x,-u)=-\bar{f}(x, u)$. Furthermore, we choose $c_{1}=c_{0}+V_{0}$. It results in

$$
\begin{aligned}
|\bar{f}(x, u)| & \leq|f(x, u)|+V_{0}|u|^{p-1} \\
& \leq c_{0}\left(|u|^{p-1}+|u|^{\alpha-1}\right)+V_{0}|u|^{p-1} \\
& \leq c_{1}\left(|u|^{p-1}+|u|^{\alpha-1}\right),
\end{aligned}
$$

so condition $\left(f_{1}\right)$ holds.
We clearly have $\bar{F}(x, u)=F(x, u)+\frac{V_{0}}{p}|u|^{p}, \bar{F}(x, 0)=0, \bar{F}(x, u) \geq 0$ for all $(x, u) \in \mathbb{R}^{N} \times \mathbb{R}$, and

$$
\lim _{|u| \rightarrow \infty} \frac{\bar{F}(x, u)}{|u|^{p}}=\lim _{|u| \rightarrow \infty} \frac{F(x, u)}{|u|^{p}}+\frac{V_{0}}{p}=+\infty .
$$

So, condition $\left(f_{2}\right)$ holds.

Meanwhile, we have

$$
\begin{aligned}
\overline{\mathcal{F}}(x, u) & =u \bar{f}(x, u)-p \bar{F}(x, u) \\
& =u\left(f(x, u)+V_{0}|u|^{p-1} u\right)-p\left(F(x, u)+\frac{V_{0}}{p}|u|^{p}\right) \\
& =u f(x, u)-p F(x, u) \\
& =\mathcal{F}(x, u) .
\end{aligned}
$$

Thus, we have

$$
\theta \overline{\mathcal{F}}(x, u)=\theta \mathcal{F}(x, u) \geq \mathcal{F}(x, s u)=\overline{\mathcal{F}}(x, s u)
$$

So, condition $\left(f_{3}\right)$ holds.
Therefore, all the assumptions of Theorem 1 have been verified. Hence, we can assume without loss of generality that $V(x) \geq 1$ for all $x \in \mathbb{R}^{N}$. Hence, we consider the following function space

$$
E=\left\{u \in W^{1, p}\left(\mathbb{R}^{N}\right): \int_{\mathbb{R}}\left(|\nabla u|^{p}+V(x)|u|^{p}\right) d x<+\infty\right\}
$$

endowed with the norm

$$
\|u\|=\left[\int_{\mathbb{R}^{N}}\left(|\nabla u|^{p}+V(x)|u|^{p}\right) d x\right]^{\frac{1}{p}}
$$

Clearly, $E$ is a reflexive, separable Banach space. Evidently, $E$ is continuously embedded into $W^{1, p}\left(\mathbb{R}^{N}\right)$ and hence continuously embedded into $L^{q}\left(\mathbb{R}^{N}\right)$ for $p \leq$ $q \leq p^{*}$, i.e., there exists $c>0$ such that

$$
\begin{equation*}
|u|_{p} \leq c\|u\|, \forall u \in E \tag{1}
\end{equation*}
$$

where $|u|_{q}$ denotes the usual norm in $L^{q}\left(\mathbb{R}^{N}\right)$ for $p \leq q<p^{*}$. In fact, we further have the following lemma due to [6].

Lemma 2 (see [6], Theorem 2.1). The embedding from $E$ into $L^{q}\left(\mathbb{R}^{N}\right)$ is compact for $p \leq q<p^{*}$.

Lemma 3 (see [15], Lemma 2.2). Assume that $p_{1}, p_{2}>1, r \geq 1$ and $\Omega \subseteq \mathbb{R}^{N}$. Let $g(x, t)$ be a Carathéodory function on $\Omega \times \mathbb{R}$ and satisfy

$$
|g(x, t)| \leq a_{1}|t|^{\frac{p_{1}-1}{r}}+a_{2}|t|^{\frac{p_{2}-1}{r}}, \forall(x, t) \in \Omega \times \mathbb{R},
$$

where $a_{1}, a_{2} \geq 0$. If $u_{n} \rightarrow u$ in $L^{p_{1}}(\Omega) \cap L^{p_{2}}(\Omega)$, and $u_{n} \rightarrow u$ a.e. $x \in \Omega$, then

$$
\lim _{n \rightarrow \infty} \int_{\Omega}\left|g\left(x, u_{n}\right)-g(x, u)\right|^{r}\left|u_{n}-u\right| d x
$$

To prove Theorem 1, we consider the $C^{1}$ functional $\varphi: E \rightarrow \mathbb{R}$ defined by

$$
\begin{equation*}
\varphi(u)=\frac{1}{p} \int_{\mathbb{R}^{N}}\left(|\nabla u|^{p}+V(x)|u|^{p}\right) d x-\lambda \int_{\mathbb{R}^{N}} F(x, u) d x . \tag{2}
\end{equation*}
$$

It is clear that critical points of $\varphi$ are weak solutions of $(P)$. In order to find a critical point of this functional, we will use the following variant fountain theorem.

Let $X$ be a Banach space with the norm $\|\cdot\|$ and $X=\overline{\oplus_{j \in \mathbb{N}} X_{j}}$ with $\operatorname{dim} X_{j}<\infty$ for any $j \in \mathbb{N}$. Set $Y_{k}=\oplus_{j=1}^{k} X_{j}$ and $Z_{k}=\overline{\oplus_{j=k}^{\infty} X_{j}}$. Consider the following $C^{1}$ -functional $\varphi_{\lambda}(u):=A(u)-\lambda B(u), \lambda \in[1,2]$. Then we have

Theorem 2 (see [18], Theorem 2.1). Assume that the above functional $\varphi_{\lambda}$ satisfies:
(1) $\varphi_{\lambda}$ maps bounded sets to bounded sets for $\lambda \in[1,2]$, and $\varphi_{\lambda}(-u)=\varphi_{\lambda}(u)$ for all $(\lambda, u) \in[1,2] \times E$.
(2) $B(u) \geq 0$ for all $u \in E$, and $A(u) \rightarrow \infty$ as $\|u\| \rightarrow \infty$.
(3) There exist $r_{k}>\rho_{k}>0$ such that

$$
a_{k}(\lambda):=\inf _{u \in Z_{k},\|u\|=\rho_{k}} \varphi_{\lambda}(u)>b_{k}(\lambda):=\inf _{u \in Y_{k},\|u\|=r_{k}} \varphi_{\lambda}(u), \forall \lambda \in[1,2] .
$$

Then

$$
a_{k}(\lambda) \leq \zeta_{k}(\lambda):=\inf _{\gamma \in \Gamma_{k}} \max _{u \in B_{k}} \varphi_{\lambda}(\gamma(u)), \forall \lambda \in[1,2],
$$

where $B_{k}:=\left\{u \in Y_{k}:\|u\| \leq r_{k}\right\}$ and $\Gamma_{k}:=\left\{\gamma \in C\left(B_{k}, X\right): \gamma\right.$ is odd, $\left.\left.\gamma\right|_{\partial B_{k}}=\mathrm{id}\right\}$. Moreover, for a.e. $\lambda \in[1,2]$, there exists a sequence $\left\{u_{m}^{k}(\lambda)\right\}_{k}^{\infty}$ such that

$$
\sup _{m}\left\|u_{m}^{k}(\lambda)\right\|<\infty, \varphi_{\lambda}^{\prime}\left(u_{m}^{k}(\lambda)\right) \rightarrow 0 \text { and } \varphi_{\lambda}\left(u_{m}^{k}(\lambda)\right) \rightarrow \zeta_{k}(\lambda) \text { as } m \rightarrow \infty
$$

## 3. Proof of Theorem 1.1

In order to apply Theorem 2 to prove our main result, we define the functionals $A, B$ and $\varphi_{\lambda}$ on our working space $E$ by

$$
\begin{align*}
A(u) & =\frac{1}{p} \int_{\mathbb{R}^{N}}\left(|\nabla u|^{p}+V(x)|u|^{p}\right) d x \\
B(u) & =\int_{\mathbb{R}^{N}} F(x, u) d x  \tag{3}\\
\varphi_{\lambda}(u) & =A(u)-\lambda B(u)
\end{align*}
$$

for all $u \in E$ and $\lambda \in[1,2]$. By condition $\left(f_{1}\right)$, we have

$$
\begin{equation*}
|F(x, u)| \leq \frac{c_{1}}{p}|u|^{p}+\frac{c_{1}}{\alpha}|u|^{\alpha} \leq c_{1}\left(|u|^{p}+|u|^{\alpha}\right), \forall(x, u) \in \mathbb{R}^{N} \times \mathbb{R} \tag{4}
\end{equation*}
$$

Consequently, from $\mathbf{H}(\mathbf{V}),\left(f_{1}\right)$ and proposition 2.3 in [13], we know that $\varphi_{\lambda} \in$ $C^{1}(E, \mathbb{R})$ for all $\lambda \in[1,2]$. Moreover, we have

$$
\begin{align*}
\left\langle A^{\prime}(u), v\right\rangle & =\int_{\mathbb{R}^{N}}\left(|\nabla u|^{p-2} \nabla u \nabla v+V(x)|u|^{p-2} u v\right) d x \\
\left\langle B^{\prime}(u), v\right\rangle & =\int_{\mathbb{R}^{N}} f(x, u) v d x \tag{5}
\end{align*}
$$

for all $u, v \in E$. Furthermore, we know that (see [5]), $A^{\prime}: E \rightarrow E^{*}$ is a mapping of $\left(S_{+}\right)$-type, i.e., if $u_{n} \rightharpoonup u$ in $E$ and $\lim \sup _{n \rightarrow \infty}\left\langle A^{\prime}\left(u_{n}\right), u_{n}-u\right\rangle \leq 0$, then $u_{n} \rightarrow u$ in $E$.

As $E$ is a separable and reflexive Banach space, there exist $\left\{e_{n}\right\}_{n=1}^{\infty} \subset E$ and $\left\{f_{n}\right\}_{n=1}^{\infty} \subset E^{*}$ such that

$$
f_{n}\left(e_{m}\right)=\left\{\begin{array}{l}
1, \text { if } n=m \\
0, \text { if } n \neq m
\end{array}\right.
$$

$E=\overline{\operatorname{span}\left\{e_{n}: n=1,2, \cdots,\right\}}$ and $E^{*}=\overline{\operatorname{span}\left\{f_{n}: n=1,2, \cdots,\right\}}$. For $k=1,2, \cdots$, denote $X_{k}=\operatorname{span}\left\{e_{k}\right\}, Y_{k}=\oplus_{j=1}^{k} X_{j}$ and $Z_{k}=\overline{\oplus_{j=k}^{\infty} X_{j}}$.

Claim 3.1. Let $l_{p}(k):=\sup _{u \in Z_{k},\|u\|=1}|u|_{p} \rightarrow 0$ and $l_{\alpha}(k):=\sup _{u \in Z_{k},\|u\|=1}|u|_{\alpha} \rightarrow 0$. Then $l_{p}(k) \rightarrow 0$ and $l_{\alpha}(k) \rightarrow 0$, as $k \rightarrow \infty$.

It is obvious that $l_{p}(k) \geq l_{p}(k+1) \geq 0$ and $l_{\alpha}(k) \geq l_{\alpha}(k+1) \geq 0$, so $l_{p}(k) \rightarrow l_{p} \geq 0$ and $l_{\alpha}(k) \rightarrow l_{\alpha} \geq 0$ as $k \rightarrow \infty$. For each $k=1,2, \cdots$, we take $u_{k} \in Z_{k},\left\|u_{k}\right\|=1$ such that

$$
0 \leq l_{p}(k)-\left|u_{k}\right|_{p}<\frac{1}{k} \text { and } 0 \leq l_{\alpha}(k)-\left|u_{k}\right|_{\alpha}<\frac{1}{k}
$$

As $E$ is reflexive, $\left\{u_{k}\right\}$ has a weakly convergent subsequence, and without loss of generality, we can suppose that $u_{k} \rightharpoonup u$. We claim that $u=0$. In fact, for any $f_{m} \in\left\{f_{n}: n=1,2, \cdots\right\}$, we have $f_{m}\left(u_{k}\right)=0$ when $k>m$, so $f_{m}\left(u_{k}\right) \rightarrow 0$ as $k \rightarrow \infty$, this concludes $f_{m}(u)=0$ for any $f_{m} \in\left\{f_{n}: n=1,2, \cdots\right\}$, therefore $u=0$. Since $E$ is compactly embedded to both $L^{p}\left(\mathbb{R}^{N}\right)$ and $L^{\alpha}\left(\mathbb{R}^{N}\right)$, we have that $u_{k} \rightarrow 0$ in $L^{p}\left(\mathbb{R}^{N}\right)$ and $u_{k} \rightarrow 0$ in $L^{\alpha}\left(\mathbb{R}^{N}\right)$. This implies $l_{p}(k) \rightarrow 0$ and $l_{\alpha}(k) \rightarrow 0$ as $k \rightarrow \infty$. This proves Claim 3.1.

Claim 3.2. There exists a positive integer $k_{1}$ and a sequence $\rho_{k} \rightarrow \infty$ as $k \rightarrow \infty$ such that

$$
\begin{equation*}
a_{k}(\lambda):=\inf _{u \in Z_{k},\|u\|=\rho_{k}} \varphi_{\lambda}(u)>0, \forall k \geq k_{1} \tag{6}
\end{equation*}
$$

By (3) and (4), we have

$$
\begin{align*}
\varphi_{\lambda}(u) & =A(u)-\lambda B(u) \\
& \geq \frac{1}{p}\|u\|^{p}-2 \int_{\mathbb{R}^{N}} F(x, u) d x \\
& \geq \frac{1}{p}\|u\|^{p}-2 c_{1}\left(|u|_{p}^{p}+|u|_{\alpha}^{\alpha}\right)  \tag{7}\\
& \geq \frac{1}{p}\|u\|^{p}-2 c_{1}\left(l_{p}^{p}(k)\|u\|^{p}+l_{\alpha}^{\alpha}(k)\|u\|^{\alpha}\right), \forall(\lambda, u) \in[1,2] \times Z_{k} .
\end{align*}
$$

By Claim 3.1, there exists a positive integer $k_{1}$ such that

$$
\begin{equation*}
2 c_{1} l_{p}^{p}(k) \leq \frac{1}{2 p}, \forall k \geq k_{1} \tag{8}
\end{equation*}
$$

For each $k \geq k_{1}$, choose

$$
\begin{equation*}
\rho_{k}:=\left[8 p c_{1} l_{\alpha}^{\alpha}(k)\right]^{\frac{1}{p-\alpha}} . \tag{9}
\end{equation*}
$$

Then $\rho_{k} \rightarrow+\infty \quad$ as $k \rightarrow \infty$, since $\alpha>p$. Using (8) and (9) in (7), for each $k \geq k_{1}$, we have

$$
a_{k}(\lambda)=\inf _{u \in Z_{k},\|u\|=\rho_{k}} \varphi_{\lambda}(u) \geq \frac{1}{4 p} \rho_{k}^{p}>0
$$

which concludes the proof of Claim 3.2.
Claim 3.3. For each $k \geq k_{1}$, there exist $r_{k}>\rho_{k}$ such that

$$
\begin{equation*}
b_{k}(\lambda):=\inf _{u \in Y_{k},\|u\|=r_{k}} \varphi_{\lambda}(u)<0, \forall k \geq k_{1} \tag{10}
\end{equation*}
$$

We first prove that for any finite dimensional subspace $F \subset E$ there exists a constant $\varepsilon>0$ such that

$$
\begin{equation*}
\operatorname{meas}\left(\left\{x \in \mathbb{R}^{N}:|u(x)| \geq \varepsilon\|u\|\right\}\right) \geq \varepsilon, \forall u \in F \backslash\{0\} \tag{11}
\end{equation*}
$$

Suppose that this is not true. Then for any $n \in \mathbb{N}$ there exists $u_{n} \in F \backslash\{0\}$ such that

$$
\operatorname{meas}\left(\left\{x \in \mathbb{R}^{N}:\left|u_{n}(x)\right| \geq \frac{\left\|u_{n}\right\|}{n}\right\}\right)<\frac{1}{n}
$$

Set $v_{n}=\frac{u_{n}}{\left\|u_{n}\right\|} \in F$ for all $n \in \mathbb{N}$. Then $\left\|v_{n}\right\|=1$ for all $n \in \mathbb{N}$, and

$$
\begin{equation*}
\operatorname{meas}\left(\left\{x \in \mathbb{R}^{N}:\left|v_{n}(x)\right| \geq \frac{1}{n}\right\}\right)<\frac{1}{n}, \forall n \in \mathbb{N} \tag{12}
\end{equation*}
$$

Passing to a subsequence if necessary, we may assume $v_{n} \rightarrow v_{0}$ in $E$ for some $v_{0} \in F$, since $F$ is of finite dimension. Evidently, $\left\|v_{0}\right\|=1$. In view of Lemma 2 and the equivalence of any two norms on $F$, we have

$$
\begin{equation*}
\int_{\mathbb{R}^{N}}\left|v_{n}-v_{0}\right|^{p} d x \rightarrow 0, \text { as } n \rightarrow \infty \tag{13}
\end{equation*}
$$

Since $v_{0} \neq 0$, there exists a constant $\delta_{0}>0$ such that

$$
\begin{equation*}
\operatorname{meas}\left(\Omega_{0}\right):=\operatorname{meas}\left(\left\{x \in \mathbb{R}^{N}:\left|v_{0}(x)\right| \geq \delta_{0}\right\}\right) \geq \delta_{0} \tag{14}
\end{equation*}
$$

For $n \in \mathbb{N}$, let

$$
\Omega_{n}:=\left\{x \in \mathbb{R}^{N}:\left|v_{n}(x)\right|<\frac{1}{n}\right\} \text { and } \Omega_{n}^{c}:=\mathbb{R}^{N} \backslash \Omega_{n}=\left\{x \in \mathbb{R}^{N}:\left|v_{n}(x)\right| \geq \frac{1}{n}\right\}
$$

Then for $n$ large enough, by (12) and (14), we have

$$
\operatorname{meas}\left(\Omega_{0} \cap \Omega_{n}\right) \geq \operatorname{meas}\left(\Omega_{0}\right)-\operatorname{meas}\left(\Omega_{n}^{c}\right) \geq \delta_{0}-\frac{1}{n} \geq \frac{\delta_{0}}{2}
$$

Consequently, for $n$ large enough, there holds

$$
\begin{aligned}
\int_{\mathbb{R}^{N}}\left|v_{n}-v_{0}\right|^{p} d x & \geq \int_{\Omega_{0} \cap \Omega_{n}}\left|v_{n}-v_{0}\right|^{p} d x \\
& \geq \int_{\Omega_{0} \cap \Omega_{n}}\left(\left|v_{0}\right|-\left|v_{n}\right|\right)^{p} d x \\
& \geq\left(\delta_{0}-\frac{1}{n}\right)^{p} \operatorname{meas}\left(\Omega_{0} \cap \Omega_{n}\right) \\
& \geq\left[\frac{\delta_{0}}{2}\right]^{p+1}
\end{aligned}
$$

This contradicts (13). Therefore, (11) holds. Note that $Y_{k}$ is finite dimensional for each $k \in \mathbb{N}$. Then by (11), for each $k \in \mathbb{N}$, there exists a constant $\varepsilon_{k}>0$ such that

$$
\begin{equation*}
\operatorname{meas}\left(\left\{x \in \mathbb{R}^{N}:|u(x)| \geq \varepsilon_{k}\|u\|\right\}\right) \geq \varepsilon_{k}, \forall u \in Y_{k} \backslash\{0\} \tag{15}
\end{equation*}
$$

By $\left(f_{2}\right)$, for each $k \in \mathbb{N}$, there exists a constant $\eta_{k}>0$ such that

$$
\begin{equation*}
F(x, u) \geq \frac{|u|^{p}}{\varepsilon_{k}^{p+1}}, \forall x \in \mathbb{R}^{N} \text { and }|u| \geq \eta_{k} \tag{16}
\end{equation*}
$$

Combining (3), (15), (16) and $\left(f_{2}\right)$, for any $k \in \mathbb{N}$ and $\lambda \in[1,2]$, we have

$$
\begin{align*}
\varphi_{\lambda}(u) & =A(u)-\lambda B(u) \\
& \leq \frac{1}{p}\|u\|^{p}-\int_{\mathbb{R}^{N}} F(x, u) d x \\
& \leq \frac{1}{p}\|u\|^{p}-\int_{\left\{x \in \mathbb{R}^{N}:|u(x)| \geq \varepsilon_{k}\|u\|\right\}} F(x, u) d x \\
& \leq \frac{1}{p}\|u\|^{p}-\int_{\left\{x \in \mathbb{R}^{N}:|u(x)| \geq \varepsilon_{k}\|u\|\right\}} \frac{|u|^{p}}{\varepsilon_{k}^{p+1}} d x  \tag{17}\\
& \leq \frac{1}{p}\|u\|^{p}-\varepsilon_{k}^{p}\|u\|^{p} \frac{\operatorname{meas}\left(\left\{x \in \mathbb{R}^{N}:|u(x)| \geq \varepsilon_{k}\|u\| \|\right)\right.}{\varepsilon_{k}^{p+1}} \\
& \leq \frac{1}{p}\|u\|^{p}-\|u\|^{p} \\
& =-\frac{p-1}{p}\|u\|^{p},
\end{align*}
$$

for all $u \in Y_{k}$ with $\|u\| \geq \frac{\eta_{k}}{\varepsilon_{k}}$. Choose $r_{k}>\max \left\{\rho_{k}, \frac{\eta_{k}}{\varepsilon_{k}}\right\}$ for all $k \geq k_{1}$. Then (17) implies

$$
b_{k}(\lambda):=\inf _{u \in Y_{k},\|u\|=r_{k}} \varphi_{\lambda}(u) \leq-\frac{p-1}{p}\left\|r_{k}\right\|^{p}<0, \forall k \geq k_{1}
$$

This proves Claim 3.3.
From (1), (3) and (4), it follows that $\varphi_{\lambda}$ maps bounded sets to bounded sets uniformly for $\lambda \in[1,2]$. By $\left(f_{4}\right), \varphi_{\lambda}(-u)=\varphi_{\lambda}(u)$ for all $(\lambda, u) \in[1,2] \times E$. Therefore, condition (1) of Theorem 2 holds. On the other hand, by (3) and ( $f_{2}$ ), condition (2) of Theorem 2 holds. In addition, Claim 3.2 and Claim 3.3 imply that condition (3) of Theorem 2 holds. Therefore, by Theorem 2, we have that for each $k \geq k_{1}$ and $\lambda \in[1,2]$, there exists a sequence $\left\{u_{m}^{k}(\lambda)\right\}_{m=1}^{\infty} \subset E$ such that

$$
\begin{equation*}
\sup _{m}\left\|u_{m}^{k}(\lambda)\right\|<\infty, \varphi_{\lambda}^{\prime}\left(u_{m}^{k}(\lambda)\right) \rightarrow 0 \text { and } \varphi_{\lambda}\left(u_{m}^{k}(\lambda)\right) \rightarrow \zeta_{k}(\lambda), \text { as } m \rightarrow \infty \tag{18}
\end{equation*}
$$

where

$$
\zeta_{k}(\lambda)=\inf _{\gamma \in \Gamma_{k}} \max _{u \in B_{k}} \varphi_{\lambda}(\gamma(u)), \forall \lambda \in[1,2],
$$

with $B_{k}:=\left\{u \in Y_{k}:\|u\| \leq r_{k}\right\}$ and $\Gamma_{k}:=\left\{\gamma \in C\left(B_{k}, X\right): \gamma\right.$ is odd, $\left.\left.\gamma\right|_{\partial B_{k}}=\mathrm{id}\right\}$.
Furthermore, it follows from the proof of Claim 3.2 that $\zeta_{k}(\lambda) \in\left[c_{k}, d_{k}\right]$ for all $k \geq k_{1}$ and $\lambda \in[1,2]$, where $d_{k}:=\max _{u \in B_{k}} \varphi_{1}(u)$ and $c_{k}:=\frac{1}{4 p} \rho_{k}^{p}$.

In view of (18), for each $k \geq k_{1}$, we can choose $\lambda_{n} \rightarrow 1$ (depending on $k$ ) and get the corresponding sequences satisfying

$$
\begin{equation*}
\sup _{m}\left\|u_{m}^{k}\left(\lambda_{n}\right)\right\|<\infty \text { and } \varphi_{\lambda_{n}}^{\prime}\left(u_{m}^{k}\left(\lambda_{n}\right)\right) \rightarrow 0, \text { as } m \rightarrow \infty \tag{19}
\end{equation*}
$$

Claim 3.4. For each $\lambda_{n}$ given above, the sequence $\left\{u_{m}^{k}\left(\lambda_{n}\right)\right\}_{m=1}^{\infty}$ has a strongly convergent subsequence. Furthermore, if we assume that $\lim _{m \rightarrow \infty} u_{m}^{k}\left(\lambda_{n}\right)=u_{n}^{k} \in E$ for all $n \in \mathbb{N}$ and $k \geq k_{1}$, then the sequence $\left\{u_{n}^{k}\right\}_{n=1}^{\infty}$ is bounded.

In fact, by (19), without loss of generality, we may assume

$$
\begin{equation*}
u_{m}^{k}\left(\lambda_{n}\right) \rightharpoonup u_{n}^{k} \text { in } E, \text { as } m \rightarrow \infty \tag{20}
\end{equation*}
$$

for some $u_{n}^{k} \in E$. By Lemma 2, we have

$$
\begin{align*}
& u_{m}^{k}\left(\lambda_{n}\right) \rightarrow u_{n}^{k} \text { in } L^{p}\left(\mathbb{R}^{N}\right), \text { as } m \rightarrow \infty  \tag{21}\\
& u_{m}^{k}\left(\lambda_{n}\right) \rightarrow u_{n}^{k} \text { in } L^{\alpha}\left(\mathbb{R}^{N}\right), \text { as } m \rightarrow \infty
\end{align*}
$$

From the choice of the sequence $\left\{u_{m}^{k}\left(\lambda_{n}\right)\right\}_{m=1}^{\infty}$, we have

$$
\begin{equation*}
\left|\left\langle\varphi_{\lambda_{n}}^{\prime}\left(u_{m}^{k}\left(\lambda_{n}\right)\right), u_{m}^{k}\left(\lambda_{n}\right)\right\rangle\right| \leq \varepsilon_{m}, \varepsilon_{m} \downarrow 0 \tag{22}
\end{equation*}
$$

which implies that

$$
\begin{equation*}
\left|\left\langle A^{\prime}\left(u_{m}^{k}\left(\lambda_{n}\right)\right), u_{m}^{k}\left(\lambda_{n}\right)-u_{n}^{k}\right\rangle\right|-\lambda_{n} \int_{\mathbb{R}^{N}} f\left(x, u_{m}^{k}\left(\lambda_{n}\right)\right)\left(u_{m}^{k}\left(\lambda_{n}\right)-u_{n}^{k}\right) d x \leq \varepsilon_{m} \tag{23}
\end{equation*}
$$

for all $m \geq 1$.
From Lemma 3, hypothesis $\mathbf{H}(\mathbf{f})\left(f_{1}\right)$ and (21), we obtain

$$
\begin{equation*}
\lim _{m \rightarrow \infty} \int_{\mathbb{R}^{N}} f\left(x, u_{m}^{k}\left(\lambda_{n}\right)\right)\left(u_{m}^{k}\left(\lambda_{n}\right)-u_{n}^{k}\right) d x=0 \tag{24}
\end{equation*}
$$

Using (22) (23) and (24), we obtain

$$
\begin{equation*}
\lim \sup _{m \rightarrow \infty}\left\langle A^{\prime}\left(u_{m}^{k}\left(\lambda_{n}\right)\right), u_{m}^{k}\left(\lambda_{n}\right)-u_{n}^{k}\right\rangle \leq 0 \tag{25}
\end{equation*}
$$

Because $A^{\prime}$ is a mapping of $\left(S_{+}\right)$-type, from (25), it follows that

$$
\begin{equation*}
u_{m}^{k}\left(\lambda_{n}\right) \rightarrow u_{n}^{k} \text { in } E, \text { as } m \rightarrow \infty \tag{26}
\end{equation*}
$$

It remains to prove that the sequence $\left\{u_{n}^{k}\right\}_{n=1}^{\infty}$ is bounded. Suppose that this is not true. Then passing to a subsequence is necessary, we can assume that $\left\|u_{n}^{k}\right\| \rightarrow \infty$ as $n \rightarrow \infty$. Set $w_{n}^{k}=\frac{u_{n}^{k}}{\left\|u_{n}^{k}\right\|}, n \geq 1$.

We may assume that

$$
\begin{align*}
& w_{n}^{k} \rightharpoonup w_{0}^{k} \text { in } E \\
& w_{n}^{k} \rightarrow w_{0}^{k} \text { in } L^{p}\left(\mathbb{R}^{N}\right) \\
& w_{n}^{k} \rightarrow w_{0}^{k} \text { in } L^{\alpha}\left(\mathbb{R}^{N}\right)  \tag{27}\\
& w_{n}^{k}(x) \rightarrow w_{0}^{k}(x) \text { a.e. on } \mathbb{R}^{N} .
\end{align*}
$$

If $w_{0}^{k}=0$, we choose a sequence $\left\{t_{n}\right\}_{n=1}^{\infty} \subset[0,1]$ such that

$$
\begin{equation*}
\varphi_{\lambda_{n}}\left(t_{n} w_{n}^{k}\right)=\max _{t \in[0,1]} \varphi_{\lambda_{n}}\left(t w_{n}^{k}\right) \tag{28}
\end{equation*}
$$

For any $R>0$, we set $\bar{w}_{n}^{k}=\sqrt[p]{p R} w_{n}^{k}$. Combining (3), (4), (27) and ( $f_{2}$ ), we have

$$
\begin{equation*}
0 \leq \int_{\mathbb{R}^{N}} F\left(x, \bar{w}_{n}^{k}\right) d x \leq c_{1} \int_{\mathbb{R}^{N}}\left(\left|\bar{w}_{n}^{k}\right|^{p}+\left|\bar{w}_{n}^{k}\right|^{\alpha}\right) d x \rightarrow 0 \text { as } n \rightarrow \infty . \tag{29}
\end{equation*}
$$

Thus, for sufficiently large $n$, we have

$$
\begin{aligned}
\varphi_{\lambda_{n}}\left(t_{n} u_{n}^{k}\right) & \geq \varphi_{\lambda_{n}}\left(\bar{w}_{n}^{k}\right) \\
& =\frac{1}{p}\left\|\bar{w}_{n}^{k}\right\|^{p}-\lambda_{n} \int_{\mathbb{R}^{N}} F\left(x, \bar{w}_{n}^{k}\right) d x \\
& =R-\lambda_{n} \int_{\mathbb{R}^{N}} F\left(x, \bar{w}_{n}^{k}\right) d x \\
& \geq \frac{R}{2}
\end{aligned}
$$

which implies that

$$
\lim _{n \rightarrow \infty} \varphi_{\lambda_{n}}\left(t_{n} u_{n}^{k}\right)=+\infty
$$

On the one hand, it is clear from (18) (19) and (26) that

$$
\begin{equation*}
\varphi_{\lambda_{n}}^{\prime}\left(u_{n}^{k}\right)=0, \varphi_{\lambda_{n}}\left(u_{n}^{k}\right) \in\left[c_{k}, d_{k}\right], \forall n \in \mathbb{N} \text { and } k \geq k_{1} \tag{30}
\end{equation*}
$$

On the other hand, from (28) it follows that

$$
\begin{equation*}
0=\left.t_{n} \frac{d}{d t}\right|_{t=t_{n}} \varphi_{\lambda_{n}}\left(t u_{n}^{k}\right)=\left\langle\varphi_{\lambda_{n}}^{\prime}\left(t_{n} u_{n}^{k}\right), t_{n} u_{n}^{k}\right\rangle \tag{31}
\end{equation*}
$$

So from (3), (5), (28) and condition $\left(f_{3}\right)$, we conclude that

$$
\begin{aligned}
\varphi_{\lambda_{n}}\left(u_{n}^{k}\right) & =\varphi_{\lambda_{n}}\left(u_{n}^{k}\right)-\frac{1}{p}\left\langle\varphi_{\lambda_{n}}^{\prime}\left(u_{n}^{k}\right), u_{n}^{k}\right\rangle \\
& =\frac{\lambda_{n}}{p} \int_{\mathbb{R}^{N}} \mathcal{F}\left(x, u_{n}^{k}\right) d x \\
& \geq \frac{\lambda_{n}}{\theta p} \int_{\mathbb{R}^{N}} \mathcal{F}\left(x, t_{n} u_{n}^{k}\right) d x \\
& =\frac{1}{\theta} \varphi_{\lambda_{n}}\left(t_{n} u_{n}^{k}\right)-\frac{1}{\theta p}\left\langle\varphi_{\lambda_{n}}^{\prime}\left(t_{n} u_{n}^{k}\right), t_{n} u_{n}^{k}\right\rangle \\
& =\frac{1}{\theta} \varphi_{\lambda_{n}}\left(t_{n} u_{n}^{k}\right) \rightarrow+\infty \text { as } n \rightarrow \infty
\end{aligned}
$$

This provides a contradiction to (30).

If $w_{0}^{k} \neq 0$, then $\Theta:=\left\{x \in \mathbb{R}^{N}: w_{0}^{k}(x) \neq 0\right\}$ has a positive Lebesgue measure. Thus for $x \in \Theta$, we have $u_{n}^{k}(x) \rightarrow \infty$. This, together with (3), (27) and $\left(f_{2}\right)$, implies

$$
\begin{aligned}
\frac{1}{p}-\frac{\varphi_{\lambda_{n}}\left(u_{n}^{k}\right)}{\left\|u_{n}^{k}\right\|^{p}} & =\lambda_{n} \int_{\mathbb{R}^{N}} \frac{F\left(x, u_{n}^{k}\right)}{\left\|u_{n}^{k}\right\|^{p}} d x \\
& \geq \lambda_{n} \int_{\Theta}\left|w_{n}^{k}\right|^{p} \frac{F\left(x, u_{n}^{k}\right)}{\left|u_{n}^{k}\right|^{p}} d x \rightarrow \infty, \text { as } n \rightarrow \infty
\end{aligned}
$$

which is a contradiction to (30) again. Therefore, the sequence $\left\{u_{n}^{k}\right\}_{n=1}^{\infty}$ is bounded. This proves that Claim 3.4 is true.

In view of Claim 3.4 and (30), for each $k \geq k_{1}$, using similar arguments in the proof of Claim 3.4, we can also prove that the sequence $\left\{u_{n}^{k}\right\}_{n=1}^{\infty}$ has a strong convergent subsequence with the limit $u^{k}$ being just a critical point of $\varphi_{1}=\varphi$. We already know that $\varphi\left(u^{k}\right) \in\left[c_{k}, d_{k}\right]$ for all $k \geq k_{1}$. Recalling that $c_{k} \rightarrow+\infty$ as $k \rightarrow \infty$, we deduce the existence of infinitely many nontrivial critical points of $\varphi$. Therefore, problem $(P)$ possesses infinitely many nontrivial solutions. The proof of Theorem 1 is complete.

## Acknowledgement

The author would like to thank the referees for their helpful suggestions.

## References

[1] T. Bartsch, Z. Q. Wang, Existence and multiplicity results for some superlinear elliptic problems on $\mathbb{R}^{N}$, Comm. Partial Diff. Eq. 20(1995), 1725-1741.
[2] T. Bartsch, Z. Q. WANG, Sign changing solutions of nonlinear Schrödinger equations, Topol. Methods Nonlinear Anal. 13(1999), 191-198.
[3] Y.CHEN, Y. Q. Li, Infinitely many solutions for a semilinear elliptic equation with sign-changing potential, Bound. Value Probl. 2009(2009), ID 532546, DOI: 10.1155/2009/532546.
[4] J. I. Daz, Nonlinear Partial Differential Equations and Free Boundaries, Vol. I.: Elliptic Equations, Research Notes in Math., no. 106, Pitman, Boston, MA, 1985.
[5] X. L. Fan, Q. H. Zhang, Existence of solutions for $p(x)$-Laplacian Dirichlet problem, Nonlinear Anal. 52(2003), 1843-1853.
[6] B. Ge, Q. M. Zhou, X. P. Xue, Infinitely many solutions for a differential inclusion problem in $\mathbb{R}^{N}$ involving $p(x)$-Laplacian and oscillatory terms, Z. Angew. Math. Phys. $\mathbf{6 3}(2012), 691-711$.
[7] S. B. Liv, On ground states of superlinear p-Laplacian equations in $\mathbb{R}^{N}$, J. Math. Anal. Appl. 361(2010), 48-58.
[8] S. B. Liv, S. J. Li, Infinitely many solutions for a superlinear elliptic equation, Acta Math. Sinica (Chin. Ser.). 46(2003), 625-630.
[9] Z. L. Liu, Z. Q. Wang, Solutions nodales de multi-bosses ayant un nombre de domaines nodales prescrites: I, Ann. I. H. Poincaré-AN. 22(2005), 597-608.
[10] Z. L. Liu, Z. Q. WANG, Solutions nodales de multi-bosses ayant un nombre de domaines nodales prescrites: II, Ann. I. H. Poincaré-AN. 22(2005), 609-631.
[11] Y. Q. Li, Z. Q. WANG, J. Zhang, Ground states of nonlinear Schrödinger equations with potentials, Ann. I. H. Poincaré-AN. 23(2006), 829-837.
[12] Z. L. Liu, Z. Q. Wang, Schrödinger equations with concave and convex nonlinearities, Z. Angew. Math. Phys. 56(2005), 609-629.
[13] C. G. Liu, Y. Q. Zheng, Existence of nontrivial solutions for p-Laplacian equations in $\mathbb{R}^{N}$, J. Math. Anal. Appl. 380(2011), 669-679.
[14] M. Schechter, W. M. Zou, Sign-changing critical points from linking type theorems, Trans. Amer. Math. Soc. 358(2006), 5293-5318.
[15] X. H. Tang, Infinitely many solutions for semilinear Schrödinger equations with signchanging potential and nonlinearity, J. Math. Anal. Appl. 401(2013), 407-415.
[16] J. Zeng, Y. Q. Li, Existence of solutions for an elliptic equation with indefinite weight, Nonlinear Anal. 66(2007), 2512-2519.
[17] Q. Y. Zhang, B. Xu, Multiplicity of solutions for a class of semilinear Schrödinger equations, J. Math. Anal. Appl. 377(2011), 834-840.
[18] W. M. Zou, Variant fountain theorems and their applications, Manuscripta Math. 104(2001), 343-358.


[^0]:    *This work is supported by the Fundamental Research Funds for the Central Universities (No. DL12BC10), the New Century Higher Education Teaching Reform Project of Heilongjiang Province in 2012 (No. JG2012010012), the Humanities and Social Sciences Foundation of the Educational Commission of Heilongjiang Province of China (No. 12544026).
    ${ }^{\dagger}$ Corresponding author. Email addresses: zhouqingmei2008@163.com (Q.M.Zhou), zdhwkq@gmail.com (K. Q. Wang)

