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Generalized perspectives of functions of several variables

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Abstract. In this paper, we introduce the notion of multivariate generalized perspectives and verify the necessary and sufficient conditions for operator convexity (resp. concavity) of this notion. We also establish the crossing of the multivariate generalized perspective of regular operator mappings under completely positive linear maps and partial traces.

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1. Introduction and preliminaries

Effros [4] considered the case where each pair in the argument of the perspective function consists of commuting operators and proved in this way that the perspective of an operator convex function is operator convex as a function. A fully non-commutative perspective of the one variable function f defined in [3] by setting

$$\mathcal{P}_f(A,B) = B^{1/2} f(B^{-1/2} A B^{-1/2}) B^{1/2}$$

and the generalized perspective of two variables (associated with f and h) defined by

$$\mathcal{P}_{f\Delta h}(A,B) = h(B)^{1/2} f(h(B)^{-1/2} A h(B)^{-1/2}) h(B)^{1/2},$$

where A is a self-adjoint operator and B is a strictly positive operator on a Hilbert space $\mathcal{H}$ with spectra in the closed interval $\mathbb{I}$ containing 0. Note that the identity $\mathcal{P}_{f\Delta h}(A,B) = \mathcal{P}_f(A,h(B))$ expresses the generalized perspective in terms of the non-commutative perspective. The main results of [4] are generalized in [3] for the non-commutative case and the necessary and sufficient conditions for joint convexity (resp. concavity) of the perspective and generalized perspective functions are provided. As an application of these results, Nikoufar et al. [10] gave the simplest proof of Lieb concavity and Ando convexity theorem (see also [11, 12, 1]).

Hansen [6] introduced the notion of regular operator mappings of several variables generalizing the notion of the spectral function of Davis for functions of one variable. Then, he generalized the notion of the perspective of a regular mapping of several variables and defined the geometric mean for any number of operator variables.

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Zhang [13] established operator concavity (resp. convexity) of some functions of two or three variables by using the perspectives of the regular operator mappings of one or several variables and specified operator concavity (resp. convexity) of the Fréchet differential mapping associated with some functions.

Let $\mathcal{B}(\mathcal{H})$ and $\mathcal{B}(\mathcal{H})_{sa}$ be the C^*-algebra of all bounded linear operators and the C^*-subalgebra of all self–adjoint bounded linear operators on a Hilbert space $\mathcal{H}$, respectively. The notion of a regular mapping generalizes the notion of the spectral function of Davis for functions of one variable, the notion of the regular matrix mapping of two variables [8], and the notion of the regular operator mapping of two variables [5, Definition 2.1]. In [6, Definition 2.1], Hansen defined this notion as follows. Let $F : \mathcal{D}_n \longrightarrow \mathcal{B}(\mathcal{H})$ be a mapping of n variables defined in a convex domain $\mathcal{D}_n \subseteq \mathcal{B}(\mathcal{H})_{sa} \times \ldots \times \mathcal{B}(\mathcal{H})_{sa}$. Then, F is regular if the domain $\mathcal{D}_n$ is invariant under unitary transformations of $\mathcal{H}$ and

$$F(u^*x_1u, \ldots, u^*x_nu) = u^*F(x_1, \ldots, x_n)u$$

for every $(x_1, \ldots, x_n) \in \mathcal{D}_n$ and every unitary u on $\mathcal{H}$. For mutually orthogonal projections p and q acting on $\mathcal{H}$ and arbitrary n-tuples $(x_1, \ldots, x_n)$ and $(y_1, \ldots, y_n)$ of operators in $\mathcal{D}_n$ such that the compressed tuples $(px_1p, \ldots, px_np)$ and $(qy_1q, \ldots, qy_nq)$ are in the domain $\mathcal{D}_n$, the n-tuple of diagonal block matrices $(px_1p+qy_1q, \ldots, px_np+qy_nq)$ is also in the domain $\mathcal{D}_n$ and

$$F(px_1p + qy_1q, \ldots, px_np + qy_nq) = pF(px_1p, \ldots, px_np)p + qF(qy_1q, \ldots, qy_nq)q.$$

Denote by $\mathcal{D}_n^+$ the positive convex domain of strictly positive operators $A_1, \ldots, A_n$ acting on a Hilbert space $\mathcal{H}$. Let $F : \mathcal{D}_n^+ \longrightarrow \mathcal{B}(\mathcal{H})$ be a regular mapping. The perspective mapping $\mathcal{P}_F$ of $n + 1$ variables is the mapping

$$\mathcal{P}_F(A_1, \ldots, A_n, B) = B^{1/2}F(B^{-1/2}A_1B^{-1/2}, \ldots, B^{-1/2}A_nB^{-1/2})B^{1/2}$$

defined in the domain $\mathcal{D}_{n+1}^+$ for strictly positive operators $A_1, \ldots, A_n$ and B acting on a Hilbert space $\mathcal{H}$.

In Section 2, we define multivariate generalized perspectives and prove a sub-homogeneous form of Jensen's inequality for regular operator mappings of several variables. This leads us under some conditions to demonstrate the multivariate generalized perspectives are operator convex (resp. concave). In Section 3, we declare the crossing of the multivariate generalized perspective of regular operator mappings through completely positive linear maps and partial traces.

2. Multivariate generalized perspectives

In this section, we define the multivariate generalized perspective of regular operator mappings of several variables. We confirm a sub-homogeneous form of Jensen's inequality for regular operator mappings of several variables. We apply this to prove operator convexity (resp. concavity) of the multivariate generalized perspectives. For other works on operator functions of several variables see [13] and references therein.

We know that the perspective of an operator convex function is operator convex as a function of two variables [3, Theorem 2.2]. Hansen proved a similar result for a regular mapping of n operator variables [6]. Since $\mathcal{P}_F(A_1,\ldots,A_n,1) = F(A_1,\ldots,A_n)$, we confirm that the converse of this result is also true.

Theorem 1. *Let $F : \mathcal{D}_n^+ \longrightarrow \mathcal{B}(\mathcal{H})$ be a regular mapping. The perspective $\mathcal{P}_F$ is operator convex if and only if F is operator convex.*

Corollary 1. *Let $F : \mathcal{D}_n^+ \longrightarrow \mathcal{B}(\mathcal{H})$ be a regular mapping. If F is operator convex (resp. concave), then $\mathcal{P}_F$ is jointly subadditive (resp. superadditive).*

Proof. We observe that $\mathcal{P}_F$ is positively homogenous in the sense that

$$\mathcal{P}_F(\alpha A_1,\ldots,\alpha A_n,\alpha B) = \alpha \mathcal{P}_F(A_1,\ldots,A_n,B)$$

for $\alpha > 0$. Let $\{A_{i1}, A_{i2},\ldots,A_{in}\}$ and $\{B_1, B_2,\ldots,B_n\}$ be two sequences of strictly positive operators on a Hilbert space $\mathcal{H}$ for $i = 1,..,n$. According to Theorem 1 and operator convexity of F, we have

$$\begin{aligned}
&\mathcal{P}_F\Big(\sum_{i=1}^n A_{i1},\ldots,\sum_{i=1}^n A_{in},\sum_{i=1}^n B_i\Big)\\
&\quad= \mathcal{P}_F\Big((n+1)\sum_{i=1}^n \frac{1}{n+1}A_{i1},\ldots,(n+1)\sum_{i=1}^n \frac{1}{n+1}A_{in},(n+1)\sum_{i=1}^n \frac{1}{n+1}B_i\Big)\\
&\quad= (n+1)\mathcal{P}_F\Big(\sum_{i=1}^n \frac{1}{n+1}A_{i1},\ldots,\sum_{i=1}^n \frac{1}{n+1}A_{in},\sum_{i=1}^n \frac{1}{n+1}B_i\Big)\\
&\quad\le (n+1)\sum_{i=1}^n \frac{1}{n+1}\mathcal{P}_F\Big(A_{i1},\ldots,A_{in},B_i\Big)\\
&\quad= \sum_{i=1}^n \mathcal{P}_F\Big(A_{i1},\ldots,A_{in},B_i\Big).
\end{aligned}$$

□

After this, throughout the paper we assume that h is a real valued and continuous function defined on $[0,\infty)$. We say that h is strictly positive if $h(A) > 0$ for every $A > 0$.

Definition 1. *Let $F : \mathcal{D}_n^+ \longrightarrow \mathcal{B}(\mathcal{H})$ be a regular mapping. The generalized perspective mapping $\mathcal{P}_{F\Delta h}$ (associated with F and h) is the mapping defined in the domain $\mathcal{D}_{n+1}^+$ by setting*

$$\begin{aligned}
&\mathcal{P}_{F\Delta h}(A_1,\ldots,A_n,B)\\
&\qquad= h(B)^{1/2}F(h(B)^{-1/2}A_1h(B)^{-1/2},\ldots,h(B)^{-1/2}A_nh(B)^{-1/2})h(B)^{1/2}
\end{aligned}$$

for strictly positive operators $A_1,\ldots,A_n$ and B acting on $\mathcal{H}$.

It should be noted that the identity $\mathcal{P}_{F\Delta h}(A_1,\dots,A_n,B)=\mathcal{P}_F(A_1,\dots,A_n,h(B))$ declares the generalized perspective in terms of Hansen's perspective.

In [6] Hansen proved an affine form of Jensen's inequality for the regular operator mappings. For our purpose we prove the following sub-homogeneous form of Jensen's inequality for the regular operator mappings. A sub-homogeneous form of Jensen's inequality for the functions of one variable is proved in [7]. The proof of the following result is similar to that of [6, Theorem 2.2] (see also [9, Theorem 2.2]) and we omit it. Here, we state the result for operators T_1 and T_2 acting on a Hilbert space $\mathcal{H}$ with $T_1^*T_1+T_2^*T_2\le 1$. In [6, Theorem 2.2] the result is proved for the case where $T_1^*T_1+T_2^*T_2=1$. We need to apply the following theorem to our main result Theorem 3.

Theorem 2. *Let $F:\mathcal{D}_n\longrightarrow\mathcal{B}(\mathcal{H})_{sa}$ be a convex regular mapping. If $F(0,\dots,0)\le 0$ and T_1,T_2 are operators acting on $\mathcal{H}$ with $T_1^*T_1+T_2^*T_2\le 1$, then*

$$\begin{aligned}&F(T_1^*A_1T_1+T_2^*A_1'T_2,\dots,T_1^*A_nT_1+T_2^*A_n'T_2)\\&\qquad\le T_1^*F(A_1,\dots,A_n)T_1+T_2^*F(A_1',\dots,A_n')T_2\end{aligned}\tag{1}$$

for n-tuples $(A_1,\dots,A_n)$ and $(A_1',\dots,A_n')$ in $\mathcal{D}_n$.

In the following theorem, we extend [4, Theorem 3.2] and [3, Theorem 2.5, Corollary 2.6] for the regular mappings of several variables.

Theorem 3. *Suppose that h is a strictly positive function on $(0,\infty)$ and $F:\mathcal{D}_n^+\longrightarrow\mathcal{B}(\mathcal{H})$ is a regular mapping.*

(*i*) *If F is operator convex and h is operator concave with $F(0,\dots,0)\le 0$, then the generalized perspective mapping $\mathcal{P}_{F\Delta h}$ is operator convex.*

(*ii*) *If the generalized perspective mapping $\mathcal{P}_{F\Delta h}$ is operator convex (resp. concave), then F is operator convex (resp. concave).*

(*iii*) *If $F(0,\dots,0)>0$ and the generalized perspective mapping $\mathcal{P}_{F\Delta h}$ is operator convex (resp. concave), then h is operator convex (resp. concave).*

(*iv*) *If $F(0,\dots,0)<0$ and the generalized perspective mapping $\mathcal{P}_{F\Delta h}$ is operator convex (resp. concave), then h is operator concave (resp. convex).*

Proof. (i): Let $(A_1,\dots,A_n,B_1)$ and $(A_1',\dots,A_n',B_2)$ be in $\mathcal{D}_{n+1}^+$ and $0\le c\le 1$. Put $B=cB_1+(1-c)B_2$. Define

$$\begin{aligned}T_1&=c^{1/2}h(B_1)^{1/2}h(B)^{-1/2},\\T_2&=(1-c)^{1/2}h(B_2)^{1/2}h(B)^{-1/2}.\end{aligned}$$

So, $T_1^*T_1+T_2^*T_2\le 1$, by operator concavity of h. From operator convexity of F and

Theorem 2, we have

$$
\begin{aligned}
&\mathcal{P}_{F\Delta h}(cA_1+(1-c)A_1',\ldots,cA_n+(1-c)A_n',cB_1+(1-c)B_2)\\
&\quad= h(B)^{1/2}F\Big(h(B)^{-1/2}(cA_1+(1-c)A_1')h(B)^{-1/2},\ldots,\\
&\qquad h(B)^{-1/2}(cA_n+(1-c)A_n')h(B)^{-1/2}\Big)h(B)^{1/2}\\
&\quad= h(B)^{1/2}F\Big(T_1^*h(B_1)^{-1/2}A_1h(B_1)^{-1/2}T_1+T_2^*h(B_2)^{-1/2}A_1'h(B_2)^{-1/2}T_2,\ldots,\\
&\qquad T_1^*h(B_1)^{-1/2}A_nh(B_1)^{-1/2}T_1+T_2^*h(B_2)^{-1/2}A_n'h(B_2)^{-1/2}T_2\Big)h(B)^{1/2}\\
&\quad\le h(B)^{1/2}\Big(T_1^*F(h(B_1)^{-1/2}A_1h(B_1)^{-1/2},\ldots,h(B_1)^{-1/2}A_nh(B_1)^{-1/2})T_1\\
&\qquad +T_2^*F(h(B_2)^{-1/2}A_1'h(B_2)^{-1/2},\ldots,h(B_2)^{-1/2}A_n'h(B_2)^{-1/2})T_2\Big)h(B)^{1/2}\\
&\quad= ch(B_1)^{1/2}F(h(B_1)^{-1/2}A_1h(B_1)^{-1/2},\ldots,h(B_1)^{-1/2}A_nh(B_1)^{-1/2})h(B_1)^{1/2}\\
&\qquad +(1-c)h(B_2)^{1/2}F(h(B_2)^{-1/2}A_1'h(B_2)^{-1/2},\ldots,\\
&\qquad h(B_2)^{-1/2}A_n'h(B_2)^{-1/2})h(B_2)^{1/2}\\
&\quad= c\mathcal{P}_{F\Delta h}(A_1,\ldots,A_n,B_1)+(1-c)\mathcal{P}_{F\Delta h}(A_1',\ldots,A_n',B_2).
\end{aligned}
$$

(ii): Note that $F(A_1,\ldots,A_n)=\frac{1}{h(1)}\mathcal{P}_{F\Delta h}(h(1)A_1,\ldots,h(1)A_n,1)$. We have

$$
\begin{aligned}
&F(cA_1+(1-c)A_1',\ldots,cA_n+(1-c)A_n')\\
&\quad=\frac{1}{h(1)}\mathcal{P}_{F\Delta h}(ch(1)A_1+(1-c)h(1)A_1',\ldots,ch(1)A_n+(1-c)h(1)A_n'),1)\\
&\quad\le\frac{1}{h(1)}(c\mathcal{P}_{F\Delta h}(h(1)A_1,\ldots,h(1)A_n,1)+(1-c)\mathcal{P}_{F\Delta h}(h(1)A_1',\ldots,h(1)A_n',1))\\
&\quad= cF(A_1,\ldots,A_n)+(1-c)F(A_1',\ldots,A_n').
\end{aligned}
$$

(iii): It is clear that $h(B)=\frac{1}{F(0,\ldots,0)}\mathcal{P}_{F\Delta h}(0,\ldots,0,B)$ and so that

$$
\begin{aligned}
h(cB_1+(1-c)B_2)&=\frac{1}{F(0,\ldots,0)}\mathcal{P}_{F\Delta h}(0,\ldots,0,cB_1+(1-c)B_2)\\
&\le\frac{c}{F(0,\ldots,0)}\mathcal{P}_{F\Delta h}(0,\ldots,0,B_1)+\frac{1-c}{F(0,\ldots,0)}\mathcal{P}_{F\Delta h}(0,\ldots,0,B_2)\\
&= ch(B_1)+(1-c)h(B_2).
\end{aligned}
$$

(iv) The proof is similar to that of (iii). □

Remark 1. *As a simple consequence of Theorem 3 (i), if F and h are operator concave with $F(0,\ldots,0)\ge 0$, then the generalized perspective mapping $\mathcal{P}_{F\Delta h}$ is operator concave.*

Corollary 2. *Suppose that h is a strictly positive function defined on $[0,\infty)$ and $F:\mathcal{D}_n^+\longrightarrow\mathcal{B}(\mathcal{H})$ is a regular mapping. Let h be operator concave and positively homogenous.*

(i) *If F is operator convex with $F(0,\ldots,0)\leq 0$, then the generalized perspective mapping $\mathcal{P}_{F\Delta h}$ is jointly subadditive.*

(ii) *If F is operator concave with $F(0,\ldots,0)\geq 0$, then the generalized perspective mapping $\mathcal{P}_{F\Delta h}$ is jointly superadditive.*

Proof. (i): Let $\{A_{i1},A_{i2},\ldots,A_{in}\}$ and $\{B_1,B_2,\ldots,B_n\}$ be two sequences of strictly positive operators on a Hilbert space $\mathcal{H}$ for $i=1,..,n$. Since h is positively homogenous, we observe that $\mathcal{P}_{F\Delta h}$ is jointly homogenous. Regarding Theorem 3 (i), $\mathcal{P}_{f\Delta h}$ is jointly convex and so

$$\begin{aligned}
&\mathcal{P}_{F\Delta h}\Big(\sum_{i=1}^{n}A_{i1},\ldots,\sum_{i=1}^{n}A_{in},\sum_{i=1}^{n}B_i\Big)\\
&\quad=\mathcal{P}_{F\Delta h}\Big((n+1)\sum_{i=1}^{n}\frac{1}{n+1}A_{i1},\ldots,(n+1)\sum_{i=1}^{n}\frac{1}{n+1}A_{in},(n+1)\sum_{i=1}^{n}\frac{1}{n+1}B_i\Big)\\
&\quad=(n+1)\mathcal{P}_{f\Delta h}\Big(\sum_{i=1}^{n}\frac{1}{n+1}A_{i1},\ldots,\sum_{i=1}^{n}\frac{1}{n+1}A_{in},\sum_{i=1}^{n}\frac{1}{n+1}B_i\Big)\\
&\quad\leq(n+1)\sum_{i=1}^{n}\frac{1}{n+1}\mathcal{P}_{F\Delta h}\Big(A_{i1},\ldots,A_{in},B_i\Big)\\
&\quad=\sum_{i=1}^{n}\mathcal{P}_{f\Delta h}(A_{i1},\ldots,A_{in},B_i).
\end{aligned}$$

(ii): It follows from Remark 1. □

Corollary 3. *Suppose that h is a strictly positive function defined on $[0,\infty)$ and $F:\mathcal{D}_n^+\longrightarrow\mathcal{B}(\mathcal{H})$ is a regular mapping.*

(i) *If the generalized perspective mapping $\mathcal{P}_{F\Delta h}$ is operator convex (resp. concave) and F is positively homogenous, then F is jointly subadditive (resp. superadditive).*

(ii) *If $F(0,\ldots,0)>0$, the generalized perspective mapping $\mathcal{P}_{F\Delta h}$ is operator convex (resp. concave), and h is positively homogenous, then h is subadditive (resp. superadditive).*

(iii) *If $F(0,\ldots,0)<0$, the generalized perspective mapping $\mathcal{P}_{F\Delta h}$ is operator convex (resp. concave), and h is positively homogenous, then h is superadditive (resp. subadditive).*

3. Generalized perspectives and completely positive linear maps

In this section, we study the filtering of the multivariate generalized perspective of a regular operator mapping through completely positive linear maps and partial traces.

Theorem 4. *Let $\Phi : \mathcal{B}(\mathcal{H}) \to \mathcal{B}(\mathcal{K})$ be a completely positive linear mapping between operators on Hilbert spaces of finite dimensions, and let $F : \mathcal{D}_n^+ \to \mathcal{B}(\mathcal{K})$ be a convex regular mapping. Then,*

$$\mathcal{P}_{F\Delta h}(\Phi(A_1), \ldots, \Phi(A_n), \Phi(B)) \leq \Phi(\mathcal{P}_{F\Delta h}(A_1, \ldots, A_n, B)) \tag{2}$$

for operators $(A_1, \ldots, A_n, B)$ in $\mathcal{D}_{n+1}^+$, where $\mathcal{P}_{F\Delta h}$ is the generalized perspective mapping (associated to F and h) and $\Phi \circ h = h \circ \Phi$ on the strictly positive operators.

Proof. Due to [9, Theorem 3.1], we get

$$\begin{aligned}
\mathcal{P}_{F\Delta h}(\Phi(A_1), \ldots, \Phi(A_n), \Phi(B)) &= \mathcal{P}_F(\Phi(A_1), \ldots, \Phi(A_n), h(\Phi(B))) \\
&= \mathcal{P}_F(\Phi(A_1), \ldots, \Phi(A_n), \Phi(h(B)) \\
&\leq \Phi(\mathcal{P}_F(A_1, \ldots, A_n, h(B))) \\
&= \Phi(\mathcal{P}_{F\Delta h}(A_1, \ldots, A_n, B)).
\end{aligned}$$

□

Remark 2. *Under the hypotheses of Theorem 4, if F is a concave regular mapping, then the reverse inequality is valid in* (2).

We justify the other conditions under which the reverse inequality in (2) is valid.

Definition 2. *Let $F : \mathcal{D}_n^+ \to \mathcal{B}(\mathcal{K})$ be a mapping. We say that F is n-monotone if it is monotone on its nth variable, in the sense that if $A_n \leq B_n$, then $F(A_1, \ldots, A_n) \leq F(B_1, \ldots, B_n)$ for $(A_1, \ldots, A_n), (B_1, \ldots, B_n)$ in $\mathcal{D}_n^+$.*

Theorem 5. *Let $\Phi : \mathcal{B}(\mathcal{H}) \to \mathcal{B}(\mathcal{K})$ be a completely positive linear mapping between operators on Hilbert spaces of finite dimensions, and let $F : \mathcal{D}_n^+ \to \mathcal{B}(\mathcal{K})$ be a concave regular mapping. If $\mathcal{P}_F$ is $(n+1)$-monotone and h is concave with $h(0) = 0$, then*

$$\mathcal{P}_{F\Delta h}(\Phi(A_1), \ldots, \Phi(A_n), \Phi(B)) \geq \Phi(\mathcal{P}_{F\Delta h}(A_1, \ldots, A_n, B)). \tag{3}$$

Proof. Applying [2, Corollary], we get $h(\Phi(B)) \geq \Phi(h(B))$. Since F is concave, the reverse inequality holds in [9, Theorem 3.1], so that by the $(n+1)$-monotonicity property of $\mathcal{P}_F$ we conclude

$$\begin{aligned}
\mathcal{P}_{F\Delta h}(\Phi(A_1), \ldots, \Phi(A_n), \Phi(B)) &= \mathcal{P}_F(\Phi(A_1), \ldots, \Phi(A_n), h(\Phi(B))) \\
&\geq \mathcal{P}_F(\Phi(A_1), \ldots, \Phi(A_n), \Phi(h(B)) \\
&\geq \Phi(\mathcal{P}_F(A_1, \ldots, A_n, h(B))) \\
&= \Phi(\mathcal{P}_{F\Delta h}(A_1, \ldots, A_n, B)).
\end{aligned}$$

□

Remark 3. *Under the hypotheses of Theorem 5, if F is convex and regular and h is convex, then the reverse inequality is valid in* (3).

We know that for a bipartite system $\mathcal{H} = \mathcal{H}_1 \otimes \mathcal{H}_2$, $Tr_2 A$ is the partial trace of A on $\mathcal{H}_1$. The function h is the partial trace preserving on $\mathcal{H}_1$ if $h(Tr_2 A) = Tr_2 h(A)$. The fact that the partial trace is completely positive leads to the following corollaries.

Corollary 4. *Consider a bipartite system $\mathcal{H} = \mathcal{H}_1 \otimes \mathcal{H}_2$ of Hilbert spaces $\mathcal{H}_1$ and $\mathcal{H}_2$ of finite dimensions. If $F : \mathcal{D}_n^+ \to \mathcal{B}(\mathcal{K})$ is a convex regular mapping and h is partial trace preserving on $\mathcal{H}_1$, then*

$$\mathcal{P}_{F\Delta h}(Tr_2 A_1, \ldots, Tr_2 A_n, Tr_2 B) \leq Tr_2 \mathcal{P}_{F\Delta h}(A_1, \ldots, A_n, B)$$

for operators $(A_1, \ldots, A_n, B)$ in $\mathcal{D}_{n+1}^+$.

Corollary 5. *Consider a bipartite system $\mathcal{H} = \mathcal{H}_1 \otimes \mathcal{H}_2$ of Hilbert spaces $\mathcal{H}_1$ and $\mathcal{H}_2$ of finite dimensions and a concave regular mapping $F : \mathcal{D}_n^+ \to \mathcal{B}(\mathcal{K})$. If $\mathcal{P}_F$ is $(n+1)$-monotone and h is concave with $h(0) = 0$, then*

$$\mathcal{P}_{F\Delta h}(Tr_2 A_1, \ldots, Tr_2 A_n, Tr_2 B) \geq Tr_2(\mathcal{P}_{F\Delta h}(A_1, \ldots, A_n, B))$$

for operators $(A_1, \ldots, A_n, B)$ in $\mathcal{D}_{n+1}^+$.

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