Matematika 2
15.4.2020.

PRIMITIVNA FUNKCIJA, NEODREĐENI i ODREĐENI INTEGRAL

* **Def** PRIMITIVNOM FUNKCIJOM funkcije $f:[a,b]\to\mathbb{R}$ nazivamo svaku funkciju $F:[a,b]\to\mathbb{R}$ sa svojstvom
$$F'(x)=f(x),\ \forall x\in[a,b].$$

* NEWTON-LEIBNIZOVA FORMULA

Neka je $f:[a,b]\to\mathbb{R}$ neprekidna funkcija na segmentu $[a,b]$. Ako je F bilo koja primitivna funkcija od f na $[a,b]$ onda je
$$\int_a^b f(x)\,dx = F(b)-F(a)$$
$$=F(x)\Big|_a^b=F(b)-F(a).$$

* NEODREĐENI INTEGRAL funkcije f
$$\int f(x)\,dx=\underbrace{F(x)}_{\text{primitivna funkcija od } f}+c,\ c\in\mathbb{R}.$$

* TABLICA NEODREĐENIH INTEGRALA (udžbenik, str. 256)

1) $\int c_1\,dx=c_1x+c,\ c_1,c\in\mathbb{R}$

$\int c_1\,dx=c_1\int 1\,dx=c_1\int x^0\,dx=c_1\cdot\dfrac{x^{0+1}}{0+1}+c=c_1x+c$

2) $\int x^{\alpha}\,dx=\dfrac{x^{\alpha+1}}{\alpha+1}+c,\ \alpha\neq -1$

3) $\int \frac{1}{x}\,dx=\ln|x|+c$

4) $\int a^x\,dx=\dfrac{a^x}{\ln a}+c$

5) $\int e^x dx = e^x + c$

6) $\int \sin x\, dx = -\cos x + c$

7) $\int \cos x\, dx = \sin x + c$

!!!

$$\boxed{\int x^\alpha dx = \frac{x^{\alpha+1}}{\alpha+1} + c \quad \alpha \neq -1}$$

<u>Zad</u> Izračunajte sljedeće integrale:

a) $\int_5^7 5\,dx = 5 \int_5^7 1\,dx = 5 \cdot \frac{x^1}{1}\Big|_5^7 = 5\cdot 7 - 5\cdot 5$ ($1 = x^0$)

$= 5(7-5) = 5\cdot 2 = 10$

b) $\int_{-3}^3 10\,dx = 10 \int_{-3}^3 1\,dx = 10 \cdot \frac{x^1}{1}\Big|_{-3}^3 = 10\cdot 3 - 10\cdot(-3)$ ($1 = x^0$)

$= 10\cdot 3 + 10\cdot 3 = 30 + 30 = 60$

c) $\int_{-2}^2 x^2\,dx = \frac{x^3}{3}\Big|_{-2}^2 = \frac{2^3}{3} - \frac{(-2)^3}{3} = \frac{2^3}{3} + \frac{2^3}{3} = \frac{8}{3} + \frac{8}{3}$

$= \frac{16}{3}$

d) $\int_0^2 13x^3\,dx = 13\int_0^2 x^3 dx = 13 \cdot \frac{x^4}{4}\Big|_0^2 = 13\left(\frac{2^4}{4} - \underbrace{\frac{0^4}{4}}_{=0}\right)$

$= 13\cdot\frac{16}{4} = 13 \cdot 4 = 52$

e) $\int_0^1 (-5x)\,dx = -5\int_0^1 x^1 dx = -5\frac{x^2}{2}\Big|_0^1 = -5\left(\frac{1^2}{2} - \frac{0^2}{2}\right)$

$= -\frac{5}{2}$

f) $\int_1^3 \frac{5}{x}\,dx = 5\int_1^3 \frac{1}{x}\,dx = 5 \cdot \ln|x|\Big|_1^3$

$= 5\cdot\left(\underbrace{\ln|3|}_{=\ln 3} - \underbrace{\ln 1}_{=0}\right) = 5 \cdot \ln 3$

g) $\int_{-1}^{5} x^{-4}\,dx = \frac{x^{-4+1}}{-4+1}\Big|_{-1}^{5} = \frac{x^{-3}}{-3}\Big|_{-1}^{5} = -\frac{1}{3x^3}\Big|_{-1}^{5}$

$= -\frac{1}{3\cdot 5^3} + \frac{1}{3(-1)^3} = -\frac{1}{375} - \frac{1}{3} = -\frac{42}{125}$

h) $\int_{1}^{2} 3x^{-1}\,dx = 3\int_{1}^{2} x^{-1}\,dx = 3\cdot\int_{1}^{2} \frac{1}{x}\,dx$

$= 3\cdot \ln|x|\Big|_{1}^{2} = 3\cdot(\ln 2 - \underbrace{\ln 1}_{=0}) = 3\cdot \ln 2$

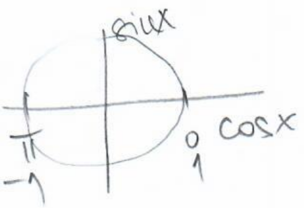
i) $\int_{3}^{4} 5^x\,dx = \frac{5^x}{\ln 5}\Big|_{3}^{4} = \frac{1}{\ln 5}(5^4 - 5^3) = \frac{500}{\ln 5}$

j) $\int_{0}^{2} 7e^x\,dx = 7\int_{0}^{2} e^x\,dx = \underbrace{7e^x\Big|_{0}^{2} = 7\cdot e^2 - 7e^0}_{=7(e^2-e^0)} = 7e^2 - 7$

$= 7(e^2-1)$

k) $\int_{0}^{\pi} (-2\sin x)\,dx = -2\int_{0}^{\pi} \sin x\,dx = -2(-\cos x)\Big|_{0}^{\pi}$

$= 2\cos x\Big|_{0}^{\pi} = 2\cos\pi - 2\cos 0 = 2(-1) - 2\cdot 1 = -2-2 = -4$



l) $\int_{-\pi}^{\pi} \cos x\,dx = \sin x\Big|_{-\pi}^{\pi} = \sin\pi - \sin(-\pi)$

$= \sin\pi - (-\sin\pi))$

$= \sin\pi + \sin\pi = 2\sin\pi = \underline{\underline{0}}$

zad Izračunajte sljedeće integrale:

a) $$\int_{-1}^{1}\left(x-\frac{3}{4}x^2\right)dx = \int_{-1}^{1}x\,dx - \frac{3}{4}\int_{-1}^{1}x^2\,dx$$

$$= \frac{x^2}{2}\Big|_{-1}^{1} \underbrace{- \frac{3}{4}}\cdot\frac{x^3}{3}\Big|_{-1}^{1} = \frac{1^2}{2} - \frac{(-1)^2}{2} \underbrace{- \frac{3}{4}}\left(\frac{1^3}{3} - \underbrace{\frac{(-1)^3}{3}}_{-\frac{1}{3}}\right)$$

$$= \cancel{\frac{1^2}{2}} - \cancel{\frac{1}{2}} - \frac{3}{4}\left(\frac{1}{3}+\frac{1}{3}\right) = -\frac{\cancel{3}}{\cancel{4}_{2}}\cdot\frac{\cancel{2}}{\cancel{3}} = -\frac{1}{2}$$

b) $$\int_{1}^{2}\left(3x^2+\sqrt[3]{x}+5\cdot 6^x\right)dx$$

$$=3\int_{1}^{2}x^2\,dx + \int_{1}^{2}x^{\frac{1}{3}}\,dx + 5\cdot\int_{1}^{2}6^x\,dx$$

$$= \cancel{3}\cdot\frac{x^3}{\cancel{3}}\Big|_{1}^{2} + \underbrace{\frac{x^{\frac{1}{3}+1}}{\frac{1}{3}+1}}\Big|_{1}^{2} + 5\cdot\frac{6^x}{\ln 6}\Big|_{1}^{2} = \underbrace{8-1} + \underbrace{\frac{3}{4}\cdot 2^{\frac{4}{3}} - \frac{3}{4}\cdot 1^{\frac{4}{3}}} + \frac{5}{\ln 6}\cdot 6^2 - \frac{5}{\ln 6}\cdot 6^1$$

$$\left(\frac{x^{\frac{4}{3}}}{\frac{4}{3}} = \frac{3x^{\frac{4}{3}}}{4}\right)$$

$$= 7 + \frac{3}{4}\left(\sqrt[3]{2^4}-1\right) + \frac{5}{\ln 6}\cdot 30$$

c) $$\int_{0}^{1}\left(3^x\cdot 4 + \frac{\sqrt{x}}{\sqrt[5]{x^2}}\right)dx = \left[\frac{\sqrt{x}}{\sqrt[5]{x^2}} = \frac{x^{\frac{1}{2}}}{(x^2)^{\frac{1}{5}}} = \frac{x^{\frac{1}{2}}}{x^{\frac{2}{5}}} = x^{\frac{1}{2}-\frac{2}{5}} = x^{\frac{5-4}{10}} = x^{\frac{1}{10}}\right]$$

$$= 4\cdot\int_{0}^{1}3^x\,dx + \int_{0}^{1}x^{\frac{1}{10}}\,dx = 4\cdot\frac{3^x}{\ln 3}\Big|_{0}^{1} + \frac{10x^{\frac{11}{10}}}{11}\Big|_{0}^{1} =$$

$$= \frac{4}{\ln 3}\left(\underbrace{3^1-3^0}_{3-1=2}\right) + \frac{10}{11}\left(\underbrace{1^{\frac{11}{10}}-0^{\frac{11}{10}}}_{=1}\right) = \frac{8}{\ln 3}+\frac{10}{11}$$

d) $\int_0^2 \left(\underbrace{\sqrt{x\sqrt{x}}} + \sqrt[4]{x^3}\right) dx$

$\sqrt{x^1 \cdot x^{\frac{1}{2}}} = \sqrt{x^{1+\frac{1}{2}}} = \sqrt{x^{\frac{3}{2}}} = \left(x^{\frac{3}{2}}\right)^{\frac{1}{2}} = x^{\frac{3}{4}}$

$\sqrt[4]{x^3} = x^{\frac{3}{4}}$

$= \int_0^2 \left(x^{\frac{3}{4}} + x^{\frac{3}{4}}\right) dx = \int_0^2 2 \cdot x^{\frac{3}{4}} dx = 2 \int_0^2 x^{\frac{3}{4}} dx$

$= 2 \cdot \dfrac{x^{\frac{3}{4}+1}}{\frac{3}{4}+1} \Big|_0^2 = 2 \cdot \frac{4}{7} x^{\frac{7}{4}} \Big|_0^2 = \frac{8}{7}\left(\cancel{2^{\frac{7}{4}} - 0^{\frac{7}{4}}}\right)$

$\frac{3}{4}+1 = \frac{7}{4}$

$= \frac{8}{7} \cdot \sqrt[4]{2^7}$

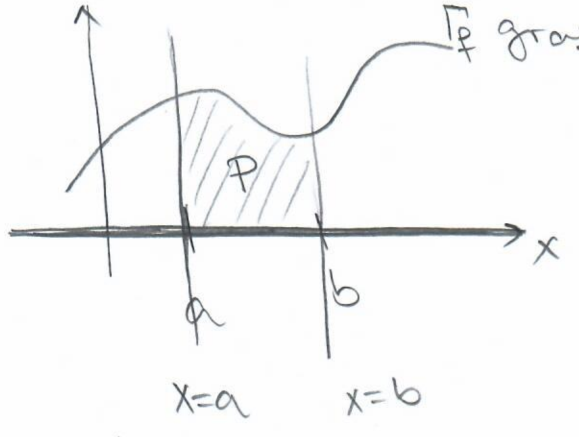
* Računanje površine lika u ravnini *

1. slučaj POVRŠINA P lika omeđenog s

① krivuljom $y = f(x)$, gdje je f neprekidna funkcija na $[a,b]$ t.d. $f(x) \geq 0 \ \forall x \in [a,b]$

② x-osi

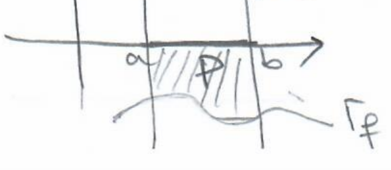
③ pravcima $x=a$ i $x=b$



jednadžba je

$$P = \int_a^b f(x)\,dx$$

* Ako je sve isto kao gore samo $f(x) \leq 0, \forall x \in [a,b]$

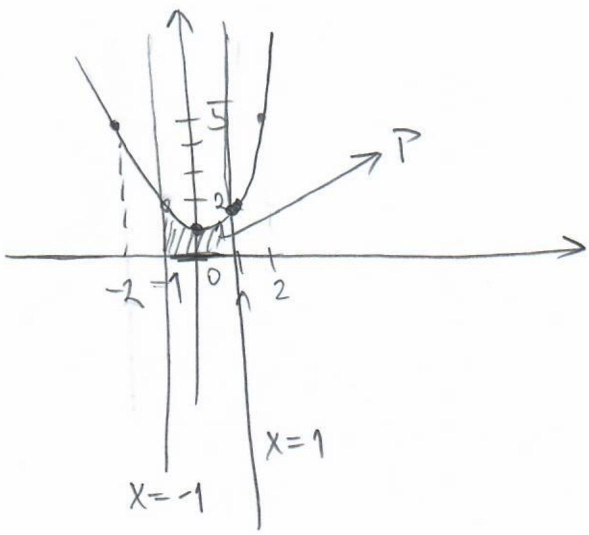


$$P = -\int_a^b f(x)\,dx$$

zad 1 Izračunajte površinu lika omeđenog grafom funkcije $f(x)=x^2+1$, x-osi i pravcima $\underbrace{x=-1 \text{ i } x=1}_{\text{granice integrala}}$.

Rj Skiciramo

x	-2	-1	0	1	2
f(x)	5	2	1	2	5



$$P=\int_{-1}^{1} f(x)\,dx=\int_{-1}^{1}(x^2+1)\,dx$$

$$=\int_{-1}^{1} x^2\,dx+\int_{-1}^{1} \underset{x^0}{1}\,dx$$

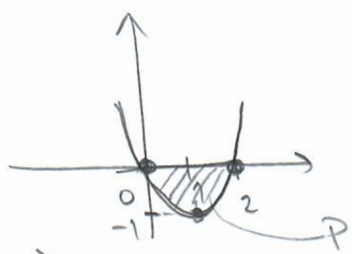
$$=\frac{x^3}{3}\Big|_{-1}^{1}+x\Big|_{-1}^{1}=\frac{1}{3}-\frac{(-1)^3}{3}+1-(-1)=\frac{1}{3}+\frac{1}{3}+1+1=\frac{8}{3}$$

zad 2 Izračunajte površinu lika omeđenog grafom f-je $f(x)=x^2-2x$, x-osi i pravcima $x=0$ i $x=2$.

Rj Skiciramo:

x	0	1	2
f(x)	0	-1	0

$f(x)=x(x-2)$



$$P=-\int_{0}^{2} f(x)\,dx=-\int_{0}^{2}(x^2-2x)\,dx$$

$$=\int_{0}^{2}(-x^2+2x)\,dx=\left(-\frac{x^3}{3}+\frac{\cancel{2}x^2}{\cancel{2}}\right)\Big|_{0}^{2}=-\frac{8}{3}+4-0$$

$$=-\frac{8}{3}+4=\frac{4}{3}$$

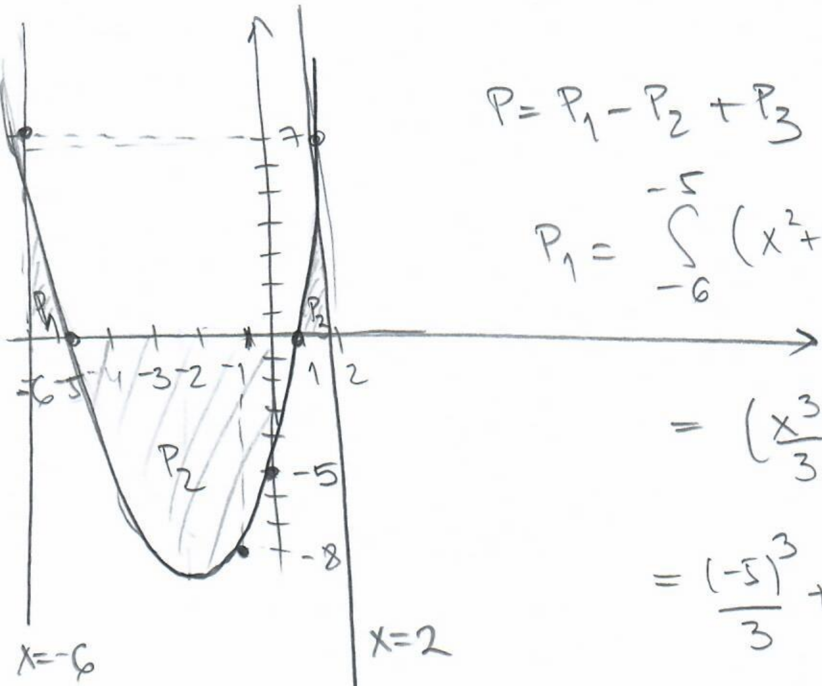
zad 3 (DZ) Izračunajte površinu lika omeđenog grafom funkcije $f(x)=\frac{1}{x}$, x-osi i pravcima $x=1$ i $x=2$.

Rj $\boxed{P=\ln 2}$

Zad 4 Izračunajte površinu lika omeđenog grafom funkcije $f(x)=x^2+4x-5$, x-osi i pravcima
$\underbrace{x=-6 \text{ i } x=2}_{\text{granice integrala}}$.

Rj: Skiciramo

x	-6	-5	-1	0	1	2
$f(x)$	7	0	-8	-5	0	7



$$P=P_1-P_2+P_3$$

$$P_1=\int_{-6}^{-5}(x^2+4x-5)\,dx$$

$$=\left(\frac{x^3}{3}+\cancel{4}^{\,2}\cdot\frac{x^2}{\cancel{2}}-5x\right)\Big|_{-6}^{-5}$$

$$=\frac{(-5)^3}{3}+2\cdot(-5)^2-5\cdot(-5)-\left(\frac{(-6)^3}{3}+2\cdot(-6)^2-5\cdot(-6)\right)$$

$$=-\frac{125}{3}+50+25-\left(-\frac{216}{3}+72+30\right)$$

$$=+\frac{100}{3}-(30)=\frac{10}{3}$$

$$P_2=\int_{-5}^{1}(x^2+4x-5)\,dx=\int_{-5}^{1}x^2\,dx+4\int_{-5}^{1}x\,dx-5\int_{-5}^{1}1\,dx$$

$$=\frac{x^3}{3}\Big|_{-5}^{1}+4\frac{x^2}{2}\Big|_{-5}^{1}-5\cdot x\Big|_{-5}^{1}=\frac{1}{3}-\frac{(-5)^3}{3}+4\cdot\frac{1}{2}-4\cdot\frac{(-5)^2}{2}-5\cdot 1+5(-5)$$

$$=-36$$

$$P_3=\int_{1}^{2}(x^2+4x-5)\,dx=\frac{10}{3}$$

Konačno:

$$P=P_1-P_2+P_3=\frac{10}{3}-(-36)+\frac{10}{3}$$

$$=\frac{128}{3}$$