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Intrinsic boundary conditions for Friedrichs systems

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Abstract

The admissible boundary conditions for symmetric positive systems of first-order linear partial differential equations, originally introduced by Friedrichs (1958), were recently related to three different sets of intrinsic geometric conditions in graph spaces (Ern, Guermond and Caplain, 2007).

We rewrite their cone formalism in terms of an indefinite inner product space, which in a quotient by its isotropic part gives a Kreĭn space. This new viewpoint allows us to show that the three sets of intrinsic boundary conditions are actually equivalent, which will hopefully facilitate further investigation of their precise relation to the original Friedrichs boundary conditions.

Keywords: symmetric positive system, first-order system of pde's, Kreĭn space, boundary operator

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1. Introduction

An overview

Symmetric positive systems of first-order linear partial differential equations were introduced by Friedrichs (1958) in order to treat the equations that change their type, like the equations modelling the transonic fluid flow.

Friedrichs showed that this class of problems encompasses a wide variety of classical and neoclassical initial and boundary value problems for various linear partial differential equations, such as boundary value problems for some elliptic systems, the Cauchy problem for linear symmetric hyperbolic systems, mixed initial and boundary value problem for hyperbolic equations, and, last but not the least, boundary value problems for equations of mixed type, such as the Tricomi equation. Friedrichs' main goal was to systematically derive the type of conditions that have to be imposed on various parts of the boundary, such as it was done by Cathleen Synge Morawetz [M] in the case of Tricomi's equation.

Inclusion of such a variety of different problems, each with technical peculiarities of its own, into a single framework naturally requires all different characteristics to be included as well. A good theory which would incorporate natural discontinuities for hyperbolic problems simultaneously with the poles in the corners for elliptic problems, remains a challenge even today, and it is still an open problem. Many authors have since tried to surmount these difficulties, like Morawetz, Lax, Rauch, Phillips and Sarason [M, FL, Ra, LP, PS].

Already Friedrichs considered the numerical solution of such systems, by a finite difference scheme. We understand that, based on the needs to adapt the finite element method to these problems, there has been a renewed interest in the theory of Friedrichs' systems during the last decade. Here we just mention [HMSW], the Ph.D. thesis of Max Jensen [J] and, most recently, [EGC, EG]. Ern, Guermond and Caplain reformulated the Friedrichs theory such that the traces on the boundary are not explicitly used. They expressed the theory in terms of operators acting in abstract Hilbert spaces, and represented the boundary conditions in an intrinsic way, thus avoiding ad-hoc matrix-valued boundary fields. In this paper we try to answer the questions that were left open there.

Our interest in this theory was originally motivated by the need for a better formulation of initial and boundary value problems for equations that change their type, in order to extend the results of [A] to the non-hyperbolic region, making use of some recent advances initiated in [AL, AL1].

In the remaining part of the Introduction we first recall the classical (i.e. Friedrichs') setting, and formulate the three equivalent sets of classical boundary conditions. Then we recall the basic definitions and results concerning the indefinite inner product spaces, and in particular the Kreĭn spaces. We also make a number of simple conclusions, based on the precisely cited results, which will be used later in the paper.

In the second section we introduce the abstract setting, as it was done in [EGC]. While in the first subsection we mostly follow their presentation [EGC, Section 2] (except that in the Example we refer to [AB1]), in the second subsection we reformulate their results and definitions in the terms of indefinite inner product spaces, reducing some of their proofs to simple well-known results in that theory. Finally, we formulate the abstract analogues of the above mentioned three sets of boundary conditions.

The main results are contained in the last section, where we start by proving the properties of the quotient of an inner product space by its isotropic part; most important is the fact that the quotient is a Kreĭn space. Then we can easily prove the equivalence of (V) and (X); in [EGC] it was only shown that (V) implies (X). The proof of the other equivalence, between (V) and (M), is more delicate. We first show by an example that some additional assumptions proposed in [EGC] are not always fulfilled, and then provide another approach, that allows us to complete the proof.

The classical setting

Let $d, r \in \mathbf{N}$, and let $\Omega \subseteq \mathbf{R}^d$ be an open and bounded set with Lipschitz boundary Γ. If real matrix functions $\mathbf{A}_k \in \mathrm{W}^{1,\infty}(\Omega; \mathrm{M}_r)$, $k \in 1..d$, and $\mathbf{C} \in \mathrm{L}^\infty(\Omega; \mathrm{M}_r)$ satisfy:

(F1) $$\mathbf{A}_k \text{ is symmetric: } \mathbf{A}_k = \mathbf{A}_k^\top \,,$$

(F2) $$(\exists\, \mu_0 > 0) \quad \mathbf{C} + \mathbf{C}^\top + \sum_{k=1}^d \partial_k \mathbf{A}_k \geqslant 2\mu_0 \mathbf{I} \qquad (\text{a.e. on } \Omega)\,,$$

then the first-order differential operator $\mathcal{L} : \mathrm{L}^2(\Omega; \mathbf{R}^r) \longrightarrow \mathcal{D}'(\Omega; \mathbf{R}^r)$ defined by

$$\mathcal{L}\mathsf{u} := \sum_{k=1}^d \partial_k(\mathbf{A}_k \mathsf{u}) + \mathbf{C}\mathsf{u}$$

is called *the Friedrichs operator* or *the positive symmetric operator*, while (for given $\mathsf{f} \in \mathrm{L}^2(\Omega; \mathbf{R}^r)$) the first-order system of partial differential equations

$$\mathcal{L}\mathsf{u} = \mathsf{f}$$

is called *the Friedrichs system* or *the positive symmetric system*.

In fact, the above can be extended [FL] to the complex case, by replacing the transpose of a matrix with the Hermitian conjugation; e.g. in (F1) one would require $\mathbf{A}_k = \mathbf{A}_k^*$.

Kurt Otto Friedrichs [F] also introduced an interesting way for representing the boundary conditions via matrix valued boundary fields. First define

$$\mathbf{A}_{\boldsymbol{\nu}} := \sum_{k=1}^d \nu_k \mathbf{A}_k \in \mathrm{L}^\infty(\Gamma; \mathrm{M}_r)\,,$$

where $\boldsymbol{\nu} = (\nu_1, \nu_2, \cdots, \nu_d)$ is the outward unit normal on Γ, and let $\mathbf{M} : \Gamma \longrightarrow \mathrm{M}_r$ be a given matrix field on the boundary. The boundary condition is then prescribed by

$$(\mathbf{A}_{\boldsymbol{\nu}} - \mathbf{M})\mathsf{u}_{|_\Gamma} = 0\,,$$

and by choosing different $\mathbf{M}$ one can enforce different boundary conditions. For the matrix field $\mathbf{M}$ Friedrichs required the following two conditions (for a.e. $\mathbf{x} \in \Gamma$) to hold:

(FM1) $$(\forall\, \boldsymbol{\xi} \in \mathbf{R}^r) \quad \mathbf{M}(\mathbf{x})\boldsymbol{\xi} \cdot \boldsymbol{\xi} \geqslant 0\,,$$

(FM2) $$\mathbf{R}^r = \ker\Big(\mathbf{A}_{\boldsymbol{\nu}}(\mathbf{x}) - \mathbf{M}(\mathbf{x})\Big) + \ker\Big(\mathbf{A}_{\boldsymbol{\nu}}(\mathbf{x}) + \mathbf{M}(\mathbf{x})\Big)\,;$$

such an $\mathbf{M}$ he called *an admissible boundary condition*.

The boundary value problem thus reads: for given $\mathsf{f} \in \mathrm{L}^2(\Omega; \mathbf{R}^r)$ find $\mathsf{u} \in \mathrm{L}^2(\Omega; \mathbf{R}^r)$ such that

$$\begin{cases} \mathcal{L}\mathsf{u} = \mathsf{f} \\ (\mathbf{A}_{\boldsymbol{\nu}} - \mathbf{M})\mathsf{u}_{|_\Gamma} = 0 \end{cases}.$$

Instead of imposing admissible boundary conditions, after [F] some authors have required different, but equivalent, boundary conditions.

For example, Peter Lax proposed *the maximal boundary condition*—we say that a family $N = \{N(\mathbf{x}) : \mathbf{x} \in \Gamma\}$ of subspaces of $\mathbf{R}^r$ defines *the maximal boundary condition* if (for a.e. $\mathbf{x} \in \Gamma$) $N(\mathbf{x})$ is maximal nonnegative with respect to $\mathbf{A}_{\boldsymbol{\nu}}(\mathbf{x})$, i.e. if

(FX1) $N(\mathbf{x})$ is nonnegative with respect to $\mathbf{A}_{\boldsymbol{\nu}}(\mathbf{x})$: $\qquad (\forall\, \boldsymbol{\xi} \in N(\mathbf{x})) \quad \mathbf{A}_{\boldsymbol{\nu}}(\mathbf{x})\boldsymbol{\xi} \cdot \boldsymbol{\xi} \geqslant 0\,,$

and

(FX2) there is no nonnegative subspace with respect to $\mathbf{A}_{\boldsymbol{\nu}}(\mathbf{x})$, which is larger than $N(\mathbf{x})$.

Now the boundary value problem takes the form

$$\begin{cases} \mathcal{L}\mathsf{u} = \mathsf{f} \\ \mathsf{u}(\mathbf{x}) \in N(\mathbf{x})\,, \quad \mathbf{x} \in \Gamma \end{cases}.$$

Instead of the above condition imposed on family N, in literature one can also find a different set of conditions [PS]: it is required that $N(\mathbf{x})$ and $\tilde{N}(\mathbf{x}) := (\mathbf{A}_{\boldsymbol{\nu}}(\mathbf{x})N(\mathbf{x}))^{\perp}$ satisfy

(FV1) $N(\mathbf{x})$ is nonnegative with respect to $\mathbf{A}_{\boldsymbol{\nu}}(\mathbf{x})$: $\quad (\forall\,\boldsymbol{\xi} \in N(\mathbf{x}))\quad \mathbf{A}_{\boldsymbol{\nu}}(\mathbf{x})\boldsymbol{\xi}\cdot\boldsymbol{\xi} \geqslant 0\,,$
$\tilde{N}(\mathbf{x})$ is nonpositive with respect to $\mathbf{A}_{\boldsymbol{\nu}}(\mathbf{x})$: $\quad (\forall\,\boldsymbol{\xi} \in \tilde{N}(\mathbf{x}))\quad \mathbf{A}_{\boldsymbol{\nu}}(\mathbf{x})\boldsymbol{\xi}\cdot\boldsymbol{\xi} \leqslant 0\,,$

(FV2) $$\tilde{N}(\mathbf{x}) = (\mathbf{A}_{\boldsymbol{\nu}}(\mathbf{x})N(\mathbf{x}))^{\perp} \quad \text{and} \quad N(\mathbf{x}) = (\mathbf{A}_{\boldsymbol{\nu}}(\mathbf{x})\tilde{N}(\mathbf{x}))^{\perp}\,,$$

for a.e. $\mathbf{x} \in \Gamma$. Note that the first condition in (FV2) is actually the definition of $\tilde{N}$, and it is stated here for the sake of completeness.

We have three sets of boundary conditions for the Friedrichs system, and are going to define three more in the abstract setting below. In order to simplify the notation, when referring to e.g. (FM1)–(FM2) we shall simply write only (FM) in the sequel. However, in order to keep a clear distinction with other conditions, like (F1)–(F2), such abbreviations will be reserved only for various forms of boundary conditions.

It is well known that these three sets of conditions: (FM), (FX) and (FV) are in fact equivalent [B], with $N(\mathbf{x}) := \ker(\mathbf{A}_{\boldsymbol{\nu}}(\mathbf{x}) - \mathbf{M}(\mathbf{x}))$. It is also well known that for a weak existence result one needs some additional assumptions—see [Ra] for additional details on assumptions (FX), and [J] for (FM). In this paper our goal is to resolve the question of equivalence for the abstract versions of these conditions.

Kreĭn spaces

The theory of linear operators on the spaces with an indefinite metric was initiated by Lev S. Pontryagin in 1944. Despite a number of books on the topic (we have primarily used [Bo] and [AI]), this theory is not well known in the mathematical community. For the convenience of the reader, here we summarise some basic notions and properties of indefinite inner product spaces, and in particular of Kreĭn spaces (named in honour of Mark Grigor'evič Kreĭn).

By W we denote *an indefinite inner product space*, i.e. a complex (or real) vector space equipped with a sesquilinear functional $[\,\cdot \mid \cdot\,] : W \times W \longrightarrow \mathbf{C}$, i.e. the one satisfying

$$\begin{aligned} (\forall\,\lambda_1,\lambda_2 \in \mathbf{C})(\forall\,x_1,x_2,y \in W) \quad & [\,\lambda_1 x_1 + \lambda_2 x_2 \mid y\,] = \lambda_1[\,x_1 \mid y\,] + \lambda_2[\,x_2 \mid y\,]\,,\\ (\forall\,x,y \in W) \quad & [\,y \mid x\,] = \overline{[\,x \mid y\,]}\,. \end{aligned}$$

Such a functional is called *the indefinite inner product* on W.

We say that two vectors $x,y \in W$ are $[\,\cdot \mid \cdot\,]$-*orthogonal* (and write $x[\perp]y$) if $[\,x \mid y\,] = 0$. Two sets $K,L \subseteq W$ are $[\,\cdot \mid \cdot\,]$-*orthogonal* if $x[\perp]y$, for each $x \in K$, $y \in L$. We then write $K[\perp]L$.

$[\,\cdot \mid \cdot\,]$-*orthogonal complement* of set $L \subseteq W$ is defined by

$$L^{[\perp]} := \{y \in W : (\forall\,x \in L) \quad [\,x \mid y\,] = 0\}\,.$$

One can easily see that $L^{[\perp]}$ is a subspace of W, and that $K \subseteq L$ implies $L^{[\perp]} \leqslant K^{[\perp]}$.

A vector $x \in L \leqslant W$ is *isotropic* in L if $x \in L^{[\perp]}$. The set of all isotropic vectors in L is denoted by $L^0 := L \cap L^{[\perp]}$, and one can easily see that $W^0 \leqslant L^{[\perp]}$, for any subspace L of W. If $L^0 = \{0\}$ we say that L is *a non-degenerate space*, while otherwise we say that it is *degenerate*.

The quotient space of W by the subspace of its isotropic vectors is denoted by $\hat{W} := W/W^0$, and its elements by $\hat{x} := x + W^0$. One can easily check that

$$[\, \hat{x} \mid \hat{y}\,]\hat{} := [\, x \mid y\,]$$

defines an indefinite inner product on $\hat{W}$ (see [AI, pp. 7–8]).

We say that a vector $x \in W$ is *positive* if $[\, x \mid x\,] > 0$, and that a subspace $L \leqslant W$ is *positive* if its each nonzero vector is positive. In the same way we define *negative* ($<$), *nonnegative* ($\geqslant$), *nonpositive* ($\leqslant$), and *neutral* ($=$) vectors and subspaces.

A subspace L of W is *maximal positive*, if it is positive and there exists no positive subspace $M \neq L$, such that $L \subset M$. In the same way we define *maximal negative*, *maximal nonnegative*, *maximal nonpositive*, and *maximal neutral* subspaces. We use the term *maximal definite* for maximal positive or maximal negative, while *maximal semidefinite* is used for maximal nonnegative or maximal nonpositive.

Lemma I. *[Bo, p. 13]* $[\,\cdot \mid \cdot\,]$*-orthogonal complement of a maximal nonnegative (nonpositive) subspace is nonpositive (nonnegative).* ∎

Lemma II. *[AI, p. 7] Each maximal semidefinite subspace contains all isotropic vectors of the space W.* ∎

Direct sum $K\dot{+}L$ of subspaces $K, L \leqslant W$, which are also $[\,\cdot \mid \cdot\,]$-orthogonal, we denote by $K[\dot{+}]L$.

Each pair consisting of a positive subspace W^+ and a negative subspace W^-, such that $W = W^+[\dot{+}]W^-$ is called *a canonical decomposition of space W*. If W is a direct sum of a positive and a negative subspace, then it is also non-degenerate [AI, p. 8].

If $(W, \langle\,\cdot \mid \cdot\,\rangle)$ is a Hilbert space and G is a bounded hermitian ($G = G^*$) operator on W, then

$$[\, x \mid y\,] := \langle\, Gx \mid y\,\rangle\,, \qquad x, y \in W\,,$$

defines an indefinite inner product on W. The operator G is called *the Gramm operator* of the space $(W, [\,\cdot \mid \cdot\,])$, and the isotropic part W^0 of W is equal to $\ker G$.

Whenever the indefinite inner product is defined by a Gramm operator, we can use the corresponding Hilbert topology on that space. While the choice of the Gramm operator and the Hilbert scalar product is not unique, the resulting topologies are equivalent [Bo, p. 63].

It is of interest to note that $L^{[\perp]}$ is closed in the mentioned topology for any subset L of W.

An indefinite inner product space that allows a canonical decomposition $W = W^+[\dot{+}]W^-$, such that $(W^+, [\,\cdot \mid \cdot\,])$ and $(W^-, -[\,\cdot \mid \cdot\,])$ are Hilbert spaces, is called *the Kreĭn space*.

It is a well known fact that the indefinite inner product on any Kreĭn space can be expressed by a Gramm operator in some Hilbert scalar product on that space [Bo, p. 101]. As this topology does not depend on the choice of the Gramm operator and the Hilbert scalar product, it is usually considered as the standard topology on the Kreĭn space.

Theorem I. *[AI, p. 40] Let G be the Gramm operator of the space W. The quotient space $\hat{W} := W/\ker G$ is then a Kreĭn space if and only if $\operatorname{im} G$ is closed.* ∎

Theorem II. *[Bo, p. 106] If L is a nonnegative (nonpositive) subspace of a Kreĭn space, such that $L^{[\perp]}$ is nonpositive (nonnegative), then its closure $\mathsf{Cl}\, L$ is a maximal nonnegative (nonpositive) subspace of the Kreĭn space.* ∎

Theorem III. *[Bo, p. 105] Each maximal semidefinite subspace of a Kreĭn space is closed.* ∎

Theorem IV. *[Bo, pp. 69, 101–102] Subspace L of a Kreĭn space is closed if and only if $L = L^{[\perp][\perp]}$.* ∎

Theorem V. *[AI, p. 44] For a subspace L of a Kreĭn space W it holds*

$$L \cap L^{[\perp]} = \{0\} \qquad \iff \qquad \mathsf{Cl}\,(L + L^{[\perp]}) = W\,.$$

∎

Theorem VI. *[Bo, p. 112] For each maximal nonnegative (nonpositive) subspace L_1 of a Kreĭn space W there is a nonpositive (nonnegative) subspace L_2, such that $L_1 \dot{+} L_2 = W$. One of the possible choices for L_2 is the space W^- (W^+) from the canonical decomposition of W.* ∎

2. An abstract setting

Abstract Hilbert space formalism

Let L be a real Hilbert space, which we identify with its dual L'; this identification will be kept throughout the paper. Furthermore, let $\mathcal{D} \subseteq L$ be a dense subspace, and $T, \tilde{T} : \mathcal{D} \longrightarrow L$ unbounded operators satisfying

$$\text{(T1)} \qquad (\forall \varphi, \psi \in \mathcal{D}) \quad \langle T\varphi \mid \psi \rangle_L = \langle \varphi \mid \tilde{T}\psi \rangle_L \,,$$

$$\text{(T2)} \qquad (\exists\, c > 0)(\forall \varphi \in \mathcal{D}) \quad \|(T + \tilde{T})\varphi\|_L \leqslant c\|\varphi\|_L \,,$$

$$\text{(T3)} \qquad (\exists\, \mu_0 > 0)(\forall \varphi \in \mathcal{D}) \quad \langle (T + \tilde{T})\varphi \mid \varphi \rangle_L \geqslant 2\mu_0 \|\varphi\|_L^2 \,.$$

By (T1) the operators T and $\tilde{T}$ have densely defined formal adjoints, and thus are closable. The closures we denote by $\bar{T}$ and $\bar{\tilde{T}}$, and the corresponding domains by $\mathcal{D}(\bar{T})$ and $\mathcal{D}(\bar{\tilde{T}})$.

The graph inner product $\langle \cdot \mid \cdot \rangle_T := \langle \cdot \mid \cdot \rangle_L + \langle T\cdot \mid T\cdot \rangle_L$ defines the *graph norm* $\| \cdot \|_T$, and it is immediate that $(\mathcal{D}, \langle \cdot \mid \cdot \rangle_T)$ is an inner product space, whose completion we denote by W_0. Analogously we could have defined $\langle \cdot \mid \cdot \rangle_{\tilde{T}}$ which, by (T2), leads to a norm that is equivalent to $\| \cdot \|_T$. W_0 is continuously imbedded in L (as T is closable); the image of W_0 being $\mathcal{D}(\bar{T}) = \mathcal{D}(\bar{\tilde{T}})$. Moreover, when equipped with the graph norm, these spaces are isometrically isomorphic.

As $T, \tilde{T} : \mathcal{D} \longrightarrow L$ are continuous, each can be extended by density to a unique operator from $\mathcal{L}(W_0; L)$ (i.e. a continuous linear operator from W_0 to L). These extension coincide with $\bar{T}$ and $\bar{\tilde{T}}$ (take into account the isomorphism between W_0 and $\mathcal{D}(\bar{T})$). For simplicity, we shall drop the bar from notation and simply write $T, \tilde{T} \in \mathcal{L}(W_0; L)$, prompted by the fact that (T1)–(T3) still hold for $\varphi, \psi \in W_0$.

We have the Gelfand triple (the imbeddings are dense and continuous)

$$W_0 \hookrightarrow L \equiv L' \hookrightarrow W_0' \,.$$

The adjoint operator $\tilde{T}^* \in \mathcal{L}(L; W_0')$ defined by

$$(\forall\, u \in L)(\forall\, v \in W_0) \quad {}_{W_0'}\langle \tilde{T}^* u, v \rangle_{W_0} = \langle u \mid \tilde{T} v \rangle_L$$

satisfies $T = \tilde{T}^*_{|W_0}$, as (T1) implies

$${}_{W_0'}\langle \tilde{T}^* u, v \rangle_{W_0} = \langle Tu \mid v \rangle_L = {}_{W_0'}\langle Tu, v \rangle_{W_0} \,, \quad u \in W_0 \,.$$

Therefore, $T : W_0 \longrightarrow L \hookrightarrow W_0'$ is a continuous linear operator from $(W_0, \| \cdot \|_L)$ to W_0', whose unique continuous extension to the whole L is the operator $\tilde{T}^*$ (the same holds for T^* and $\tilde{T}$ instead of T and $\tilde{T}^*$). In order to further simplify the notation we shall use T and $\tilde{T}$ also to denote their extensions $\tilde{T}^*$ and T^*. By using this convention, for $u \in L$ and $\varphi \in W_0$ we have

$${}_{W_0'}\langle Tu, \varphi \rangle_{W_0} = \langle u \mid \tilde{T}\varphi \rangle_L \qquad \text{and} \qquad {}_{W_0'}\langle \tilde{T}u, \varphi \rangle_{W_0} = \langle u \mid T\varphi \rangle_L \,.$$

Note that the above construction of operators $T, \tilde{T} \in \mathcal{L}(L; W_0')$ has been achieved independently of the validity of (T3).

Lemma 1. *([EGC, Lemma 2.2]) The properties (T1)–(T2) imply $T+\tilde{T} \in \mathcal{L}(L; L)$ and $(T+\tilde{T})^* = T+\tilde{T}$. In particular, (T2)–(T3) hold for $\varphi \in L$.* ∎

By

$$W := \{u \in L : Tu \in L\} = \{u \in L : \tilde{T}u \in L\}\,,$$

we denote the *graph space* which, equipped with the graph norm, is an inner product space, containing W_0. Furthermore, for $u \in W$ and $\varphi \in W_0$ we have

$$\langle\, Tu \mid \varphi \,\rangle_L = \langle\, u \mid \tilde{T}\varphi \,\rangle_L \qquad \text{and} \qquad \langle\, \tilde{T}u \mid \varphi \,\rangle_L = \langle\, u \mid T\varphi \,\rangle_L\,.$$

Lemma 2. *([EGC, Lemma 2.1]) Under the assumptions (T1)–(T2), $(W, \langle\, \cdot \mid \cdot \,\rangle_T)$ is a Hilbert space.* ∎

The goal is to solve the following problem:

for given $f \in L$ find $u \in W$ such that $Tu = f$.

To be more precise, the goal is to find sufficient conditions on subspace $V \subseteq W$, such that the operator $T_{|_V} : V \longrightarrow L$ is an isomorphism. In order to find such sufficient conditions, we first introduce a *boundary operator* $D \in \mathcal{L}(W; W')$ defined by

$${}_{W'}\langle\, Du, v \,\rangle_W := \langle\, Tu \mid v \,\rangle_L - \langle\, u \mid \tilde{T}v \,\rangle_L\,, \qquad u, v \in W\,.$$

The following lemma justifies the usage of the term *boundary operator.*

Lemma 3. *([EGC, Lemmas 2.3 and 2.4]) Under assumptions (T1)–(T2), the operator D satisfies*

$$(\forall\, u, v \in W) \quad {}_{W'}\langle\, Du, v \,\rangle_W = {}_{W'}\langle\, Dv, u \,\rangle_W\,,$$
$$\ker D = W_0 \qquad \textit{and} \qquad \operatorname{im} D = \{g \in W' : (\forall\, u \in W_0) \quad {}_{W'}\langle\, g, u \,\rangle_W = 0\}\,.$$

In particular, $\operatorname{im} D$ *is closed in W'.* ∎

Example. (Friedrichs' operator) As in the Introduction, let $d, r \in \mathbf{N}$, and $\Omega \subseteq \mathbf{R}^d$ be an open and bounded set with Lipschitz boundary Γ. Furthermore, assume that the matrix functions $\mathbf{A}_k \in \mathrm{W}^{1,\infty}(\Omega; \mathrm{M}_r)$, $k \in 1..d$, and $\mathbf{C} \in \mathrm{L}^\infty(\Omega; \mathrm{M}_r)$ satisfy (F1)–(F2). If we denote $\mathcal{D} := \mathrm{C}_c^\infty(\Omega; \mathbf{R}^r)$, $L = \mathrm{L}^2(\Omega; \mathbf{R}^r)$, and define operators $T, \tilde{T} : \mathcal{D} \longrightarrow L$ by formulaæ

$$T\mathsf{u} := \sum_{k=1}^d \partial_k(\mathbf{A}_k \mathsf{u}) + \mathbf{C}\mathsf{u}\,,$$
$$\tilde{T}\mathsf{u} := -\sum_{k=1}^d \partial_k(\mathbf{A}_k^\top \mathsf{u}) + (\mathbf{C}^\top + \sum_{k=1}^d \partial_k \mathbf{A}_k^\top)\mathsf{u}\,,$$

where ∂_k stands for *the classical* derivate, then one can easily see that T and $\tilde{T}$ satisfy (T1)–(T3) ($\tilde{T}$ is defined so that (T1) holds, (F1) implies (T2), and (T3) follows from (F2)). Therefore, T and $\tilde{T}$ can be uniquely extended to respective operators from $\mathcal{L}(L; W_0')$.

Note that the classical Friedrichs operator $\mathcal{L}$ was defined (cf. Introduction) in a slightly different way than the operator T above: $\mathcal{L}$ was formally defined by the same formula as T, but with distributional derivatives instead of classical ones, and it was considered as a continuous linear operator from $\mathrm{L}^2(\Omega; \mathbf{R}^r)$ to $\mathcal{D}'(\Omega; \mathbf{R}^r)$. However, as W_0' is continuously imbedded in $\mathcal{D}'(\Omega; \mathbf{R}^r)$ (because $\mathrm{C}_c^\infty(\Omega; \mathbf{R}^r)$ with the strict inductive limit topology is continuously and densely imbedded in W_0 with the graph norm topology), we can consider T as an operator from $\mathcal{L}(\mathrm{L}^2(\Omega; \mathbf{R}^r); \mathcal{D}'(\Omega; \mathbf{R}^r))$. As such, it is equal to the operator $\mathcal{L}$, because they coincide on a dense subspace $\mathcal{D} = \mathrm{C}_c^\infty(\Omega; \mathbf{R}^r)$ (to conclude that, we use the fact that the classical derivative of a Lipschitz function equals its distributional derivative almost everywhere).

The graph space W in this example is given by

$$W = \Big\{ \mathsf{u} \in \mathrm{L}^2(\Omega; \mathbf{R}^r) : \sum_{k=1}^d \partial_k(\mathbf{A}_k \mathsf{u}) + \mathbf{C}\mathsf{u} \in \mathrm{L}^2(\Omega; \mathbf{R}^r) \Big\} ,$$

where we have to take the distributional derivatives in the above formula. If we denote by $\boldsymbol{\nu} = (\nu_1, \nu_2, \dots, \nu_d) \in \mathrm{L}^\infty(\Gamma; \mathbf{R}^d)$ the unit outward normal on Γ, and define a matrix field on Γ by

$$\mathbf{A}_{\boldsymbol{\nu}} := \sum_{k=1}^d \nu_k \mathbf{A}_k ,$$

then for $\mathsf{u}, \mathsf{v} \in \mathrm{C}_c^\infty(\mathbf{R}^d; \mathbf{R}^r)$ the boundary operator D is given by

$$_{W'}\langle D\mathsf{u}, \mathsf{v} \rangle_W = \int_\Gamma \mathbf{A}_{\boldsymbol{\nu}}(\mathbf{x}) \mathsf{u}_{|_\Gamma}(\mathbf{x}) \cdot \mathsf{v}_{|_\Gamma}(\mathbf{x}) dS(\mathbf{x}) .$$

Thus we can say that, in the abstract setting, the operator D plays the role of the matrix function $\mathbf{A}_{\boldsymbol{\nu}}$ in the classical Friedrichs theory. To be more precise, it replaces the trace operator defined on the graph space; for the details concerning the definition and properties of the trace operator on graph spaces see [AB1]. ∎

Formulation in terms of indefinite inner product spaces

Before we write down sufficient conditions on a subspace V that ensure the well-posedness result, still following the main ideas from [EGC], which they baptised *the cone formalism*, we shall introduce a new notation which is more appropriate for describing the desired subspace V. The conditions on V [EGC, EG] can naturally be written in terms of an indefinite inner product on W, defined by the Gramm operator $G = J \circ D$, where $J : W' \longrightarrow W$ is the canonical isomorphism (cf. the proof of Lemma 8 below):

$$[\, u \mid v \,] := \langle\, Gu \mid v \,\rangle_T = {}_{W'}\langle\, Du, v \,\rangle_W = \langle\, Tu \mid v \,\rangle_L - \langle\, u \mid \tilde{T}v \,\rangle_L , \qquad u, v \in W .$$

Note that the isotropic part W^0 of W is just $\ker G = \ker D = W_0$. The usage of basic geometric properties in indefinite inner product spaces can simplify (in some cases even trivialise) many proofs from [EGC] (such as Lemma 3.3 there).

Let us denote

$$\begin{aligned} C^+ &:= \{ u \in W : [\, u \mid u \,] \geqslant 0 \} , \\ C^- &:= \{ u \in W : [\, u \mid u \,] \leqslant 0 \} , \\ C^0 &:= C^+ \cap C^- , \end{aligned}$$

i.e. C^+ is the nonnegative, while C^- the nonpositive cone in $(W, [\,\cdot \mid \cdot\,])$. Further, let V and $\tilde{V}$ be subspaces of W satisfying the following conditions

(V1) $$V \subseteq C^+ , \qquad \tilde{V} \subseteq C^- ;$$

(V2) $$V = \tilde{V}^{[\perp]} , \qquad \tilde{V} = V^{[\perp]} ,$$

where by $^{[\perp]}$ we denote the $[\,\cdot \mid \cdot\,]$-orthogonal complement.

Remark. If we write down the above conditions using the operator D, as in [EGC], then (V1) reads

$$\begin{aligned} (\forall\, u \in V) &\qquad {}_{W'}\langle\, Du, u \,\rangle_W \geqslant 0 , \\ (\forall\, v \in \tilde{V}) &\qquad {}_{W'}\langle\, Dv, v \,\rangle_W \leqslant 0 , \end{aligned}$$

while (V2) becomes (note that here ^{0} stands for the annihilator)

$$V = D(\tilde{V})^0 , \qquad \tilde{V} = D(V)^0 .$$

The analogy with conditions (FV) in the classical Friedrichs' system theory is obvious. ∎

The following lemma is then immediate.

Lemma 4. *If (T1)–(T2) and (V2) hold, then V and $\tilde V$ are closed, and* $\ker D = W_0 \subseteq V \cap \tilde V$. ∎

The proof of well-posedness result uses the coercivity property of T and $\tilde T$ stated in the following lemma.

Lemma 5. *([EGC, Lemma 3.2]) Under assumptions (T1)–(T3) and (V), the operators T and $\tilde T$ are L–coercive on V and $\tilde V$, respectively; in other words:*

$$\begin{aligned}(\forall u \in V) \qquad & \langle Tu \mid u \rangle_L \geqslant \mu_0 \|u\|_L^2 \,, \\ (\forall v \in \tilde V) \qquad & \langle Tv \mid v \rangle_L \geqslant \mu_0 \|v\|_L^2 \,.\end{aligned}$$

∎

Theorem 1. *([EGC, Theorem 3.1]) If (T1)–(T3) and (V) hold, then the restrictions of operators $T_{|_V} : V \longrightarrow L$ and $\tilde T_{|_{\tilde V}} : \tilde V \longrightarrow L$ are isomorphisms.* ∎

The above theorem gives sufficient conditions on subspaces V and $\tilde V$ that ensure well-posedness of the following problems:

1) for given $f \in L$ find $u \in V$ such that $Tu = f$;
2) for given $f \in L$ find $v \in \tilde V$ such that $\tilde T v = f$.

Its importance also arises from relative simplicity of geometric conditions (V), which do not involve the question of trace for functions in the graph space.

Different formulations of boundary conditions

We have already seen that properties (V) in the abstract setting are related to (FV) for the Friedrichs system. It is only natural to search for appropriate analogues of (FX) and (FM) in the abstract setting. Actually, the notion of the maximal nonnegative subspace is well known in the theory of indefinite inner product spaces: we say that subspace V of $(W, [\,\cdot \mid \cdot\,])$ is *maximal nonnegative* if

(X1) $\qquad V$ is nonnegative in $(W, [\,\cdot \mid \cdot\,])$, i.e $\qquad V \subseteq C^+ \,,$

(X2) $\qquad$ there is no nonnegative subspace of $(W, [\,\cdot \mid \cdot\,])$ which is true superset of V.

The analogy of (FX) and (X) is obvious.

It remains to write down the conditions corresponding to (FM) (see [EGC, EG]): let $M \in \mathcal{L}(W; W')$ be an operator satisfying

(M1) $$(\forall u \in W) \quad {}_{W'}\langle Mu, u \rangle_W \geqslant 0 \,,$$

(M2) $$W = \ker(D - M) + \ker(D + M) \,.$$

The equivalence of (FV), (FX) and (FM) motivates the investigation of the correspondence between the properties (V), (X) and (M), which is the subject of the next section. First we write down some properties of operator M (having strong analogy with properties of matrix boundary field $\mathbf{M}$ in the case of Friedrichs' system). In the rest we assume that (T1)–(T3) hold.

Let $M^* \in \mathcal{L}(W; W')$ be the adjoint of operator M (with identification of spaces W and W''), given by

$$(\forall u, v \in W) \quad {}_{W'}\langle M^* u, v \rangle_W = {}_{W'}\langle Mv, u \rangle_W \,.$$

Lemma 6. *([EGC, Lemma 4.1]) If M satisfies (M), then*

$$\ker D = \ker M = \ker M^* \,, \quad \textit{and}$$

$$\operatorname{im} D = \operatorname{im} M = \operatorname{im} M^* \,.$$

∎

Remark. Since $\ker M = \ker D = W_0$, it makes sense to call M *the boundary operator* as well. ∎

3. On equivalence of various boundary conditions

Quotient as a Kreĭn space

We have already mentioned that in the case of Friedrichs' systems it is known that the conditions (FM), (FV) and (FX) are mutually equivalent. In this section we shall investigate the relationship between conditions (M), (V) and (X), mainly using the well known geometric properties of Kreĭn spaces. These spaces have a structure that is close enough to that of the Hilbert space, which leads to many similar properties. However, the space $(W, [\cdot \mid \cdot])$ is not a Kreĭn space (as it is degenerated), which motivates us to look at the quotient space of W and its isotropic part W_0. We will show that this quotient space is a Kreĭn space (the closedness of $\operatorname{im} D$ appears to be crucial in this conclusion), and use its properties to investigate the relations among (M), (V) and (X). As far as we know, such an approach is novel in the theory of indefinite inner product spaces.

As the indefinite inner product on W is defined by a Gram operator, W can be considered also as a Hilbert space, and we can define its quotient by W_0 in the framework of Hilbert spaces. More precisely, let us denote by $Q : W \longrightarrow W_0^\perp$ the orthogonal projector on the subspace $W_0^\perp$ of W. $W_0^\perp$ is unitary isomorphic to the quotient space $\hat{W} := W/W_0$ (the isomorphism being given by $\hat{x} \mapsto Qx$, for $x \in W$, where $\hat{x} = x + W_0$). Thus $\hat{W}$ is a Hilbert space; by closedness of W_0 in W, we have the following lemma (see [K, p. 140]).

Lemma 7. *A subspace V of W, containing W_0, is closed in W if and only if $\hat{V} := \{\hat{v} : v \in V\}$ is closed in the quotient space $\hat{W}$.*

∎

The closedness of $\operatorname{im} D$ in the dual space W' implies the following result.

Lemma 8. *The space $(\hat{W}, [\cdot \mid \cdot]\hat{})$, where $[\cdot \mid \cdot]\hat{} : \hat{W} \times \hat{W} \longrightarrow \mathbf{R}$ denotes the quotient indefinite inner product defined by*

$$[\,\hat{u} \mid \hat{v}\,]\hat{} := [\,u \mid v\,] , \qquad u, v \in W ,$$

is a Kreĭn space.

Dem. Let $J : W' \longrightarrow W$ stand for the isomorphism (which exists by the Riesz representation theorem) satisfying

$$(\forall\, f \in W')(\forall\, u \in W) \quad {}_{W'}\langle\, f, u\,\rangle_W = \langle\, J(f) \mid u\,\rangle_T .$$

Then $G := J \circ D : W \longrightarrow W$ is a continuous linear operator, and for $u, v \in W$ we have

$$[\,u \mid v\,] = {}_{W'}\langle\, Du, v\,\rangle_W = \langle\, J(Du) \mid v\,\rangle_T = \langle\, Gu \mid v\,\rangle_T .$$

As $[\,u \mid v\,] = [\,v \mid u\,]$, it follows that $G = G^*$, which implies that G is the Gramm operator for the space $(W, [\cdot \mid \cdot])$. Furthermore, $\ker G = \ker D = W_0$, and $\operatorname{im} G = J(\operatorname{im} D)$ is closed ($\operatorname{im} D$ is closed and J is an isomorphism of Hilbert spaces), so Theorem I implies that $(\hat{W}, [\cdot \mid \cdot]\hat{}\,)$ is a Kreĭn space.

Q.E.D.

In the next two lemmas we address how the properties of orthogonality and maximality transfer from W to $\hat{W}$ and vice versa.

Lemma 9. *For any subspace U of W we have*

$$(\hat{U})^{[\perp]\hat{}} = \widehat{U^{[\perp]}} .$$

Dem. Let $\hat{v} \in \widehat{U^{[\perp]}}$ for some $v \in U^{[\perp]}$; the following equivalences hold:

$$\begin{aligned} v \in U^{[\perp]} \quad &\iff \quad (\forall\, u \in U) \quad [\,u \mid v\,] = 0 \\ &\iff \quad (\forall\, \hat{u} \in \hat{U}) \quad [\,\hat{u} \mid \hat{v}\,]\hat{} = 0 \quad \iff \quad \hat{v} \in (\hat{U})^{[\perp]\hat{}} . \end{aligned}$$

Q.E.D.

Lemma 10. *For any subspace V of W:*

a) if V is maximal nonnegative (nonpositive) in W, then $\hat{V}$ is maximal nonnegative (nonpositive) in $\hat{W}$;

b) if $W_0 \subseteq V$ and $\hat{V}$ is maximal nonnegative (nonpositive) in $\hat{W}$, then V is maximal nonnegative (nonpositive) in W.

Dem. **a)** Let V be a maximal nonnegative subspace of W. Then for $\hat{u} \in \hat{V}$ (for some $u \in V$) we have

$$[\,\hat{u} \mid \hat{u}\,]\hat{} = [\,u \mid u\,] \geqslant 0\,,$$

which implies that $\hat{V}$ is nonnegative in $\hat{W}$. Suppose that $\hat{V}$ is not maximal nonnegative. Then there exists $\hat{v} \in \hat{W} \setminus \hat{V}$ (for some $v \notin V$), such that for arbitrary $\alpha, \beta \in \mathbf{R}$ and $u \in V$ we have

$$[\,\alpha\hat{u} + \beta\hat{v} \mid \alpha\hat{u} + \beta\hat{v}\,]\hat{} \geqslant 0\,.$$

As $(\widehat{\alpha u + \beta v}) = \alpha\hat{u} + \beta\hat{v}$, we get

$$[\,\alpha u + \beta v \mid \alpha u + \beta v\,] = [\,(\widehat{\alpha u + \beta v}) \mid (\widehat{\alpha u + \beta v})\,]\hat{} = [\,\alpha\hat{u} + \beta\hat{v} \mid \alpha\hat{u} + \beta\hat{v}\,]\hat{} \geqslant 0\,,$$

which implies that the subspace $[v] + V \supset V$ is nonnegative in W, which contradicts the maximality.

In an analogous way one can prove the statement when V is maximal nonpositive subspace of W.

b) Let now $\hat{V}$ be a maximal nonnegative subspace of the quotient space $\hat{W}$ and $W_0 \subseteq V$. For any $u \in V$ we have

$$[\,u \mid u\,] = [\,\hat{u} \mid \hat{u}\,]\hat{} \geqslant 0\,,$$

which implies the non-negativity of V in W. If V were not maximal nonnegative, then there would exist a $v \in W \setminus V$, such that for arbitrary $\alpha, \beta \in \mathbf{R}$ and $u \in V$ it would be

$$[\,\alpha u + \beta v \mid \alpha u + \beta v\,] \geqslant 0\,.$$

Then from

$$[\,\alpha\hat{u} + \beta\hat{v} \mid \alpha\hat{u} + \beta\hat{v}\,]\hat{} = [\,(\widehat{\alpha u + \beta v}) \mid (\widehat{\alpha u + \beta v})\,]\hat{} = [\,\alpha u + \beta v \mid \alpha u + \beta v\,] \geqslant 0\,,$$

it would follow that $[\hat{v}] + \hat{V} \supseteq \hat{V}$ is nonnegative in W. However, $\hat{V}$ being maximal nonnegative gives us the equality $\hat{V} = [\hat{v}] + \hat{V}$ or, in other words, that $\hat{v} \in \hat{V}$. Then there exists $u \in V$, such that $v \in u + W_0$. Since $W_0 \subseteq V$, we have $v \in V$, which is a contradiction.

In an analogous way one can prove the statement when $\hat{V}$ is maximal nonpositive.

Q.E.D.

Remark. From the proof of previous lemma it follows that a subspace V of W is nonnegative (nonpositive) if and only if $\hat{V}$ is nonnegative (nonpositive) in $\hat{W}$. ∎

Conditions (V) are equivalent to (X)

In [EGC, Theorem 3.3] it has been shown that the conditions (V) imply that V is a maximal nonnegative subspace of W. In this subsection we shall give a quite different (and simpler) proof of that statement and prove the converse as well.

Theorem 2.

a) Let subspaces V and $\tilde{V}$ of W satisfy (V). Then V is maximal nonnegative in W (i.e. it satisfies (X)) and $\tilde{V}$ is maximal nonpositive in W.

b) Let a subspace V be maximal nonnegative in W. Then V and $\tilde{V} := V^{[\perp]}$ satisfy (V).

Dem. **a)** From (V1) it follows that $\hat{V}$ is nonnegative and $\hat{\tilde{V}}$ nonpositive in $\hat{W}$, while (V2) and Lemma 9 imply

$$\hat{\tilde{V}} = \widehat{V^{[\perp]}} = \hat{V}^{[\perp]\hat{}}.$$

An application of Theorem II gives that its closure $\mathsf{Cl}\,\hat{V}$ is maximal nonnegative in $\hat{W}$. Since V is closed and $W_0 \subseteq V$, from Lemma 7 it follows that $\hat{V}$ is closed in $\hat{W}$. Therefore, $\hat{V} = \mathsf{Cl}\,\hat{V}$, and it is maximal nonnegative in the quotient, so by Lemma 10(b) V is a maximal nonnegative subspace of W. The proof that $\tilde{V}$ is maximal nonpositive is analogous.

b) By Lemma I the orthogonal complement of a maximal nonnegative subspace is nonpositive, which proves (V1). Lemma 10(a) implies then that $\hat{V}$ is maximal nonnegative in $\hat{W}$, and therefore by theorems III and IV it is closed and equal to $(\hat{V}^{[\perp]\hat{}})^{[\perp]\hat{}}$. Now we apply Lemma 9 and obtain

$$\hat{V} = (\hat{V}^{[\perp]\hat{}})^{[\perp]\hat{}} = (\widehat{V^{[\perp]}})^{[\perp]\hat{}} = (\widehat{V^{[\perp][\perp]}})\,.$$

Since $W_0 \subseteq V$ (V is maximal nonnegative and therefore contains the isotropic part of W, by Lemma II), and also $W_0 \subseteq V^{[\perp][\perp]}$, it follows that $V = V^{[\perp][\perp]} = \tilde{V}^{[\perp]}$, which proves (V2).

Q.E.D.

As a consequence of theorems 1 and 2 we have the following well-posedness result.

Corollary 1. *If (T1)–(T3) hold, V is a maximal nonnegative subspace of W, and if we define $\tilde{V} := V^{[\perp]}$, then the operators*

$$T_{|_V} : V \longrightarrow L \qquad \text{and} \qquad \tilde{T}_{|_{\tilde{V}}} : \tilde{V} \longrightarrow L$$

are isomorphisms. ∎

Interdependence between (V) and (M)

Theorem 3. *([EGC, Theorem 4.2]) Let (T1)–(T3) hold and let $M \in \mathcal{L}(W; W')$ satisfy (M). Then the subspaces*

$$V := \ker(D - M) \quad \text{and} \quad \tilde{V} := \ker(D + M^*)$$

satisfy (V). ∎

As a direct consequence, we have

Corollary 2. *Under assumptions of previous theorem. the restrictions of operators*

$$T_{|_{\ker(D-M)}} : \ker(D - M) \longrightarrow L \qquad i \qquad \tilde{T}_{|_{\ker(D+M^*)}} : \ker(D + M^*) \longrightarrow L$$

are isomorphisms. ∎

The above theorem states that the conditions (M) imply (V) (with $V := \ker(D - M)$ and $\tilde{V} := \ker(D + M^*)$). The question of the converse: *For given V and $\tilde{V}$ that satisfy (V), is there an operator $M \in \mathcal{L}(W; W')$ satisfying (M) and such that $V = \ker(D - M)$?* appears to be more challenging. The following results give only a partial answer.

Theorem 4. *([EGC, Theorem 4.3]) Let V and $\tilde{V}$ be two subspaces of W satisfying (V), and let us assume that there exist operators $P \in \mathcal{L}(W; V)$ and $Q \in \mathcal{L}(W, \tilde{V})$ such that*

$$\begin{aligned} (\forall\, v \in V) \quad & D(v - Pv) = 0\,, \\ (\forall\, v \in \tilde{V}) \quad & D(v - Qv) = 0\,, \\ & DPQ = DQP\,. \end{aligned}$$

If we define $M \in \mathcal{L}(W; W')$ by

$$\begin{aligned} {}_{W'}\langle\, Mu, v \,\rangle_W = & {}_{W'}\langle\, DPu, Pv \,\rangle_W - {}_{W'}\langle\, DQu, Qv \,\rangle_W \\ & + {}_{W'}\langle\, D(P + Q - PQ)u, v \,\rangle_W - {}_{W'}\langle\, Du, (P + Q - PQ)v \,\rangle_W\,, \end{aligned}$$

for $u, v \in W$, then $V = \ker(D - M)$, $\tilde{V} = \ker(D + M^*)$ and M satisfies (M). ∎

The proof of the above theorem was inspired by Friedrichs' proof of the corresponding result in the finite-dimensional (i.e. the classical) case. However, while in the finite-dimensional case the existence of operators P and Q is guaranteed, here it cannot be taken for granted (see [EGC]). One simple situation when those operators do exist is given by the following lemma.

Lemma 11. *[EGC, Lemma 4.4] Suppose additionally that $V+\tilde{V}$ is closed in W. Then there exist projectors $P: W \longrightarrow V$ and $Q: W \longrightarrow \tilde{V}$ such that $PQ = QP$, and $M \in \mathcal{L}(W; W')$ can be constructed as in the previous theorem.* ∎

In general, the question of closedness of $V+\tilde{V}$, as well as a more general question of existence of operators P and Q from the above lemma was left open in [EGC]. However, in a number of examples we either have $V+\tilde{V} = W$ or $V = \tilde{V}$, and the closedness of $V+\tilde{V}$ is then fulfilled (see [B, EGC]).

On closedness of $V+\tilde{V}$

Using some well-known properties of Kreĭn spaces we shall construct an example that shows that $V+\tilde{V}$ does not need to be closed in W, and briefly discuss some cases when it is closed.

One can easily see that any two subspaces $V_1, V_2 \subseteq W$ satisfy

$$\widehat{(V_1+V_2)} = \{u+v+W_0 : u \in V_1, v \in V_2\} = \hat{V}_1 + \hat{V}_2 \,.$$

Therefore, if additionally $W_0 \subseteq V_1+V_2$, by Lemma 7 it follows that V_1+V_2 is closed if and only if $\hat{V}_1+\hat{V}_2$ is closed.

Theorem 5. *Let subspaces V and $\tilde{V}$ of W satisfy (V), $V\cap\tilde{V} = W_0$ and $W \neq V+\tilde{V}$. Then $V+\tilde{V}$ is not closed in W.*

Dem. From $V\cap\tilde{V} = W_0$ it follows that $\hat{V}\cap\hat{\tilde{V}} = \{\hat{0}\}$, while by using (V2) and Lemma 9 we get $\hat{\tilde{V}} = \widehat{V^{[\perp]}} = (\hat{V})^{[\perp]}$. By Theorem V we have that $\mathsf{Cl}\,(\hat{V}+\hat{\tilde{V}}) = \hat{W}$. From $W \neq V+\tilde{V}$ and $V\cap\tilde{V} = W_0$, it is immediate that $\hat{V}+\hat{\tilde{V}} \neq \hat{W}$, which implies $\hat{V}+\hat{\tilde{V}} \neq \mathsf{Cl}\,(\hat{V}+\hat{\tilde{V}})$, and therefore $\hat{V}+\hat{\tilde{V}}$ is not closed in $\hat{W}$. Thus $V+\tilde{V}$ is not closed in W, by Lemma 7.

Q.E.D.

The above theorem gives a basis for construction of an example of subspaces satisfying (V) whose sum is not closed. We shall use the same elliptic equation as in [EGC, 5.3], but with the Robin boundary condition, instead of the Dirichlet one. In our case we do not get $V = \tilde{V}$. We choose to work out the details, in order to illustrate the use of indefinite inner product space formalism.

Example. (elliptic equation) Let $\Omega \subseteq \mathbf{R}^d$ $(d>1)$ be an open and bounded set with Lipschitz boundary Γ, and function $\mu \in \mathrm{L}^\infty(\Omega)$ away from zero (i.e. $|\mu(\mathbf{x})| \geqslant \alpha_0 > 0$ (a.e. $\mathbf{x}\in\Omega$)). A scalar elliptic equation

$$-\triangle u + \mu u = f \,,$$

where $f \in \mathrm{L}^2(\Omega)$ is given, can be written as an equivalent first order system

$$\begin{cases} \mathsf{p} + \nabla u = 0 \\ \mu u + \operatorname{div}\mathsf{p} = f \end{cases} ,$$

which is a Friedrichs system (i.e. (T1)–(T3) hold) with the following choice of matrix functions $\mathbf{A}_k$ $(k \in 1..d)$ and $\mathbf{C}$:

$$(\mathbf{A}_k)_{ij} = \begin{cases} 1, & (i,j) \in \{(k,d+1),(d+1,k)\} \\ 0, & \text{otherwise} \end{cases} ,$$

$$(\mathbf{C})_{ij} = \begin{cases} \mu(x), & i=j=d+1 \\ 1, & i=j\neq d+1 \\ 0, & \text{otherwise} \end{cases} .$$

If we define (for more details v. [AB1]) the spaces:

$$\begin{aligned} \mathrm{L}^2_{\mathrm{div}}(\Omega) &:= \{\mathsf{u} \in \mathrm{L}^2(\Omega; \mathbf{R}^d) : \operatorname{div} \mathsf{u} \in \mathrm{L}^2(\Omega)\} , \\ \mathrm{L}^2_{\mathrm{div},0}(\Omega) &:= \mathsf{Cl}_{\,\mathrm{L}^2_{\mathrm{div}}(\Omega)} \mathrm{C}^\infty_c(\Omega; \mathbf{R}^d) ; \end{aligned}$$

then one can easily see [EGC] that

$$\begin{aligned} W &= \mathrm{L}^2_{\mathrm{div}}(\Omega) \times \mathrm{H}^1(\Omega) , \\ W_0 &= \mathrm{L}^2_{\mathrm{div},0}(\Omega) \times \mathrm{H}^1_0(\Omega) = \mathsf{Cl}_{\,W} \mathrm{C}^\infty_c(\Omega; \mathbf{R}^{d+1}) , \end{aligned}$$

and

$$[\, (\mathsf{p}, u)^\top \mid (\mathsf{r}, v)^\top \,] = {}_{\mathrm{H}^{-\frac12}(\Gamma)}\langle\, \mathcal{T}_{\mathrm{div}} \mathsf{p}, \mathcal{T}_{\mathrm{H}^1} v \,\rangle_{\mathrm{H}^{\frac12}(\Gamma)} + {}_{\mathrm{H}^{-\frac12}(\Gamma)}\langle\, \mathcal{T}_{\mathrm{div}} \mathsf{r}, \mathcal{T}_{\mathrm{H}^1} u \,\rangle_{\mathrm{H}^{\frac12}(\Gamma)} ,$$

where $\mathcal{T}_{\mathrm{div}} : \mathrm{L}^2_{\mathrm{div}}(\Omega) \longrightarrow \mathrm{H}^{-\frac12}(\Gamma)$ and $\mathcal{T}_{\mathrm{H}^1} : \mathrm{H}^1(\Omega) \longrightarrow \mathrm{H}^{\frac12}(\Gamma)$ are the trace operators [AB1].

For a fixed $\alpha > 0$ we define subspaces V and $\tilde V$ defining the Robin boundary condition for the original elliptic equation:

$$\begin{aligned} V &:= \{(\mathsf{p}, u)^\top \in W : \mathcal{T}_{\mathrm{div}} \mathsf{p} = \alpha \mathcal{T}_{\mathrm{H}^1} u\} , \\ \tilde V &:= \{(\mathsf{r}, v)^\top \in W : \mathcal{T}_{\mathrm{div}} \mathsf{r} = -\alpha \mathcal{T}_{\mathrm{H}^1} v\} . \end{aligned}$$

Let us show that these subspaces satisfy the conditions of the above theorem. We need to prove:

a) V and $\tilde V$ satisfy (V);
b) $V \cap \tilde V = W_0$;
c) $V + \tilde V \neq W$.

a) For $(\mathsf{p}, u)^\top \in V$ we have

$$\begin{aligned} [\, (\mathsf{p}, u)^\top \mid (\mathsf{p}, u)^\top \,] &= 2 {}_{\mathrm{H}^{-\frac12}(\Gamma)}\langle\, \mathcal{T}_{\mathrm{div}} \mathsf{p}, \mathcal{T}_{\mathrm{H}^1} u \,\rangle_{\mathrm{H}^{\frac12}(\Gamma)} \\ &= 2\alpha {}_{\mathrm{H}^{-\frac12}(\Gamma)}\langle\, \mathcal{T}_{\mathrm{H}^1} u, \mathcal{T}_{\mathrm{H}^1} u \,\rangle_{\mathrm{H}^{\frac12}(\Gamma)} = 2\alpha \int_\Gamma (\mathcal{T}_{\mathrm{H}^1} u)^2 dS \geqslant 0 , \end{aligned}$$

which implies $V \subseteq C^+$. In the same way one can prove that $\tilde V \subseteq C^-$, which completes the proof of (V1).

Let us show that $\tilde V = V^{[\perp]}$: the condition $(\mathsf{r}, v)^\top \in V^{[\perp]}$ is by definition

$$(\forall\, (\mathsf{p}, u)^\top \in V) \quad [\, (\mathsf{p}, u)^\top \mid (\mathsf{r}, v)^\top \,] = 0 ,$$

which is equivalent to

$$(\forall\, (\mathsf{p}, u)^\top \in V) \qquad {}_{\mathrm{H}^{-\frac12}(\Gamma)}\langle\, \alpha \mathcal{T}_{\mathrm{H}^1} u, \mathcal{T}_{\mathrm{H}^1} v \,\rangle_{\mathrm{H}^{\frac12}(\Gamma)} + {}_{\mathrm{H}^{-\frac12}(\Gamma)}\langle\, \mathcal{T}_{\mathrm{div}} \mathsf{r}, \mathcal{T}_{\mathrm{H}^1} u \,\rangle_{\mathrm{H}^{\frac12}(\Gamma)} = 0 .$$

Since

$${}_{\mathrm{H}^{-\frac12}(\Gamma)}\langle\, \alpha \mathcal{T}_{\mathrm{H}^1} u, \mathcal{T}_{\mathrm{H}^1} v \,\rangle_{\mathrm{H}^{\frac12}(\Gamma)} = \int_\Gamma \alpha \mathcal{T}_{\mathrm{H}^1} u \mathcal{T}_{\mathrm{H}^1} v dS = {}_{\mathrm{H}^{-\frac12}(\Gamma)}\langle\, \alpha \mathcal{T}_{\mathrm{H}^1} v, \mathcal{T}_{\mathrm{H}^1} u \,\rangle_{\mathrm{H}^{\frac12}(\Gamma)} ,$$

it follows

$$\begin{aligned} (\mathsf{r}, v)^\top \in V^{[\perp]} \quad &\iff \quad (\forall\, (\mathsf{p}, u)^\top \in V) \quad {}_{\mathrm{H}^{-\frac12}(\Gamma)}\langle\, \mathcal{T}_{\mathrm{div}} \mathsf{r} + \alpha \mathcal{T}_{\mathrm{H}^1} v, \mathcal{T}_{\mathrm{H}^1} u \,\rangle_{\mathrm{H}^{\frac12}(\Gamma)} = 0 \\ &\iff \quad (\forall\, z \in \mathrm{H}^{\frac12}(\Gamma)) \quad {}_{\mathrm{H}^{-\frac12}(\Gamma)}\langle\, \mathcal{T}_{\mathrm{div}} \mathsf{r} + \alpha \mathcal{T}_{\mathrm{H}^1} v, z \,\rangle_{\mathrm{H}^{\frac12}(\Gamma)} = 0 \\ &\iff \quad \mathcal{T}_{\mathrm{div}} \mathsf{r} + \alpha \mathcal{T}_{\mathrm{H}^1} v = 0 \quad \iff \quad (\mathsf{r}, v)^\top \in \tilde V . \end{aligned}$$

In an analogous way one could see that $V = \tilde V^{[\perp]}$, which proves (V2).

b) By Lemma 4 $W_0 \subseteq V \cap \tilde{V}$. Vice versa, if $(\mathsf{p}, u)^\top \in V \cap \tilde{V}$, then from the system

$$\begin{aligned} \mathcal{T}_{\mathrm{div}}\mathsf{p} &= \alpha \mathcal{T}_{\mathrm{H}^1} u \\ \mathcal{T}_{\mathrm{div}}\mathsf{p} &= -\alpha \mathcal{T}_{\mathrm{H}^1} u \end{aligned}$$

we easily get $\mathcal{T}_{\mathrm{div}}\mathsf{p} = \mathcal{T}_{\mathrm{H}^1} u = 0$, or in other words $(\mathsf{p}, u)^\top \in \mathrm{L}^2_{\mathrm{div},0}(\Omega) \times \mathrm{H}^1_0(\Omega) = W_0$.

c) Let $(\mathsf{s}, w)^\top \in W$ be such that $\mathcal{T}_{\mathrm{div}}\mathsf{s} \in \mathrm{H}^{-\frac12}(\Gamma) \setminus \mathrm{L}^2(\Gamma)$ (since $\operatorname{im}\mathcal{T}_{\mathrm{div}} = \mathrm{H}^{-\frac12}(\Gamma)$ and $d > 1$, such s does exist). If there exist $(\mathsf{p}, u)^\top \in V$ and $(\mathsf{r}, v)^\top \in \tilde{V}$ such that $(\mathsf{s}, w)^\top = (\mathsf{p}, u)^\top + (\mathsf{r}, v)^\top$, then from

$$\mathcal{T}_{\mathrm{div}}\mathsf{s} = \mathcal{T}_{\mathrm{div}}\mathsf{p} + \mathcal{T}_{\mathrm{div}}\mathsf{r} = \alpha(\mathcal{T}_{\mathrm{H}^1} u - \mathcal{T}_{\mathrm{H}^1} v) \in \mathrm{L}^2(\Gamma)\,,$$

we have contradiction with choice of s.

Theorem 5 implies that for such V and $\tilde{V}$ the space $V + \tilde{V}$ is not closed. Therefore, in general, the properties (V1)–(V2) do not imply closedness of $V + \tilde{V}$. However, we will see that in this example the operator M that satisfy (M1)–(M2) and $V = \ker(D - M)$ still exist. Such operator $M \in \mathcal{L}(W; W')$ can be defined with following expression:

$$\begin{aligned} {}_{W'}\langle\, M(\mathsf{p}, u)^\top, (\mathsf{r}, v)^\top \,\rangle_W = &- {}_{\mathrm{H}^{-\frac12}(\Gamma)}\langle\, \mathcal{T}_{\mathrm{div}}\mathsf{p}, \mathcal{T}_{\mathrm{H}^1} v \,\rangle_{\mathrm{H}^{\frac12}(\Gamma)} \\ &+ {}_{\mathrm{H}^{-\frac12}(\Gamma)}\langle\, \mathcal{T}_{\mathrm{div}}\mathsf{r}, \mathcal{T}_{\mathrm{H}^1} u \,\rangle_{\mathrm{H}^{\frac12}(\Gamma)} \\ &+ 2\alpha \int_\Gamma \mathcal{T}_{\mathrm{H}^1} u \mathcal{T}_{\mathrm{H}^1} v dS\,. \end{aligned}$$

Let us prove that statement: from

$${}_{W'}\langle\, M(\mathsf{p}, u)^\top, (\mathsf{p}, u)^\top \,\rangle_W = 2\alpha \int_\Gamma (\mathcal{T}_{\mathrm{H}^1} u)^2 dS \geqslant 0$$

we have that property (M1) holds.

In order to check (M2), we have to first identify the spaces $\ker(D - M)$ and $\ker(D + M)$. From

$${}_{W'}\langle\, (D - M)(\mathsf{p}, u)^\top, (\mathsf{r}, v)^\top \,\rangle_W = 2\, {}_{\mathrm{H}^{-\frac12}(\Gamma)}\langle\, \mathcal{T}_{\mathrm{div}}\mathsf{p} - \alpha\mathcal{T}_{\mathrm{H}^1} u, \mathcal{T}_{\mathrm{H}^1} v \,\rangle_{\mathrm{H}^{\frac12}(\Gamma)}$$

one can easily see that

$$(D - M)(\mathsf{p}, u)^\top = \mathsf{0} \qquad \Longleftrightarrow \qquad \mathcal{T}_{\mathrm{div}}\mathsf{p} - \alpha\mathcal{T}_{\mathrm{H}^1} u = 0 \qquad \Longleftrightarrow \qquad (\mathsf{p}, u)^\top \in V\,,$$

which means that $\ker(D - M) = V$. We also have

$${}_{W'}\langle\, (D + M)(\mathsf{p}, u)^\top, (\mathsf{r}, v)^\top \,\rangle_W = 2\, {}_{\mathrm{H}^{-\frac12}(\Gamma)}\langle\, \mathcal{T}_{\mathrm{div}}\mathsf{r} + \alpha\mathcal{T}_{\mathrm{H}^1} v, \mathcal{T}_{\mathrm{H}^1} u \,\rangle_{\mathrm{H}^{\frac12}(\Gamma)}\,,$$

and then get (having in mind that $\operatorname{im}\mathcal{T}_{\mathrm{div}} = \mathrm{H}^{-\frac12}(\Gamma)$)

$$(D + M)(\mathsf{p}, u)^\top = \mathsf{0} \qquad \Longleftrightarrow \qquad \mathcal{T}_{\mathrm{H}^1} u = 0 \qquad \Longleftrightarrow \qquad u \in \mathrm{H}^1_0(\Omega)\,,$$

which implies $\ker(D + M) = \mathrm{L}^2_{\mathrm{div}}(\Omega) \times \mathrm{H}^1_0(\Omega)$.

Let us now show that $W = V + \mathrm{L}^2_{\mathrm{div}}(\Omega) \times \mathrm{H}^1_0(\Omega)$. Indeed, for an arbitrary $(\mathsf{s}, w)^\top \in W$, we choose $u = 0$, $v = w$, $\mathsf{r} \in \mathrm{L}^2_{\mathrm{div}}(\Omega)$ such that $\mathcal{T}_{\mathrm{div}}\mathsf{r} = \alpha\mathcal{T}_{\mathrm{H}^1} v$, and $\mathsf{p} = \mathsf{s} - \mathsf{r}$. Now one can easily check that $(\mathsf{s}, w)^\top = (\mathsf{p}, u)^\top + (\mathsf{r}, v)^\top$, and $(\mathsf{r}, v)^\top \in V$, $(\mathsf{p}, u)^\top \in \mathrm{L}^2_{\mathrm{div}}(\Omega) \times \mathrm{H}^1_0(\Omega)$, which proves (M2).

It remains to show that $\ker(D + M^*) = \tilde{V}$. Since

$$\begin{aligned} &{}_{W'}\langle\, M^*(\mathsf{p}, u)^\top, (\mathsf{r}, v)^\top \,\rangle_W = {}_{W'}\langle\, M(\mathsf{r}, v)^\top, (\mathsf{p}, u)^\top \,\rangle_W \\ &= -{}_{\mathrm{H}^{-\frac12}(\Gamma)}\langle\, \mathcal{T}_{\mathrm{div}}\mathsf{r}, \mathcal{T}_{\mathrm{H}^1} u \,\rangle_{\mathrm{H}^{\frac12}(\Gamma)} + {}_{\mathrm{H}^{-\frac12}(\Gamma)}\langle\, \mathcal{T}_{\mathrm{div}}\mathsf{p}, \mathcal{T}_{\mathrm{H}^1} v \,\rangle_{\mathrm{H}^{\frac12}(\Gamma)} + 2\alpha \int_\Gamma \mathcal{T}_{\mathrm{H}^1} v \mathcal{T}_{\mathrm{H}^1} u dS\,, \end{aligned}$$

it follows

$${}_{W'}\langle\, (D + M^*)(\mathsf{p}, u)^\top, (\mathsf{r}, v)^\top \,\rangle_W = 2\, {}_{\mathrm{H}^{-\frac12}(\Gamma)}\langle\, \mathcal{T}_{\mathrm{div}}\mathsf{p} + \alpha\mathcal{T}_{\mathrm{H}^1} u, \mathcal{T}_{\mathrm{H}^1} v \,\rangle_{\mathrm{H}^{\frac12}(\Gamma)}\,,$$

and now, as before, we easily get $\ker(D + M^*) = \tilde{V}$.

∎

The above example shows that neither (V), nor (M) is sufficient to imply that $V+\tilde{V}$ is closed.

One particular situation when $V+\tilde{V}$ is closed is in the case of finite codimension of W_0 in W. For classical Friedrichs' systems this appears when $d=1$, i.e. when dealing with a system of ordinary differential equations (see [AB2] for details).

The question of existence for P and Q

We shall show that the closedness of $V+\tilde{V}$ is actually equivalent to the existence of operators P and Q from Theorem 4. As a first step in that direction we first show that, although looking more general at a first glance, the existence of operators P and Q is actually equivalent to existence of certain projectors on V and $\tilde{V}$ (see the theorem below). In other words we can say that conditions on operators P and Q from Theorem 4 can be simplified.

Our original approach was indirect: we first noticed that the existence of operators P and Q implies the existence of certain projectors in the quotient Kreĭn space; more precisely, by formulæ:

$$\hat{P}\hat{w} := \widehat{Pw}\,, \quad \hat{Q}\hat{w} := \widehat{Qw}\,, \quad w \in W$$

the projectors $\hat{P}, \hat{Q} : \hat{W} \longrightarrow \hat{W}$ are defined, satisfying

$$\begin{gathered}\hat{P}^2 = \hat{P} \quad \text{and} \quad \hat{Q}^2 = \hat{Q}\,,\\ \operatorname{im}\hat{P} = \hat{V} \quad \text{and} \quad \operatorname{im}\hat{Q} = \hat{\tilde{V}}\,,\\ \hat{P}\hat{Q} = \hat{Q}\hat{P}\,.\end{gathered}$$

Then, in the second step, this allowed us to prove the existence of corresponding projectors on W.

However, we are providing a direct (and simpler) proof below.

Theorem 6. *If V and $\tilde{V}$ are two closed subspaces of W that satisfy $W_0 \subseteq V \cap \tilde{V}$, then the following statements are equivalent:*

a) There exist operators $P \in \mathcal{L}(W;V)$ and $Q \in \mathcal{L}(W;\tilde{V})$, such that

$$\begin{aligned}(\forall\, v \in V) &\quad D(v - Pv) = 0\,,\\ (\forall\, v \in \tilde{V}) &\quad D(v - Qv) = 0\,,\\ &\quad DPQ = DQP\,.\end{aligned} \tag{3.1}$$

b) There exist projectors $P', Q' \in \mathcal{L}(W;W)$, such that

$$\begin{gathered}P'^2 = P' \quad \textit{and} \quad Q'^2 = Q'\,,\\ \operatorname{im} P' = V \quad \textit{and} \quad \operatorname{im} Q' = \tilde{V}\,,\\ P'Q' = Q'P'\,.\end{gathered} \tag{3.2}$$

Dem. The second statement trivially implies the first one, and it remains to show the converse. Let us denote by $Q_0 : W \longrightarrow W_0^\perp$ the orthogonal projector on $W_0^\perp$, and define P' and Q' by

$$P' := I - Q_0 + Q_0 P Q_0\,, \qquad Q' := I - Q_0 + Q_0 Q Q_0\,.$$

It is clear that $P', Q' \in \mathcal{L}(W;W)$, and we want to prove that they satisfy (3.2). Most of the statements will be proved only for the operator P', as the proofs for the operator Q' are similar.

Let us first show that $P'u = u$, for $u \in V$: as $I - Q_0$ is an orthogonal projector on W_0 and $W_0 \subseteq V$, it follows $(I - Q_0)u \in V$, and therefore

$$Q_0 u = u - (I - Q_0)u \in V\,. \tag{3.3}$$

By (3.1_1) is then $Q_0u-PQ_0u \in W_0$, which implies existence of $z \in W_0$ such that $PQ_0u = Q_0u+z$. Using that $Q_0^2 = Q_0$ and $\ker Q_0 = W_0$ we finally get

$$P'u = (I - Q_0)u + Q_0PQ_0u = u - Q_0u + Q_0(Q_0u + z) = u - Q_0u + Q_0u = u\,. \tag{3.4}$$

We now prove that $\operatorname{im} P' = V$: from (3.4) it follows that $V \subseteq \operatorname{im} P'$. In order to prove converse inclusion note that $\operatorname{im} P \subseteq V$, which together with (3.3) imply that $\operatorname{im} Q_0PQ_0 \subseteq V$. As $\operatorname{im}(I - Q_0) = W_0 \subseteq V$, from the definition of P' we get $\operatorname{im} P' \subseteq V$.

Using (3.4) and $\operatorname{im} P' = V$ we easily get $P'^2 = P'$, and it remains to prove $P'Q' - Q'P' = 0$. In order to do that let us first show that

$$Q_0PQ_0 = Q_0P \qquad \text{and} \qquad Q_0QQ_0 = Q_0Q\,. \tag{3.5}$$

As before, we only prove the statement for the operator P: since

$$Q_0Pw = Q_0P(I - Q_0)w + Q_0PQ_0w\,, \quad w \in W\,,$$

it is sufficient to show that $Q_0P(I - Q_0) = 0$. As $(I - Q_0)w \in W_0 \subseteq V$, by (3.1_1) there exist $z \in W_0$, such that $P(I - Q_0)w - (I - Q_0)w = z$. This implies

$$Q_0P(I - Q_0)w = Q_0(I - Q_0)w + Q_0z = 0\,.$$

Using (3.5) and $\operatorname{im}(PQ - QP) \subseteq W_0$, we finally get

$$\begin{aligned}
P'Q' - Q'P' &= (I - Q_0 + Q_0PQ_0)(I - Q_0 + Q_0QQ_0) - (I - Q_0 + Q_0QQ_0)(I - Q_0 + Q_0PQ_0)\\
&= (I - Q_0)^2 + Q_0PQ_0^2QQ_0 - (I - Q_0)^2 - Q_0QQ_0^2PQ_0\\
&= Q_0PQ_0QQ_0 - Q_0QQ_0PQ_0\\
&= Q_0PQQ_0 - Q_0QPQ_0\\
&= Q_0(PQ - QP)Q_0 = 0\,,
\end{aligned}$$

which completes the proof.

Q.E.D.

We have already noted that the closedness of $V + \tilde{V}$ implies the existence of operators P and Q. Using the above theorem we can now easily prove the converse statement.

Theorem 7. *If V and $\tilde{V}$ are two closed subspaces of W, then the following statements are equivalent:*

a) $V + \tilde{V}$ is closed;

b) There exist projectors $P', Q' \in \mathcal{L}(W; W)$ such that

$$\begin{gathered}
P'^2 = P' \quad \text{and} \quad Q'^2 = Q'\,,\\
\operatorname{im} P' = V \quad \text{and} \quad \operatorname{im} Q' = \tilde{V}\,,\\
P'Q' = Q'P'\,.
\end{gathered} \tag{3.6}$$

Dem. It has already been proved in [EGC, Lemma 4.4] that the first statement implies the second one. We briefly repeat the construction of P' and Q' for completeness: let $V_3 := (V + \tilde{V})^\perp$, $V_0 := V \cap \tilde{V}$, and we denote by V_1 and V_2 the orthogonal complement of V_0 in V and $\tilde{V}$, respectively. By closedness of V, $\tilde{V}$ and $V + \tilde{V}$ we have

$$W = V_0 \dot{+} V_1 \dot{+} V_2 \dot{+} V_3\,.$$

If we denote by $w = w_0 + w_1 + w_2 + w_3$ decomposition of $w \in W$ that corresponds to above direct sum and define operators P' and Q' with

$$P'w := w_0 + w_1\,, \qquad Q'w := w_0 + w_2\,,$$

one can easily check that those P' and Q' satisfy required properties.

In order to prove the other implication, let us assume that projectors P' and Q' do exist, while $V + \tilde{V}$ is not closed in W. Then there exist $w \in \mathsf{Cl}\,(V + \tilde{V}) \setminus (V + \tilde{V})$ and sequence (w_n) in $V + \tilde{V}$, such that $w_n \longrightarrow w$ in W. For each $n \in \mathbf{N}$, there exist $u_n \in V$ and $v_n \in \tilde{V}$, such that $w_n = u_n + v_n$. Since, by $(3.6_{1,2})$, $P'u_n = u_n$ and $Q'v_n = v_n$ we get

$$
\begin{aligned}
(P' + Q')w_n &= P'u_n + P'v_n + Q'u_n + Q'v_n \\
&= u_n + P'v_n + Q'u_n + v_n \\
&= w_n + P'v_n + Q'u_n \, .
\end{aligned} \tag{3.7}
$$

Using again $P'u_n = u_n$ and (3.6_3) we have

$$Q'u_n = Q'P'u_n = P'Q'u_n \in \operatorname{im} P' = V \, .$$

As $\operatorname{im} Q' = \tilde{V}$, it follows $Q'u_n \in V \cap \tilde{V}$. Analogously we could prove that $P'v_n \in V \cap \tilde{V}$, and therefore $z_n := P'v_n + Q'u_n \in V \cap \tilde{V}$. Now (3.7) becomes

$$(P' + Q')w_n = w_n + z_n \, , \tag{3.8}$$

and from convergence of (w_n) and $(P' + Q')w_n$ (the operator $P' + Q'$ is continuous) it follows that z_n converges to some limit z which belongs also to $V \cap \tilde{V}$ by closedness. Using this fact and $\operatorname{im} P' = V$, $\operatorname{im} Q' = \tilde{V}$, after passing to the limit in (3.8) we achieve

$$w = P'w + Q'w - z \in V + \tilde{V} \, ,$$

which is a contradiction.

Q.E.D.

As we have already constructed an example where $V + \tilde{V}$ is not closed, this implies that operators P and Q do not always exist. However, we have also seen that in this example the operator M does exist. Therefore, the construction of operator M from Theorem 4 is not the only possible way to do that, and the question of equivalence of conditions (V) and (M) is still open.

Another approach to the equivalence of (V) and (M)

We now present a slightly different approach to the question of existence of operator M (under the assumptions (V)), and reduce it to some geometric conditions in the graph space. We shall actually prove that the existence of operator M is equivalent to the existence of a closed nonpositive subspace that, together with V, spans the whole graph space. If such a space is given, we show how to explicitly construct an operator M.

Theorem 8.

a) *If V and $\tilde{V}$ are two subspaces of W that satisfy (V), and if there exists a closed subspace $W_2 \subseteq C^-$ of W, such that $V \dot{+} W_2 = W$, then there exist an operator $M \in \mathcal{L}(W; W')$ that satisfy (M) and $V = \ker(D - M)$. If we define W_1 as orthogonal complement of W_0 in V, so that $W = W_1 \dot{+} W_0 \dot{+} W_2$, and denote by R_1, R_0, R_2 projectors that correspond to above direct sum, then one such operator is given with $M = D(R_1 - R_2)$.*

b) *Let $M \in \mathcal{L}(W; W')$ be an operator that satisfy (M1)–(M2), and denote $V = \ker(D - M)$. If we denote by W_2 the orthogonal complement of W_0 in $\ker(D + M)$, then $W_2 \subseteq C^-$ is closed, $V \dot{+} W_2 = W$, and M coincide with the operator constructed as in (a).*

Dem. **a)** Let $w = w_1 + w_0 + w_2$ denotes decomposition of an arbitrary $w \in W$ that corresponds to $W_1 \dot{+} W_0 \dot{+} W_2$. Let us first prove that M satisfy (M1): Using symmetry of D, $w_1 \in C^+$, and $w_2 \in C^-$ we get

$$
\begin{aligned}
{}_{W'}\langle\, Mw, w \,\rangle_W &= {}_{W'}\langle\, D(R_1 - R_2)w, w \,\rangle_W = {}_{W'}\langle\, Dw, (R_1 - R_2)w \,\rangle_W \\
&= {}_{W'}\langle\, Dw_1 + Dw_2, w_1 - w_2 \,\rangle_W = [\, w_1 \mid w_1 \,] - [\, w_2 \mid w_2 \,] \geqslant 0 \, .
\end{aligned}
$$

Let us now prove that $V = \ker(D - M)$: for given $w = w_1 + w_0 \in V$ we have

$$(D - M)(w_1 + w_0) = D(w_1 + w_0) - D(R_1 - R_2)(w_1 + w_0) = Dw_1 - Dw_1 = 0\,,$$

which proves one inclusion. In order to prove the second one we first note that

$$D - M = D(I - R_1 + R_2) = D(I - R_1 - R_2 + 2R_2) = D(R_0 + 2R_2) = 2DR_2\,.$$

Therefore, for $w \in \ker(D - M)$ we have $DR_2w = 0$, which implies $R_2w \in W_0$. Since $\operatorname{im} R_2 = W_2$ and $W_0 \cap W_2 = \{0\}$ it follows $R_2w = 0$, or in other words $w \in \ker R_2 = V$.

In order to prove (M2) it is sufficient to show that $W_2 \subseteq \ker(D + M)$: for $w_2 \in W_2$ we have

$$(D + M)w_2 = Dw_2 + DR_1w_2 - DR_2w_2 = Dw_2 - Dw_2 = 0\,,$$

which completes the proof of statement (a).

b) First we note that $\ker(D - M) \cap \ker(D + M) = W_0$: indeed, from

$$\begin{aligned} Dw - Mw &= 0 \\ Dw + Mw &= 0 \end{aligned}$$

we easily get $Dw = 0$, meaning that $w \in W_0$, while the second inclusion follows from $\ker D = \ker M = W_0$. Let W_1 and W_2 be the orthogonal complements of W_0 in $V = \ker(D - M)$ and $\ker(D + M)$, respectively. Then by (M2) it holds $W_1 \dot{+} W_0 \dot{+} W_2$. Let, as before, R_1, R_0, R_2 stand for projectors associated to this direct sum of closed subspaces, and $w = w_1 + w_0 + w_2$ corresponding decomposition of an arbitrary $w \in W$. Let us first show that $M = D(R_1 - R_2)$: since $w_1 + w_0 \in \ker(D - M)$ and $w_0 + w_2 \in \ker(D + M)$ it follows

$$\begin{aligned} (D - M)w &= (D - M)w_2 \\ (D + M)w &= (D + M)w_1 \end{aligned}\,.$$

Subtracting these equations (and having in mind that $\ker M = W_0$) we get

$$2Mw = Dw_1 - Dw_2 + Mw_1 + Mw_2 = Dw_1 - Dw_2 + Mw\,,$$

which imply

$$Mw = D(w_1 - w_2) = D(R_1 - R_2)w\,,$$

and therefore $M = D(R_1 - R_2)$.

It remains to show that $W_2 \subseteq C^-$: using (M1) and symmetry of D for arbitrary $w_2 \in W_2$ we get

$$\begin{aligned} 0 &\leqslant {}_{W'}\langle\, Mw_2, w_2 \,\rangle_W = {}_{W'}\langle\, D(R_1 - R_2)w_2, w_2 \,\rangle_W \\ &= {}_{W'}\langle\, Dw_2, (R_1 - R_2)w_2 \,\rangle_W = {}_{W'}\langle\, Dw_2, -w_2 \,\rangle_W = -[\, w_2 \mid w_2 \,]\,, \end{aligned}$$

which completes the proof.

Q.E.D.

By the above theorem, under assumptions (V), the existence of an operator $M \in \mathcal{L}(W, W')$ that satisfy (M) and $V = \ker(D - M)$ is equivalent to the existence of a closed subspace $W_2 \subseteq C^-$ that satisfies $W_2 \dot{+} V = W$. Next we prove that conditions on W_2 can be reduced. In the rest we assume that conditions (V) hold (or, equivalently, that V is a maximal nonnegative subspace of W).

Lemma 12. *Let a subspace W_2'' of W satisfies $W_2'' \subseteq C^-$ and $W_2'' + V = W$. Then there is a closed subspace W_2 of W, such that $W_2 \subseteq C^-$ and $W_2 \dot{+} V = W$.*

Dem. Let us denote by W_2' a maximal nonpositive subspace that contains W_2'' (such exists by Zorn's lemma), and note that $W_0 \subseteq W_2'$ (Lemma II). From Lemma 10 it follows that $\hat{W}_2'$ is maximal nonpositive in $\hat{W}$, and therefore closed, as $\hat{W}$ is a Kreĭn space (Theorem III). Lemma 7 implies that W_2' is closed, and so is $W_2' \cap V$. If we denote by W_2 the orthogonal complement of $W_2' \cap V$ in W_2', then it easily follows that W_2 is closed, $W_2 \subseteq C^-$ and $W_2 \dot{+} V = W$.

Q.E.D.

The question of equivalence between (V) and (M) is therefore reduced to the question of existence of a nonpositive subspace W_2 such that $W_2 + V = W$, for given maximal nonnegative subspace V. We do not know the answer to this question in a general indefinite inner product space, but we do know that the existence of such W_2 is ensured in the case of a Kreĭn space (Theorem VI). Since W is not a Kreĭn space, we would once more use the same idea and try to interpret our question in terms of the quotient Kreĭn space. First we prove one technical lemma.

Lemma 13. *If $U_1+U_2 = W$ for some subspaces $U_1 \subseteq C^+$ and $U_2 \subseteq C^-$ of W, then $U_1 \cap U_2 \subseteq W_0$. If additionally U_1 is maximal nonnegative and U_2 maximal nonpositive, then $U_1 \cap U_2 = W_0$.*

Dem. In order to prove the first statement, let us suppose that there is $v \in (U_1 \cap U_2) \setminus W_0$. Next we prove that $v[\perp]U_1$: for an arbitrary $u \in U_1$ and $\lambda \in \mathbf{R}$ we have $u + \lambda v \in U_1$. Using non-negativity of U_1 and $v \in C^0$ we derive

$$0 \leqslant [\, u + \lambda v \mid u + \lambda v\,] = [\, u \mid u\,] + 2\lambda[\, u \mid v\,]\,.$$

Since $\lambda \in \mathbf{R}$ is arbitrary, we easily get $[\, u \mid v\,] = 0$, which proves that $v[\perp]U_1$. Similarly one can prove that $v[\perp]U_2$, and therefore $v[\perp](U_1 + U_2) = W$. Then from $v \in W \cap W^{[\perp]} = W_0$ we have contradiction with assumption, which proves the first statement.

The second statement now easily follows from Lemma II.

Q.E.D.

Theorem 9. *For a maximal nonnegative subspace V of W, the following statements are equivalent:*

a) There is a maximal nonpositive subspace W_2 of W, such that $W_2 + V = W$;

b) There is a nonpositive subspace $W_{\hat 2}$ of $\hat W$, such that $W_{\hat 2} + \hat V = \hat W$.

Dem. First we prove that the first statement implies the second one: clearly $\hat W_2 + \hat V = \widehat{W_2 + V} = \hat W$, and $\hat W_2$ is nonpositive.

Let us now prove the converse implication: first we define a subspace

$$W_2 := \{v \in W : \hat w \in W_{\hat 2}\}$$

of W, so that $W_2 \in C^-$, $\hat W_2 = W_{\hat 2}$ and $\widehat{W_2 + V} = \hat W_2 + \hat V = \hat W$. Since $W_0 \subseteq W_2 + V$ it follows that $W_2 + V = W$. Finally, we extend W_2 to maximal nonpositive subspace, which then satisfies our requirements.

Q.E.D.

The above theorem reduces the question of equivalence between (V) and (M) to the question of existence of a nonpositive subspace $W_{\hat 2}$ such that $W_{\hat 2} + \hat V = \hat W$, for a given maximal nonnegative subspace $\hat V$ of the quotient Kreĭn space $\hat W$. As existence of such $W_{\hat 2}$ is ensured by Theorem VI, we have the following corollary.

Corollary 3. *The conditions (V) and (M) are equivalent.* ∎

Concluding remarks

Significant advance to the theory of Friedrichs' systems has been made in [EGC], and our contribution is mostly in answering the questions they left open. The next step in this research would be an attempt to better connect the abstract results to those in the classical Friedrichs setting [Ra, J].

We have also indirectly proved that a number of properties of Kreĭn spaces is shared by general indefinite inner product spaces whose quotient by its degenerate part is also a Kreĭn space. As much as we know, such an approach is new, and we hope that some experts on the Kreĭn space theory might find our results a starting point for further research, either on general theory of such spaces, or in the direction of its potential further application to Friedrichs' systems.

References

[A] Nenad Antonić: *H-measure applied to symmetric systems*, *Proc. Roy. Soc. Edinburgh* **126A** (1996) 1133–1155.

[AB1] Nenad Antonić, Krešimir Burazin: *Graph spaces of first-order linear partial differential operators*, *Math. Communications* **14**(1) (2009) 135–155.

[AB2] Nenad Antonić, Krešimir Burazin: *On equivalent descriptions of boundary conditions for Friedrichs systems*, submitted.

[AL] Nenad Antonić, Martin Lazar: *H-measures and variants applied to parabolic equations*, *J. Math. Anal. Appl.* **343** (2008) 207–225.

[AL1] Nenad Antonić, Martin Lazar: *A parabolic variant of H-measures*, *Annali Univ. Ferrara* **54** (2008) 183–201.

[AI] Tomas Jakovlevič Azizov, Iosif Semenovič Iohvidov: *Linear operators in spaces with an indefinite metric,* Wiley, 1989.

[Bo] János Bognár: *Indefinite inner product spaces,* Springer, 1974.

[B] Krešimir Burazin: *Contribution to the theory of Friedrichs' and hyperbolic systems*, Ph.D. thesis, University of Zagreb, 2008.

[EG] Alexandre Ern, Jean-Luc Guermond: *Discontinuous Galerkin methods for Friedrichs' systems. I. General theory, SIAM J. Numer. Anal.* **44**(2) (2006) 753–778.

[EGC] Alexandre Ern, Jean-Luc Guermond, Gilbert Caplain: *An intrinsic criterion for the bijectivity of Hilbert operators related to Friedrichs' systems, Comm. Partial Diff. Eq.* **32** (2007) 317–341.

[F] Kurt O. Friedrichs: *Symmetric positive linear differential equations, Comm. Pure Appl. Math.* **11** (1958) 333–418.

[FL] Kurt O. Friedrichs, Peter D. Lax: *Boundary value problems for first order operators, Comm. Pure Appl. Math.* **18** (1965) 355–388.

[HMSW] Paul Houston, John A. Mackenzie, Endre Süli, Gerald Warnecke: *A posteriori error analysis for numerical approximation of Friedrichs systems, Numer. Math.* **82** (1999) 433–470.

[J] Max Jensen: *Discontinuous Galerkin methods for Friedrichs systems with irregular solutions*, Ph.D. thesis, University of Oxford, 2004.
`http://www.comlab.ox.ac.uk/research/na/thesis/thesisjensen.pdf`

[K] Tosio Kato: *Perturbation theory for linear operators,* Springer, 1980.

[LP] Peter D. Lax, Ralph S. Phillips: *Local boundary conditions for dissipative symmetric linear differential operators, Comm. Pure Appl. Math.* **13** (1960) 427–455.

[M] Cathleen S. Morawetz: *The Dirichlet problem for the Tricomi equation, Comm. Pure Appl. Math.* **23** (1970) 587–601.

[PS] Ralph S. Phillips, Leonard Sarason: *Singular symmetric positive first order differential operators, Journal of Mathematics and Mechanics* **15** (1966) 235–271.

[Ra] Jeffrey Rauch: *Boundary value problems with nonuniformly characteristic boundary, Journal de mathèmatiques pures et appliques* **73** (1994) 347–353.