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Homogenisation theory for Friedrichs systems

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Abstract

We develop a general homogenisation procedure for Friedrichs systems. Under reasonable assumptions, the concepts of G and H-convergence are introduced. As Friedrichs systems can be used to represent various boundary or initial-boundary value problems for partial differential equations, some additional assumptions are needed for compactness results. These assumptions are particularly examined for the stationary diffusion equation, the heat equation and a model example of a first order equation leading to memory effects. In the first two cases, the equivalence with the original notion of H-convergence is proved.

Keywords: homogenisation, symmetric positive systems, H-convergence, G-convergence, compensated compactness

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1. Introduction

The development of homogenisation theory has been motivated by the necessity to describe relations between different scales: a smaller scale (microscale) and a bigger one (macroscale). Knowing the equations (physical laws) which describe the behaviour of a heterogeneous material on a microscale, one is interested in a good approximation at the macroscale.

Historically, these questions were introduced with mathematical rigour by Sergio Spagnolo in 1967 through the concept of G-convergence for stationary diffusion equation, although they were considered phenomenologically much earlier by Poisson, Faraday, Maxwell, Rayleigh, Einstein etc. The letter G stands for Green, as Spagnolo was motivated by the weak convergence of corresponding Green kernels. The notion of H-convergence was also originally introduced for the stationary diffusion equation [22] (it is also known under the name strong G-convergence [31]). It differs from the concept of G-convergence, as it treats the convergence of coefficients appearing in the equation, instead of the convergence of operators. However, for symmetric coefficients these two notions are equivalent. Similar homogenisation results are also derived for other elliptic equations and systems, including linearised elasticity (for a nice introduction see [1]), but also for parabolic and hyperbolic problems [25, 26, 32, 11], and one can naturally generalise this concept to Friedrichs systems. One of the important gains of general theories of homogenisation is the justification of multiple scale expansion method used in periodic homogenisation where the coefficients are periodic with period which tends to zero [8, 1]. Having such a sequence of problems in mind, one considers a sequence of corresponding solutions, trying to determine the limit and the corresponding boundary value problem. If the limiting equation is of the same form, the question is how to calculate the effective coefficients, expected to be constant in the periodic case.

Symmetric positive systems (also known as Friedrichs systems) form a class of boundary value problems which allow the study of a wide range of differential equations in a unified framework. They were introduced by Kurt Otto Friedrichs [19] in an attempt to handle transonic flow problems, which are partially hyperbolic and partially elliptic in different parts of the domain. Many advances have been made since, overcoming numerous difficulties Friedrichs encountered in his seminal paper [19], such as the question of traces at the boundary for functions from the corresponding graph space [23, 2, 20].

During the last decade, a renewed interest in this theory was initiated by numerical mathematicians (already Friedrichs considered numerical solution procedures for such systems, by a finite difference scheme). A number of recent results on discontinuous Galerkin methods for Friedrichs systems can be found in [10, 12, 15, 16, 17, 20]. We understood that the unification of equations of different type (elliptic/parabolic/hyperbolic) within the single framework of Friedrichs systems has practical benefits in their numerical treatment, as both convergence analysis and numerical code can *be shared*.

A good well-posedness result is essential for the justification of a numerical procedure, which motivated the development of an abstract theory of Friedrichs systems by Ern, Guermond and Caplain [15, 18]. They suggested an approach in which the theory of Friedrichs systems is written in terms of operators acting on Hilbert spaces, and gave an intrinsic description of boundary conditions. In this way they achieved an abstract well-posedness result which can be applied to the classical Friedrichs setting.

In this paper we develop a homogenisation theory for Friedrichs systems, and by applying our results to the stationary diffusion equation and the heat equation we rediscover some homogenisation results for these problems and give a new perspective on the homogenisation theory for these equations. We believe that this approach can also be applied to some other linear equations of interest, hopefully leading to some new homogenisation results for these equations.

The idea of homogenisation of a general class of problems that encompass a wide variety of partial differential equations was also studied by some other authors (see [30] and some references therein), and appears to be promising.

The paper is organised as follows: in the rest of this introductory section we first recall the concepts of G and H-convergence for the stationary diffusion equation and the heat equation. Next, we briefly describe the abstract theory of Friedrichs systems, stating the well-posedness result and applying it to the classical Friedrichs partial differential operator. We finish this section by an example, showing how the stationary diffusion equation can be written as the Friedrichs system. In the second section we identify the dual of the graph space of an abstract Friedrichs operator, and give a characterisation of the weak convergence in the graph space. The third section can be considered as the main part of this paper. Here we describe the setting for development of homogenisation theory for Friedrichs systems, introduce the concepts of G and H-convergence, give compactness theorems under some compactness assumptions, and discuss some other interesting topics, such as the convergence of the adjoint operator and the topology of H-convergence. In the fourth section we apply these results to the stationary diffusion equation and the heat equation, showing how some *classical* homogenisation results for these equations can be derived from our theory. Here the Quadratic theorem of compensated compactness is used in order to verify our compactness assumptions. We also show that these compactness assumptions are not satisfied in the simple case of equation that exhibits memory effects in homogenisation. Finally, we close the paper with some concluding remarks.

Homogenisation

Spagnolo [24, 25] considered a sequence of boundary value problems for the stationary diffusion equation in an open and bounded set $\Omega \subseteq \mathbf{R}^d$:

$$\begin{cases} -\mathsf{div}\,(\mathbf{A}^n \nabla u) = f \\ u \in \mathrm{H}^1_0(\Omega)\,, \end{cases} \tag{1}$$

with symmetric matrix-valued functions $\mathbf{A}^n$. He studied the convergence of corresponding Green functions implying the weak convergence of sequence (u_n) of solutions, which led him to the notion of G-convergence. Tartar and Murat [27, 22] considered the general, nonsymmetric case, introducing the concept of H-convergence. They found the bounds on coefficients which were stable under the homogenisation process:

$$\begin{aligned} &\mathbf{A}(\mathbf{x})\boldsymbol{\xi}\cdot\boldsymbol{\xi} \geqslant \alpha|\boldsymbol{\xi}|^2\,, \\ &\mathbf{A}(\mathbf{x})\boldsymbol{\xi}\cdot\boldsymbol{\xi} \geqslant \frac{1}{\beta}|\mathbf{A}(\mathbf{x})\boldsymbol{\xi}|^2\,, \end{aligned} \tag{2}$$

for every $\boldsymbol{\xi} \in \mathbf{R}^d$ and almost every $\mathbf{x} \in \Omega$. The set of all such matrix-valued functions $\mathbf{A} \in \mathrm{L}^\infty(\Omega; \mathrm{M}_d(\mathbf{R}))$ we denote by $\mathcal{M}_d(\alpha, \beta; \Omega)$. In particular, if $\mathbf{A}(\mathbf{x})$ is a symmetric matrix for almost every $\mathbf{x} \in \Omega$, the conditions (2) can be written in a simpler form as $\alpha\mathbf{I} \leqslant \mathbf{A}(\mathbf{x}) \leqslant \beta\mathbf{I}$.

Assuming that a sequence of coefficients $(\mathbf{A}^n)$ in $\mathcal{M}_d(\alpha, \beta; \Omega)$ is given, the corresponding sequence of the solutions (u_n) of (1) is bounded in $\mathrm{H}^1_0(\Omega)$, so it weakly converges (up to a subsequence) to some $u \in \mathrm{H}^1_0(\Omega)$. The main question in homogenisation is to determine the equation satisfied by this u. If it is the equation of the same type, as it is the case for the stationary diffusion equation, one is interested in the corresponding conductivity matrix. In other words, one is trying to define a topology on the set of admissible coefficients, such that the mapping $\mathbf{A} \mapsto u$ determined by the boundary value problem (1) is continuous (with respect to the weak topology on $\mathrm{H}^1_0(\Omega)$), for any right-hand side. Such topology is described by the following definition.

Definition. (**H-convergence for stationary diffusion equation**) We say that a sequence $(\mathbf{A}^n)$ in $\mathcal{M}_d(\alpha, \beta; \Omega)$ *H-converges* to $\mathbf{A} \in \mathcal{M}_d(\alpha', \beta'; \Omega)$ if for arbitrary $f \in \mathrm{H}^{-1}(\Omega)$ the corresponding sequence of the solutions (u_n) of (1) satisfies the following weak convergences

$$\begin{aligned} u_n &\rightharpoonup u \quad \text{in} \quad \mathrm{H}^1_0(\Omega)\,, \\ \mathbf{A}^n\nabla u_n &\rightharpoonup \mathbf{A}\nabla u \quad \text{in} \quad \mathrm{L}^2(\Omega; \mathbf{R}^d). \end{aligned}$$

The last convergence in particular implies that u is the solution of (1) with $\mathbf{A}$ instead of $\mathbf{A}^n$. The following compactness result can be shown: For any sequence $(\mathbf{A}^n)$ in $\mathcal{M}_d(\alpha, \beta; \Omega)$ there exists a H-convergent subsequence whose limit also belongs to $\mathcal{M}_d(\alpha, \beta; \Omega)$.

In this introduction we shall go no further into details concerning the stationary diffusion setting (for details see [29, 1]), but rather pass to the heat equation, and present it more thoroughly, since it fits better to what we have done in the case of Friedrichs systems. The parabolic homogenisation was originally developed by Sergio Spagnolo [26]. We also mention the survey [32] in the case of more general parabolic operators, and [11, 7].

Consider the heat equation in a domain $\Omega_T = \Omega \times \langle 0, T\rangle$, where $\Omega \subseteq \mathbf{R}^d$ is open and bounded, as before, and $T > 0$:

$$\begin{cases} \partial_t u - \mathsf{div}\,(\mathbf{A}\nabla u) = f \\ \qquad\qquad u(\cdot, 0) = u_0 \,. \end{cases} \tag{3}$$

Here we assume that the coefficient matrix $\mathbf{A}$, depending on both t and $\mathbf{x}$, belongs to the set $\mathcal{M}_d(\alpha, \beta; \Omega_T)$. We denote $V := \mathrm{H}^1_0(\Omega)$, so that $V' = \mathrm{H}^{-1}(\Omega)$ and $H := \mathrm{L}^2(\Omega)$, which gives us a Gel'fand triple: $V \hookrightarrow H \hookrightarrow V'$ (the continuous and dense inclusions). For the time dependent functions we define $\mathcal{V} := \mathrm{L}^2(0, T; V)$, so that $\mathcal{V}' = \mathrm{L}^2(0, T; V')$ and $\mathcal{H} := \mathrm{L}^2(0, T; H)$, obtaining again a Gel'fand triple $\mathcal{V} \hookrightarrow \mathcal{H} \hookrightarrow \mathcal{V}'$. The appropriate evolution space for a solution is $\mathcal{W} = \{u \in \mathcal{V} : \partial_t u \in \mathcal{V}'\}$. The corresponding parabolic operator $\mathcal{P} \in \mathcal{L}(\mathcal{W}; \mathcal{V}')$ defined by

$$\mathcal{P}u := \partial_t u - \mathsf{div}\,(\mathbf{A}\nabla u)$$

is an isomorphism of $\mathcal{W}_0 = \{u \in \mathcal{W} : u(\cdot, 0) = 0\}$ onto $\mathcal{V}'$.

Spagnolo introduced the notion of *G-convergence* for more general parabolic operators of the form:

$$\mathcal{P}_{\mathcal{A}} := \partial_t + \mathcal{A} : \mathcal{W} \longrightarrow \mathcal{V}' \,,$$

where $(\mathcal{A}u)(t) := A(t)u(t)$, with $A(t) \in \mathcal{L}(V; V')$ such that for any $\varphi, \psi \in V$

$$\begin{gathered} t \mapsto {}_{V'}\langle\, A(t)\varphi, \psi \,\rangle_V \quad \text{is measurable}\,, \\ \lambda_0 \|\varphi\|^2_V \leqslant {}_{V'}\langle\, A(t)\varphi, \varphi \,\rangle_V \leqslant \Lambda_0 \|\varphi\|^2_V \,, \\ |{}_{V'}\langle\, A(t)\varphi, \psi \,\rangle_V| \leqslant M \sqrt{{}_{V'}\langle\, A(t)\varphi, \varphi \,\rangle_V} \sqrt{{}_{V'}\langle\, A(t)\psi, \psi \,\rangle_V} \,, \end{gathered} \tag{4}$$

where λ_0, Λ_0 and M are some positive constants. The set of all such operators $\mathcal{P}_{\mathcal{A}}$ we denote by $\mathcal{P}(\lambda_0, \Lambda_0, M)$.

Definition. (**G-convergence for parabolic operators**) A sequence of parabolic operators $\mathcal{P}_{\mathcal{A}^n} \in \mathcal{P}(\lambda_0, \Lambda_0, M)$ *G-converges* to a parabolic operator $\mathcal{P}_{\mathcal{A}} \in \mathcal{P}(\lambda_0', \Lambda_0', M')$ if $\mathcal{P}_{\mathcal{A}^n}$ weakly converges to $\mathcal{P}_{\mathcal{A}}$, i.e. if for any $f \in \mathcal{V}'$

$$\mathcal{P}^{-1}_{\mathcal{A}^n} f \rightharpoonup \mathcal{P}^{-1}_{\mathcal{A}} f \quad \text{in } \mathcal{W}_0 \,.$$

In other words, the sequence of the solutions to parabolic problems determined by $\mathcal{P}_{\mathcal{A}^n}$ converges weakly to the solution of the problem determined by $\mathcal{P}_{\mathcal{A}}$ (with the homogeneous initial condition).

In the case of the Gel'fand triple $V \hookrightarrow H \hookrightarrow V'$, where the Hilbert space H is identified with its dual and the inclusions are continuous, under the additional assumption that they are also compact, Spagnolo proved [26, Theorem 1] the following compactness of G-convergence: For any sequence $(\mathcal{P}_{\mathcal{A}^n})$ in $\mathcal{P}(\lambda_0, \Lambda_0, M)$ there is a subsequence $\mathcal{P}_{\mathcal{A}^{n'}}$ and a parabolic operator $\mathcal{P}_{\mathcal{A}} \in \mathcal{P}(\lambda_0, M^2\Lambda_0, \sqrt{\Lambda_0/\lambda_0}M)$, such that $\mathcal{P}_{\mathcal{A}^{n'}} \xrightharpoonup{G} \mathcal{P}_{\mathcal{A}}$.

Moreover, if each $\mathcal{A}^n$ is of the form:

$$\mathcal{A}^n(t)u = -\mathsf{div}\,(\mathbf{A}^n(\cdot, t)\nabla u) \,,$$

then the limit is of the same form, where the matrix coefficients $\mathbf{A}$ satisfy the same type of bounds, but with different constants. To be more precise, in this situation the bounds in (4) are rewritten as

$$\begin{gathered} \lambda_0 |\boldsymbol{\xi}|^2 \leqslant \mathbf{A}\boldsymbol{\xi} \cdot \boldsymbol{\xi} \leqslant \Lambda_0 |\boldsymbol{\xi}|^2 \,, \\ |\mathbf{A}\boldsymbol{\xi} \cdot \boldsymbol{\eta}|^2 \leqslant M \sqrt{\mathbf{A}\boldsymbol{\xi} \cdot \boldsymbol{\xi}} \sqrt{\mathbf{A}\boldsymbol{\eta} \cdot \boldsymbol{\eta}} \,, \end{gathered} \tag{5}$$

which follow from the bounds (2) with $\lambda_0 := \alpha, \Lambda_0 := \beta$ and $M := \sqrt{\beta/\alpha}$, and conversely, from (5) we get the bounds (2) with $\alpha := \lambda_0$ and $\beta := M^2\Lambda_0$. Also, in such a case, on a subsequence we have the convergence

$$\mathbf{A}^{n'}\nabla u_{n'} \rightharpoonup \mathbf{A}\nabla u \quad \text{in } \mathrm{L}^2(\Omega_T;\mathbf{R}^d)\ ,$$

which motivates the following definition [11, 32]:

Definition. (**H-convergence for the heat equation**) A sequence of matrix-valued functions $\mathbf{A}^n \in \mathcal{M}(\alpha,\beta;\Omega_T)$ *H-converges* to $\mathbf{A} \in \mathcal{M}(\alpha',\beta';\Omega_T)$ if for any $f \in \mathcal{V}'$ and $u_0 \in H$ the solutions of parabolic problems

$$\begin{cases} \partial_t u_n - \mathsf{div}\,(\mathbf{A}^n\nabla u_n) = f \\ u_n(\cdot,0) = 0 \end{cases} \tag{6}$$

satisfy

$$\begin{aligned} u_n &\rightharpoonup u \quad \text{in } \mathcal{V}\,, \\ \mathbf{A}^n\nabla u_n &\rightharpoonup \mathbf{A}\nabla u \quad \text{in } \mathrm{L}^2(\Omega_T;\mathbf{R}^d)\ . \end{aligned}$$

Although they are equivalent, the advantage of bounds (2) over those from (5) is that the limiting coefficients still belong to the same set: $\mathbf{A} \in \mathcal{M}_d(\alpha,\beta;\Omega_T)$, while for (5) the constants Λ_0 and M change, in the same manner as for the sequence $\mathcal{P}_{\mathcal{A}^n}$ of corresponding parabolic operators. Instead of (5), in [32] the authors used another set of equivalent bounds, also stable under the homogenisation process:

$$\begin{aligned} \mathbf{A}\boldsymbol{\xi}\cdot\boldsymbol{\xi} &\geqslant \lambda_0|\boldsymbol{\xi}|^2\,, \\ |\mathbf{A}\boldsymbol{\xi}\cdot\boldsymbol{\eta}| &\leqslant \lambda_1\sqrt{\mathbf{A}\boldsymbol{\xi}\cdot\boldsymbol{\xi}}\,|\boldsymbol{\eta}|\ . \end{aligned}$$

Friedrichs systems

In the sequel we recall the main results of [18] (see also [3]): let L be a real Hilbert space, which we identify with its dual L', $\mathcal{D} \subseteq L$ its dense subspace, and $\mathcal{L}, \tilde{\mathcal{L}} : \mathcal{D} \longrightarrow L$ unbounded linear operators satisfying

$$(\forall\,\varphi,\psi\in\mathcal{D})\quad \langle\,\mathcal{L}\varphi \mid \psi\,\rangle_L = \langle\,\varphi \mid \tilde{\mathcal{L}}\psi\,\rangle_L\,, \tag{T1}$$

$$(\exists\,c>0)(\forall\,\varphi\in\mathcal{D})\quad \|(\mathcal{L}+\tilde{\mathcal{L}})\varphi\|_L \leqslant c\|\varphi\|_L\,. \tag{T2}$$

By (T1), the operators $\mathcal{L}$ and $\tilde{\mathcal{L}}$ are formally adjoint to each another, and thus, as they are densely defined, are closable. The closures we denote by $\bar{\mathcal{L}}$ and $\bar{\tilde{\mathcal{L}}}$, and the corresponding domains by $\mathcal{D}(\bar{\mathcal{L}})$ and $\mathcal{D}(\bar{\tilde{\mathcal{L}}})$.

The graph inner product $\langle\,\cdot \mid \cdot\,\rangle_{\mathcal{L}} := \langle\,\cdot \mid \cdot\,\rangle_L + \langle\,\mathcal{L}\cdot \mid \mathcal{L}\cdot\,\rangle_L$ defines the *graph norm* $\|\cdot\|_{\mathcal{L}}$, and it is immediate that $(\mathcal{D},\langle\,\cdot \mid \cdot\,\rangle_{\mathcal{L}})$ is an inner product space, whose completion we denote by W_0. Analogously we could have defined $\langle\,\cdot \mid \cdot\,\rangle_{\tilde{\mathcal{L}}}$ which, by (T2), leads to a norm that is equivalent to $\|\cdot\|_{\mathcal{L}}$. W_0 is continuously imbedded in L (as $\mathcal{L}$ is closable); the image of W_0 being $\mathcal{D}(\bar{\mathcal{L}}) = \mathcal{D}(\bar{\tilde{\mathcal{L}}})$. Moreover, when equipped with the graph norm, these spaces are isometrically isomorphic.

As $\mathcal{L}, \tilde{\mathcal{L}} : \mathcal{D} \longrightarrow L$ are continuous (with the graph topology on $\mathcal{D}$), each can be extended by density to a unique operator from $\mathcal{L}(W_0;L)$ (i.e. a continuous linear operator from W_0 to L). These extensions coincide with $\bar{\mathcal{L}}$ and $\bar{\tilde{\mathcal{L}}}$ (after taking into account the isomorphism between W_0 and $\mathcal{D}(\bar{\mathcal{L}})$). For simplicity, we shall drop the bar from notation and simply write $\mathcal{L}, \tilde{\mathcal{L}} \in \mathcal{L}(W_0;L)$, prompted by the fact that (T1)–(T2) still hold for $\varphi,\psi \in W_0$.

Having in mind the Gelfand triple (the imbeddings are dense and continuous) $W_0 \hookrightarrow L \equiv L' \hookrightarrow W_0'$, it turns out [18, 3] that the adjoint operator $\tilde{\mathcal{L}}^* \in \mathcal{L}(L;W_0')$ (in the sense of Banach spaces) satisfies $\mathcal{L} = \tilde{\mathcal{L}}^*_{|W_0}$. Therefore, $\mathcal{L} : W_0 \longrightarrow L \hookrightarrow W_0'$ is a continuous linear operator from $(W_0,\|\cdot\|_L)$ to W_0', whose unique continuous extension to the whole L is operator $\tilde{\mathcal{L}}^*$ (the same holds for $\tilde{\mathcal{L}}$ and $\mathcal{L}^*$ instead of $\mathcal{L}$ and $\tilde{\mathcal{L}}^*$). In order to further simplify the notation we shall use $\mathcal{L}$ and $\tilde{\mathcal{L}}$ also to denote their extensions $\tilde{\mathcal{L}}^*$ and $\mathcal{L}^*$.

We can now define the *graph space*

$$W := \{\mathsf{u} \in L : \mathcal{L}\mathsf{u} \in L\} = \{\mathsf{u} \in L : \tilde{\mathcal{L}}\mathsf{u} \in L\}\,,$$

which, equipped with the graph norm, is a Hilbert space, containing W_0.

The goal of the abstract theory of Friedrichs systems is to solve the following problem:

for given $\mathsf{f} \in L$ *find* $\mathsf{u} \in W$ *such that* $\mathcal{L}\mathsf{u} = f$.

To be more precise, the goal is to find sufficient conditions on a subspace $V \subseteq W$, such that the operator $\mathcal{L}_{|_V} : V \longrightarrow L$ is an isomorphism. In order to describe such sufficient conditions, a *boundary operator* $D \in \mathcal{L}(W; W')$ is introduced by

$${}_{W'}\langle\, D\mathsf{u}, \mathsf{v}\,\rangle_W := \langle\, \mathcal{L}\mathsf{u} \mid \mathsf{v}\,\rangle_L - \langle\, \mathsf{u} \mid \tilde{\mathcal{L}}\mathsf{v}\,\rangle_L\,, \qquad \mathsf{u},\mathsf{v} \in W\,.$$

Ern at al. [18] proved the following weak well-posedness result in this abstract setting, under the additional assumption

$$\text{(T3)} \qquad (\exists\, \alpha > 0)(\forall\, \varphi \in \mathcal{D}) \quad \langle\, (\mathcal{L} + \tilde{\mathcal{L}})\varphi \mid \varphi\,\rangle_L \geqslant 2\alpha\|\varphi\|_L^2\,.$$

In the sequel (T) stands for all properties (T1)–(T3); we shall use the same convention in other places as well.

Theorem 1. *Let (T) hold, and let subspaces V and $\widetilde{V}$ of W satisfy*

$$\text{(V1)} \qquad \begin{aligned} &(\forall\, \mathsf{u} \in V) \qquad {}_{W'}\langle\, D\mathsf{u}, \mathsf{u}\,\rangle_W \geqslant 0\,,\\ &(\forall\, \mathsf{v} \in \widetilde{V}) \qquad {}_{W'}\langle\, D\mathsf{v}, \mathsf{v}\,\rangle_W \leqslant 0\,, \end{aligned}$$

$$\text{(V2)} \qquad V = D(\widetilde{V})^0\,, \qquad \widetilde{V} = D(V)^0\,,$$

where ^{0} *stands for the annihilator.*

Then the restrictions of the operators $\mathcal{L}_{|_V} : V \longrightarrow L$ and $\tilde{\mathcal{L}}_{|_{\widetilde{V}}} : \widetilde{V} \longrightarrow L$ are isomorphisms, and for every $\mathsf{u} \in V$ the following estimate holds:

$$\text{(7)} \qquad \|\mathsf{u}\|_{\mathcal{L}} \leqslant \sqrt{\frac{1}{\alpha^2} + 1}\, \|\mathcal{L}\mathsf{u}\|_L\,.$$

A similar estimate holds for $\tilde{\mathcal{L}}$ and $\widetilde{V}$ instead of $\mathcal{L}$ and V. ∎

We are interested in situations where $\mathcal{L}$ is a partial differential operator. In this case the information about boundary (and initial) conditions will be contained in the structure of the subspace V.

Classical Friedrichs' operator

We now describe the classical Friedrichs' setting, which turns out to fit in the abstract setting of Ern at al. [18]. Let $d, r \in \mathbf{N}$, and let $\Omega \subseteq \mathbf{R}^d$ be an open and bounded set with Lipschitz boundary Γ (we shall denote its closure by $\mathsf{Cl}\,\Omega = \Omega \cup \Gamma$). Furthermore, let us suppose that the matrix-valued functions $\mathbf{A}_k \in \mathrm{W}^{1,\infty}(\Omega; \mathrm{M}_r(\mathbf{R}))$, $k \in 1..d$, and $\mathbf{C} \in \mathrm{L}^\infty(\Omega; \mathrm{M}_r(\mathbf{R}))$ satisfy:

$$\text{(F1)} \qquad \mathbf{A}_k \text{ is symmetric: } \mathbf{A}_k = \mathbf{A}_k^\top\,,$$

$$\text{(F2)} \qquad (\exists\, \alpha > 0) \quad \mathbf{C} + \mathbf{C}^\top + \sum_{k=1}^d \partial_k \mathbf{A}_k \geqslant 2\alpha\mathbf{I} \qquad \text{(a.e. on } \Omega)\,.$$

If we denote $\mathcal{D} := \mathrm{C}_c^\infty(\Omega; \mathbf{R}^r)$, $L = \mathrm{L}^2(\Omega; \mathbf{R}^r)$, and define the operators $\mathcal{L}, \tilde{\mathcal{L}} : \mathcal{D} \longrightarrow L$ by the formulaæ

$$
\begin{aligned}
\mathcal{L}\mathsf{u} &:= \sum_{k=1}^d \partial_k(\mathbf{A}_k \mathsf{u}) + \mathbf{C}\mathsf{u}\,, \\
\tilde{\mathcal{L}}\mathsf{u} &:= -\sum_{k=1}^d \partial_k(\mathbf{A}_k \mathsf{u}) + (\mathbf{C}^\top + \sum_{k=1}^d \partial_k \mathbf{A}_k)\mathsf{u}\,,
\end{aligned} \tag{8}
$$

where ∂_k stands for *the classical* derivative, then $\mathcal{L}$ and $\tilde{\mathcal{L}}$ satisfy (T), since (8) and (F1) imply (T1), (T2) follows from the regularity assumptions on $\mathbf{A}_k$ and $\mathbf{C}$, and (T3) follows from (F2). Therefore, $\mathcal{L}$ and $\tilde{\mathcal{L}}$ can be uniquely extended to the respective operators from $\mathcal{L}(L; W_0')$. Actually, those extensions can be represented by the same formulae as in (8), with the distributional derivatives instead of the classical ones [3].

The operator $\mathcal{L}$ is called *the Friedrichs operator* or *the symmetric positive operator*, and the corresponding first-order system of partial differential equations $\mathcal{L}\mathsf{u} = \mathsf{f}$, for a given function $\mathsf{f} \in \mathrm{L}^2(\Omega; \mathbf{R}^r)$, is called *the Friedrichs system* or *the symmetric positive system*.

The graph space W is here given by

$$
W = \Big\{ \mathsf{u} \in \mathrm{L}^2(\Omega; \mathbf{R}^r) : \sum_{k=1}^d \partial_k(\mathbf{A}_k \mathsf{u}) + \mathbf{C}\mathsf{u} \in \mathrm{L}^2(\Omega; \mathbf{R}^r) \Big\}\,,
$$

and the space $\mathrm{C}^\infty(\mathsf{Cl}\,\Omega; \mathbf{R}^r)$ is dense in W. Actually, the definition and basic properties of the graph space of the operator $\mathcal{L}$ given by (8) do not depend on the conditions (F) (for more details regarding these spaces see [2, 20]).

If we denote by $\boldsymbol{\nu} = (\nu_1, \nu_2, \ldots, \nu_d) \in \mathrm{L}^\infty(\Gamma; \mathbf{R}^d)$ the unit outward normal on Γ, and define a matrix field on Γ by

$$
\mathbf{A}_{\boldsymbol{\nu}} := \sum_{k=1}^d \nu_k \mathbf{A}_k\,,
$$

then for $\mathsf{u}, \mathsf{v} \in \mathrm{C}^\infty(\mathsf{Cl}\,\Omega; \mathbf{R}^r)$ the boundary operator D is given by

$$
{}_{W'}\langle\, D\mathsf{u}, \mathsf{v}\,\rangle_W = \int_\Gamma \mathbf{A}_{\boldsymbol{\nu}}(\mathbf{x}) \mathsf{u}_{|_\Gamma}(\mathbf{x}) \cdot \mathsf{v}_{|_\Gamma}(\mathbf{x}) dS(\mathbf{x})\,. \tag{9}
$$

Thus we can say that, in the abstract setting, the operator D plays the role of the matrix-valued function $\mathbf{A}_{\boldsymbol{\nu}}$ in the classical Friedrichs theory. To be more precise, it *replaces* the trace operator defined on the graph space; for the details concerning the definition and the properties of the trace operator on graph spaces see [2].

Actually, the matrix-valued function $\mathbf{C}$ can be replaced by a continuous linear operator $\mathcal{C} \in \mathcal{L}(L)$ satisfying

$$
(\exists\, \alpha > 0)(\forall\, \mathsf{u} \in L) \quad \langle\, (\mathcal{C} + \mathcal{C}^\top + \sum_{k=1}^d \partial_k \mathbf{A}_k)\mathsf{u} \mid \mathsf{u}\,\rangle_L \geqslant 2\alpha \|\mathsf{u}\|_L \tag{F2$\mathcal{C}$}
$$

instead of (F2), leading to the operators $\mathcal{L}$ and $\tilde{\mathcal{L}}$ with the properties (T) (this type of operators we shall also call *Friedrichs operators*) .

It is important to emphasise that this setting covers a large number of equations of continuum physics. Different types of (initial) boundary conditions (Dirichlet, Neumann, Robin) can be treated in this way as well. For the motivation, let us present our model example. Further examples are given in the section Examples.

Example

Let $\Omega \subseteq \mathbf{R}^d$ be an open and bounded set with the Lipschitz boundary Γ, as before. We consider the following elliptic equation

$$-\mathsf{div}\,(\mathbf{A}\nabla u) + \mathbf{b}\cdot\nabla u + cu = f\,,$$

where $f \in \mathrm{L}^2(\Omega)$, $c \in \mathrm{L}^\infty(\Omega)$, $\mathbf{b} \in \mathrm{L}^\infty(\Omega;\mathbf{R}^d)$ and $\mathbf{A} \in \mathcal{M}_d(\alpha',\beta';\Omega)$, for some constants $\beta' > \alpha' > 0$.

This equation can be rewritten as a Friedrichs system, by considering a vector function taking values in $\mathbf{R}^{d+1}$ of the form

$$\mathsf{u} = \begin{bmatrix} -\mathbf{A}\nabla u \\ u \end{bmatrix}\,,$$

and $\mathbf{A}_k = \mathbf{e}_k \otimes \mathbf{e}_{d+1} + \mathbf{e}_{d+1} \otimes \mathbf{e}_k \in \mathrm{M}_{d+1}(\mathbf{R})$, for $k \in 1..d$ (vectors $\mathbf{e}_1, \dots \mathbf{e}_{d+1}$ form the standard basis for $\mathbf{R}^{d+1}$) and a block matrix-valued function

$$\mathbf{C} = \begin{bmatrix} \mathbf{A}^{-1} & \mathbf{0} \\ -(\mathbf{A}^{-1}\mathbf{b})^\top & c \end{bmatrix} \in \mathrm{L}^\infty(\Omega;\mathrm{M}_{d+1}(\mathbf{R}))\,.$$

The positivity condition (F2) holds if and only if $2c - \frac{1}{2}\mathbf{A}^{-1}\mathbf{b}\cdot\mathbf{b} \geq \gamma$ on Ω, for some positive constant γ [5]. The graph space is given by

$$W = \mathrm{L}^2_{\mathrm{div}}(\Omega) \times \mathrm{H}^1(\Omega)\,,$$

where

$$\mathrm{L}^2_{\mathrm{div}}(\Omega) = \{\mathsf{u} \in \mathrm{L}^2(\Omega;\mathbf{R}^d) : \mathsf{div}\,\mathsf{u} \in \mathrm{L}^2(\Omega)\}$$

is the graph space of the operator div, with the normal trace $\mathsf{u} \mapsto \boldsymbol{\nu}\cdot\mathsf{u} \in \mathrm{H}^{-\frac{1}{2}}(\Gamma)$ defined on this graph space.

The homogeneous Dirichlet boundary condition $u_{|_\Gamma} = 0$ for the original equation can be imposed if we choose [4, 5, 9]

$$V = \widetilde{V} = \mathrm{L}^2_{\mathrm{div}}(\Omega) \times \mathrm{H}^1_0(\Omega)\,,$$

the Neumann boundary condition $\boldsymbol{\nu}\cdot\mathbf{A}\nabla u_{|_\Gamma} = 0$ with

$$V = \widetilde{V} = \{(\mathsf{u}^{\boldsymbol{\sigma}}, u^u)^\top \in \mathrm{L}^2_{\mathrm{div}}(\Omega) \times \mathrm{H}^1(\Omega) : \boldsymbol{\nu}\cdot\mathsf{u}^{\boldsymbol{\sigma}} = 0\}\,,$$

and the Robin boundary condition $\boldsymbol{\nu}\cdot\mathbf{A}\nabla u_{|_\Gamma} + au_{|_\Gamma} = 0$, for $a > 0$, with

$$\begin{aligned} V &:= \{(\mathsf{u}^{\boldsymbol{\sigma}}, u^u)^\top \in \mathrm{L}^2_{\mathrm{div}}(\Omega) \times \mathrm{H}^1(\Omega) : \boldsymbol{\nu}\cdot\mathsf{u}^{\boldsymbol{\sigma}} = au^u_{|_\Gamma}\}\,, \\ \widetilde{V} &:= \{(\mathsf{u}^{\boldsymbol{\sigma}}, u^u)^\top \in \mathrm{L}^2_{\mathrm{div}}(\Omega) \times \mathrm{H}^1(\Omega) : \boldsymbol{\nu}\cdot\mathsf{u}^{\boldsymbol{\sigma}} = -au^u_{|_\Gamma}\}\,. \end{aligned}$$

∎

For more examples of (initial) boundary value problems that can be treated via the theory of Friedrichs systems we refer to [5, 6, 10, 14, 18, 20].

2. Weak convergence in the graph space

In order to characterise the weak convergence in the graph space W we shall first identify its dual W' (here we only assume that $\mathcal{L}$ and $\tilde{\mathcal{L}}$ satisfy (T1)–(T2)). We proceed similarly as in the case of the graph space of the first order partial differential operator [9, 20]: if we take arbitrary $\mathsf{w}_1, \mathsf{w}_2 \in L$, then the expression

$$S(\mathsf{u}) := \langle\, \mathsf{w}_1 \mid \mathsf{u}\,\rangle_L + \langle\, \mathsf{w}_2 \mid \mathcal{L}\mathsf{u}\,\rangle_L\,, \quad \mathsf{u} \in W\,,$$

clearly defines an element S of W'. Let us now prove that every continuous linear functional on W can be represented in this way.

Theorem 2. *For each $S \in W'$ there are $\mathsf{w}_1, \mathsf{w}_2 \in L$ such that*

$$(\forall\, \mathsf{u} \in W) \qquad S(\mathsf{u}) = \langle\, \mathsf{w}_1 \mid \mathsf{u}\,\rangle_L + \langle\, \mathsf{w}_2 \mid \mathcal{L}\mathsf{u}\,\rangle_L\,. \tag{10}$$

If we denote by V_S the set of all $\tilde{\mathsf{w}} = (\mathsf{w}_1, \mathsf{w}_2) \in L \times L$ satisfying (10), then there exists a unique $\tilde{\mathsf{w}} \in V_S$ such that

$$\|S\|_{W'} = \|\tilde{\mathsf{w}}\|_{L\times L} = \min\{\|\mathsf{w}\|_{L\times L} : \mathsf{w} \in V_S\}\,.$$

Dem. After denoting

$$\Gamma_{\mathcal{L}} := \{(\mathsf{u}, \mathcal{L}\mathsf{u}) \,:\, \mathsf{u} \in W\} \subseteq L \times L\,,$$

it is obvious that W is isometrically isomorphic to $\Gamma_{\mathcal{L}}$ (with the norm inherited from $L\times L$). Then any $S \in W'$ can be considered as a continuous linear functional on $\Gamma_{\mathcal{L}}$. By the Hahn-Banach theorem for Hilbert spaces, there is a unique continuous extension of S to the whole $L \times L$, with the same norm as S. Let us denote this extension by $\tilde{\mathsf{w}} = (\mathsf{w}_1, \mathsf{w}_2) \in L \times L$. It is clear that $\tilde{\mathsf{w}} \in V_S$ and as any other $\mathsf{w} \in V_S$ is also an extension of S, we have

$$\|\tilde{\mathsf{w}}\|_{L\times L} = \|S\|_{W'} \leqslant \|\mathsf{w}\|_{L\times L}\,.$$

Therefore

$$\|\tilde{\mathsf{w}}\|_{L\times L} = \min\{\|\mathsf{w}\|_{L\times L} \,:\, \mathsf{w} \in V_S\}\,,$$

which concludes the proof.

Q.E.D.

Theorem 3. *A sequence (u_n) converges weakly to u in W if and only if*

$$\begin{aligned} \mathsf{u}_n &\rightharpoonup \mathsf{u} \quad \textit{in } L\,,\\ \mathcal{L}\mathsf{u}_n &\rightharpoonup \mathcal{L}\mathsf{u} \quad \textit{in } L\,. \end{aligned}$$

Dem. Let $\mathsf{u}_n \rightharpoonup \mathsf{u}$ in W and let us take arbitrary $\mathsf{w}_1, \mathsf{w}_2 \in L$. Then S defined by (10) belongs to W' and therefore

$$\langle\, \mathsf{w}_1 \mid \mathsf{u}_n\,\rangle_L + \langle\, \mathsf{w}_2 \mid \mathcal{L}\mathsf{u}_n\,\rangle_L = {}_{W'}\langle\, S, \mathsf{u}_n\,\rangle_W \longrightarrow {}_{W'}\langle\, S, \mathsf{u}\,\rangle_W = \langle\, \mathsf{w}_1 \mid \mathsf{u}\,\rangle_L + \langle\, \mathsf{w}_2 \mid \mathcal{L}\mathsf{u}\,\rangle_L\,.$$

If we take $\mathsf{w}_2 = 0$ in the above formula, by arbitrariness of w_1, we get $\mathsf{u}_n \rightharpoonup \mathsf{u}$ in L. Similarly, by taking $\mathsf{w}_1 = 0$ we achieve $\mathcal{L}\mathsf{u}_n \rightharpoonup \mathcal{L}\mathsf{u}$ in L.

The converse statement can be proved in a similar way by using the above representation of continuous linear functionals on W.

Q.E.D.

3. Homogenisation of Friedrichs systems

In the rest of the paper we shall work in the setting of classical partial differential operators, i.e. $\mathcal{D} := \mathrm{C}_c^\infty(\Omega; \mathbf{R}^r)$, $L := \mathrm{L}^2(\Omega; \mathbf{R}^r)$ (we shall use either notation, depending on the situation). Let us now assume that we are given an operator $\mathcal{L}_0$ of the form

$$\mathcal{L}_0\mathsf{u} := \sum_{k=1}^d \partial_k(\mathbf{A}_k\mathsf{u}) = \sum_{k=1}^d \mathbf{A}_k\partial_k\mathsf{u}\,,$$

where, in addition to (F1), we have assumed that

(F3) all matrices $\mathbf{A}_k \in \mathrm{M}_r(\mathbf{R})$ are constant.

Then, clearly, the operators $\mathcal{L}_0$ and $\tilde{\mathcal{L}}_0 := -\mathcal{L}_0$ satisfy (T1)–(T2).

Remark. Some conclusions of this section can be derived without assuming that $\mathbf{A}_k$ are constant matrices. However, as this appears to be the case for the majority of examples of particular interest (see Section 4 and [5, 6, 10, 14, 18, 20]), for the clarity of presentation we choose to impose this condition. ∎

Let us, for given $\beta > \alpha > 0$, denote

$$\mathcal{F}(\alpha,\beta;\Omega) := \left\{ \mathcal{C} \in \mathcal{L}(L) : (\forall \mathsf{u} \in L) \quad \langle \mathcal{C}\mathsf{u} \mid \mathsf{u} \rangle_L \geq \alpha \|\mathsf{u}\|_L^2 \quad \& \quad \langle \mathcal{C}\mathsf{u} \mid \mathsf{u} \rangle_L \geq \frac{1}{\beta}\|\mathcal{C}\mathsf{u}\|_L^2 \right\}.$$

Then, for $\mathcal{C} \in \mathcal{F}(\alpha,\beta;\Omega)$, the second inequality in the above expression implies $\|\mathcal{C}\|_{\mathcal{L}(L)} \leqslant \beta$, while the first inequality implies that the operator

$$\mathcal{L} := \mathcal{L}_0 + \mathcal{C}\,,$$

in addition to (F1), satisfies (F2$\mathcal{C}$), and therefore all properties (T).

If we denote by W the graph space of the operator $\mathcal{L}_0$, then one can easily see that the graph space of the operator $\mathcal{L}$ coincides with W. Indeed, since for $\mathsf{u} \in L$ we have $\mathcal{C}\mathsf{u} \in L$, it follows that $\mathcal{L}\mathsf{u} \in L$ if and only if $\mathcal{L}_0\mathsf{u} \in L$.

Let us now prove that on W norms $\|\cdot\|_{\mathcal{L}}$ and $\|\cdot\|_{\mathcal{L}_0}$ are equivalent: for $\mathsf{u} \in W$ we have

$$\begin{aligned}\|\mathsf{u}\|_{\mathcal{L}}^2 &= \|\mathsf{u}\|_L^2 + \|\mathcal{L}\mathsf{u}\|_L^2 \\ &\leqslant \|\mathsf{u}\|_L^2 + (\|\mathcal{L}_0\mathsf{u}\|_L + \|\mathcal{C}\mathsf{u}\|_L)^2 \\ &\leqslant \|\mathsf{u}\|_L^2 + 2\|\mathcal{L}_0\mathsf{u}\|_L^2 + 2\|\mathcal{C}\mathsf{u}\|_L^2\,.\end{aligned}$$

From boundedness of $\mathcal{C}$ it follows that the last term in the above sum is bounded by $2\beta^2\|\mathsf{u}\|_L^2$, which implies

$$\|\mathsf{u}\|_{\mathcal{L}} \leqslant \gamma \|\mathsf{u}\|_{\mathcal{L}_0}\,,$$

where $\gamma = \sqrt{\max\{2, 1+2\beta^2\}}$ depends only on β. Similarly, starting from

$$\|\mathsf{u}\|_{\mathcal{L}_0}^2 = \|\mathsf{u}\|_L^2 + \|\mathcal{L}\mathsf{u} - \mathcal{C}\mathsf{u}\|_L^2 \leqslant \|\mathsf{u}\|_L^2 + (\|\mathcal{L}\mathsf{u}\|_L + \|\mathcal{C}\mathsf{u}\|_L)^2\,,$$

one derives the converse inequality (with the same constant γ), which proves the equivalence of these norms.

Also note that the characterisation (9) of the boundary operator D corresponding to the operator $\mathcal{L}$ implies that it does not depend on particular $\mathcal{C}$ from $\mathcal{F}(\alpha,\beta;\Omega)$.

Let us now assume that V is a subspace of W satisfying (V). Theorem 1 implies that the operator $\mathcal{L}$ restricted to V is an isomorphism from V to L, with

$$\|\mathsf{u}\|_{\mathcal{L}} \leqslant \sqrt{\frac{1}{\alpha^2}+1}\,\|\mathcal{L}\mathsf{u}\|_L\,, \quad \mathsf{u} \in V\,.$$

Therefore,

$$\|\mathsf{u}\|_{\mathcal{L}_0} \leqslant \gamma\sqrt{\frac{1}{\alpha^2}+1}\,\|\mathcal{L}\mathsf{u}\|_L\,, \quad \mathsf{u} \in V\,,$$

and finally, for fixed $\mathcal{L}_0$ and V satisfying (V), we have a priori bound

$$(\exists\, c > 0)(\forall\, \mathcal{C} \in \mathcal{F}(\alpha,\beta;\Omega))(\forall\, \mathsf{u} \in V) \quad \|\mathsf{u}\|_{\mathcal{L}_0} \leqslant c\|(\mathcal{L}_0 + \mathcal{C})\mathsf{u}\|_L\,. \tag{11}$$

Note that the constant c depends only on α and β. For the corresponding adjoint operator the analogous conclusion holds, with $\widetilde{V}$ instead of V.

Remark. If $\mathbf{C} \in \mathcal{M}_r(\alpha,\beta;\Omega)$, where, as before

$$\mathcal{M}_r(\alpha,\beta;\Omega) = \left\{ \mathbf{C} \in \mathrm{L}^\infty(\Omega;\mathrm{M}_r(\mathbf{R})) : (\forall \boldsymbol{\xi} \in \mathbf{R}^r) \quad \mathbf{C}(\mathbf{x})\boldsymbol{\xi}\cdot\boldsymbol{\xi} \geqslant \alpha|\boldsymbol{\xi}|^2 \,, \mathbf{C}(\mathbf{x})\boldsymbol{\xi}\cdot\boldsymbol{\xi} \geqslant \frac{1}{\beta}|\mathbf{C}(\mathbf{x})\boldsymbol{\xi}|^2 \right\}.$$

then one can easily see that the mapping $\mathsf{u} \mapsto \mathbf{C}\mathsf{u}$, $\mathsf{u} \in L$, defines an operator $\mathcal{C} \in \mathcal{F}(\alpha,\beta;\Omega)$. Therefore, in this case one has a classical Friedrichs operator $\mathcal{L} = \mathcal{L}_0 + \mathbf{C}$, and from (11) an a priori bound

$$(12) \qquad (\exists\, c > 0)(\forall\, \mathbf{C} \in \mathcal{M}_r(\alpha,\beta;\Omega))(\forall\, \mathsf{u} \in V) \quad \|\mathsf{u}\|_{\mathcal{L}_0} \leq c\|(\mathcal{L}_0 + \mathbf{C})\mathsf{u}\|_L$$

follows, with the constant c depending only on α and β.

∎

G and H-convergence of Friedrichs operators

In the rest of the paper we shall denote $\mathcal{L}_0 = \sum_{k=1}^d \mathbf{A}_k\partial_k$, with $\mathbf{A}_k$ satisfying (F1) and (F3), so that $\mathcal{L}_0$ and its formal adjoint $\tilde{\mathcal{L}}_0 = -\mathcal{L}_0$ satisfy (T1)–(T2), W will be its graph space, D the corresponding boundary operator and V, $\widetilde{V}$ some subspaces of W satisfying (V).

The concept of G-convergence can be applied to Friedrichs operators in the same manner as it was originally introduced by Spagnolo for the heat equation [26].

To define G-convergence of Friedrichs operators, with the given $\mathcal{L}_0$ part and the subspace V as above, we consider a sequence $(\mathcal{C}_n)$ of operators from $\mathcal{F}(\alpha,\beta;\Omega)$, so the corresponding Friedrichs operators $\mathcal{L}_n := \mathcal{L}_0 + \mathcal{C}_n$ are isomorphisms of V to L.

Definition. (**G-convergence for Friedrichs operators**) For a sequence $(\mathcal{C}_n)$ in $\mathcal{F}(\alpha,\beta;\Omega)$, we say that the sequence of isomorphisms $\mathcal{L}_n := \mathcal{L}_0 + \mathcal{C}_n : V \to L$ G-converges to an isomorphism $\mathcal{L} := \mathcal{L}_0 + \mathcal{C} : V \to L$, for some $\mathcal{C} \in \mathcal{F}(\alpha',\beta';\Omega)$, if the inverse operators $\mathcal{L}_n^{-1} : L \to V$ converge in the weak sense:

$$(\forall\, \mathsf{f} \in \mathrm{L}^2(\Omega;\mathbf{R}^r)) \quad \mathcal{L}_n^{-1}\mathsf{f} \rightharpoonup \mathcal{L}^{-1}\mathsf{f} \text{ in } W\,.$$

On the other side, as mentioned in the Introduction, H-convergence describes the properties of the coefficients in the equation, instead of the corresponding operators.

Definition. (**H-convergence for Friedrichs systems**) We say that a sequence $(\mathbf{C}_n)$ in $\mathcal{M}_r(\alpha,\beta;\Omega)$ H-converges to $\mathbf{C} \in \mathcal{M}_r(\alpha',\beta';\Omega)$ with respect to $\mathcal{L}_0$ and V if for any $\mathsf{f} \in \mathrm{L}^2(\Omega;\mathbf{R}^r)$ the sequence (u_n) defined by $\mathsf{u}_n := \mathcal{L}_n^{-1}\mathsf{f} \in V$, with $\mathcal{L}_n = \mathcal{L}_0 + \mathbf{C}_n$, satisfies

$$\begin{aligned} \mathsf{u}_n &\rightharpoonup \mathsf{u} \quad \text{in } \mathrm{L}^2(\Omega;\mathbf{R}^r)\,, \\ \mathbf{C}_n\mathsf{u}_n &\rightharpoonup \mathbf{C}\mathsf{u} \quad \text{in } \mathrm{L}^2(\Omega;\mathbf{R}^r)\,, \end{aligned}$$

where $\mathsf{u} := \mathcal{L}^{-1}\mathsf{f} \in V$, with $\mathcal{L} = \mathcal{L}_0 + \mathbf{C}$.

The H-convergence will be denoted by $\mathbf{C}_n \xrightharpoonup{H} \mathbf{C}$.

Remark. If $\mathbf{C}_n \xrightharpoonup{H} \mathbf{C}$, then from $\mathcal{L}_0\mathsf{u}_n + \mathbf{C}_n\mathsf{u}_n = f = \mathcal{L}_0\mathsf{u} + \mathbf{C}\mathsf{u}$ and the second convergence in the definition of H-convergence, it follows that

$$\mathcal{L}_0\mathsf{u}_n \rightharpoonup \mathcal{L}_0\mathsf{u} \quad \text{in } \mathrm{L}^2(\Omega;\mathbf{R}^r)\,,$$

which gives us the weak convergence $\mathsf{u}_n \rightharpoonup \mathsf{u}$ in W, i.e. the sequence $\mathcal{L}_n = \mathcal{L}_0 + \mathbf{C}_n$ G-convergences to $\mathcal{L} = \mathcal{L}_0 + \mathbf{C}$.

∎

The H-convergence actually defines a topology on the set of all admissible coefficients $\mathbf{C}$. The following proof is adapted from [1] (see also [29]).

Theorem 4. *Let $F = \{\mathsf{f}_n : n \in \mathbf{N}\}$ be a dense countable family in $\mathrm{L}^2(\Omega;\mathbf{R}^r)$, $\mathbf{C},\mathbf{D} \in \mathcal{M}_r(\alpha,\beta;\Omega)$, and $\mathsf{u}_n,\mathsf{v}_n \in V$ the solutions of $(\mathcal{L}_0 + \mathbf{C})\mathsf{u}_n = \mathsf{f}_n$ and $(\mathcal{L}_0 + \mathbf{D})\mathsf{v}_n = \mathsf{f}_n$, respectively. Furthermore, let*

$$d(\mathbf{C},\mathbf{D}) := \sum_{n=1}^\infty 2^{-n}\frac{\|\mathsf{u}_n - \mathsf{v}_n\|_{\mathrm{H}^{-1}(\Omega;\mathbf{R}^r)} + \|\mathbf{C}\mathsf{u}_n - \mathbf{D}\mathsf{v}_n\|_{\mathrm{H}^{-1}(\Omega;\mathbf{R}^r)}}{\|\mathsf{f}_n\|_{\mathrm{L}^2(\Omega;\mathbf{R}^r)}}\,, \quad \mathbf{C},\mathbf{D} \in \mathcal{M}_r(\alpha,\beta;\Omega)\,.$$

Then d is a metric on $\mathcal{M}_r(\alpha,\beta;\Omega)$, and the above defined H-convergence is equivalent to the convergence in this metric space.

Dem. Using the continuity of the imbedding of $L^2(\Omega;\mathbf{R}^r)$ into $H^{-1}(\Omega;\mathbf{R}^r)$, the a priori estimate (12) and the upper boundedness of $\mathcal{M}_r(\alpha,\beta;\Omega)$, we conclude that there is a constant $c_1>0$ such that

$$(\forall\, n\in\mathbf{N})\qquad \|\mathsf{u}_n-\mathsf{v}_n\|_{H^{-1}(\Omega;\mathbf{R}^r)}+\|\mathbf{C}\mathsf{u}_n-\mathbf{D}\mathsf{v}_n\|_{H^{-1}(\Omega;\mathbf{R}^r)}\leqslant c_1\|\mathsf{f}_n\|_{L^2(\Omega;\mathbf{R}^r)}\,.$$

This implies that the series in the definition of d is uniformly convergent (with respect to $\mathbf{C}$ and $\mathbf{D}$), and thus d is well-defined. Clearly, d is symmetric, nonnegative, satisfies the triangle inequality and $d(\mathbf{C},\mathbf{C})=0$. Therefore, in order to complete the proof that it is a metric, it remains to show that $d(\mathbf{C},\mathbf{D})=0$ implies $\mathbf{C}=\mathbf{D}$.

Assume $d(\mathbf{C},\mathbf{D})=0$, and let us first prove that for an arbitrary $\mathsf{f}\in L^2(\Omega;\mathbf{R}^r)$ and $\mathsf{u},\mathsf{v}\in V$ defined by $(\mathcal{L}_0+\mathbf{C})\mathsf{u}=\mathsf{f}$ and $(\mathcal{L}_0+\mathbf{D})\mathsf{v}=\mathsf{f}$ we have $\mathsf{u}=\mathsf{v}$ and $\mathbf{C}\mathsf{u}=\mathbf{D}\mathsf{v}$. For $\mathsf{f}\in F$ this statement follows from the definition of d, and for the general case we use the density of F in $L^2(\Omega;\mathbf{R}^r)$: for an arbitrary $\mathsf{f}\in L^2(\Omega;\mathbf{R}^r)$ let $(\mathsf{f}_{n'})$ be a sequence in F that strongly converges to f, and let $\mathsf{u}_{n'},\mathsf{v}_{n'}\in V$ be such that $(\mathcal{L}_0+\mathbf{C})\mathsf{u}_{n'}=\mathsf{f}_{n'}$ and $(\mathcal{L}_0+\mathbf{D})\mathsf{v}_{n'}=\mathsf{f}_{n'}$. Subtracting the equations for u and $\mathsf{u}_{n'}$, and similarly for v and $\mathsf{v}_{n'}$ we get

$$\begin{aligned}(\mathcal{L}_0+\mathbf{C})(\mathsf{u}-\mathsf{u}_{n'})&=\mathsf{f}-\mathsf{f}_{n'}\,,\\(\mathcal{L}_0+\mathbf{D})(\mathsf{v}-\mathsf{v}_{n'})&=\mathsf{f}-\mathsf{f}_{n'}\,.\end{aligned}$$

From the convergence of the sequence $(\mathsf{f}_{n'})$ to f and (12) it follows $\mathsf{u}_{n'}\longrightarrow\mathsf{u}$ and $\mathsf{v}_{n'}\longrightarrow\mathsf{v}$ in $L^2(\Omega;\mathbf{R}^r)$, and using $\mathsf{u}_{n'}=\mathsf{v}_{n'}$ and $\mathbf{C}\mathsf{u}_{n'}=\mathbf{D}\mathsf{v}_{n'}$ we conclude $\mathsf{u}=\mathsf{v}$ and $\mathbf{C}\mathsf{u}=\mathbf{D}\mathsf{v}$.

In order to complete the proof that $\mathbf{C}=\mathbf{D}$, let us take $K\subseteq\Omega$ compact, $\boldsymbol{\xi}\in\mathbf{R}^r$, and $\varphi\in C_c^\infty(\Omega)$, such that $\varphi\equiv1$ on K. Then the function $\mathsf{u}(\mathbf{x}):=\varphi(\mathbf{x})\boldsymbol{\xi}$ belongs to $C_c^\infty(\Omega;\mathbf{R}^r)\subseteq V$. Let us take $\mathsf{f}:=(\mathcal{L}_0+\mathbf{C})\mathsf{u}$, and $\mathsf{v}\in V$, the solution of $(\mathcal{L}_0+\mathbf{D})\mathsf{v}=\mathsf{f}$. Then it follows $\mathsf{u}=\mathsf{v}$ and $\mathbf{C}\mathsf{u}=\mathbf{D}\mathsf{v}$, and since $\mathsf{u}\equiv\boldsymbol{\xi}$ in K, we conclude $\mathbf{C}\boldsymbol{\xi}=\mathbf{D}\boldsymbol{\xi}$ on K. Now, from the arbitrariness of K and $\boldsymbol{\xi}$ it follows $\mathbf{C}=\mathbf{D}$.

It remains to prove that H-convergence is equivalent to the sequential convergence with respect to d: let a sequence $(\mathbf{C}_m)$ in $\mathcal{M}_r(\alpha,\beta;\Omega)$ H-converges to $\mathbf{C}\in\mathcal{M}_r(\alpha,\beta;\Omega)$, and let $\mathsf{u}_n,\mathsf{u}_n^m\in V$ be such that $(\mathcal{L}_0+\mathbf{C})\mathsf{u}_n=\mathsf{f}_n$ and $(\mathcal{L}_0+\mathbf{C}_m)\mathsf{u}_n^m=\mathsf{f}_n$, for $\mathsf{f}_n\in F$. From the definition of H-convergence it follows (for every fixed $n\in\mathbf{N}$)

$$\begin{aligned}\mathsf{u}_n^m&\rightharpoonup\mathsf{u}_n\quad\text{in }L^2(\Omega;\mathbf{R}^r)\,,\\\mathbf{C}_m\mathsf{u}_n^m&\rightharpoonup\mathbf{C}\mathsf{u}_n\quad\text{in }L^2(\Omega;\mathbf{R}^r)\,.\end{aligned}$$

By the Rellich compactness theorem this implies the strong convergence $\mathsf{u}_n^m\longrightarrow\mathsf{u}_n$ and $\mathbf{C}_m\mathsf{u}_n^m\longrightarrow\mathbf{C}\mathsf{u}_n$ in $H^{-1}(\Omega;\mathbf{R}^r)$, which together with the uniform convergence of the series in the definition of d implies $d(\mathbf{C},\mathbf{C}_m)\longrightarrow0$

In order to prove the converse statement, we take a sequence $(\mathbf{C}_m)$ converging to $\mathbf{C}$ in $\mathcal{M}_r(\alpha,\beta;\Omega)$, i.e. such that $d(\mathbf{C},\mathbf{C}_m)\longrightarrow0$. For an arbitrary $\mathsf{f}\in L^2(\Omega;\mathbf{R}^r)$, let $(\mathsf{f}_{n'})$ be a sequence in F strongly converging to f, and let $\mathsf{u},\mathsf{u}^m,\mathsf{u}_{n'},\mathsf{u}_{n'}^m\in V$ satisfy $(\mathcal{L}_0+\mathbf{C})\mathsf{u}=\mathsf{f}$, $(\mathcal{L}_0+\mathbf{C}_m)\mathsf{u}^m=\mathsf{f}$, $(\mathcal{L}_0+\mathbf{C})\mathsf{u}_{n'}=\mathsf{f}_{n'}$ and $(\mathcal{L}_0+\mathbf{C}_m)\mathsf{u}_{n'}^m=\mathsf{f}_{n'}$. From $d(\mathbf{C},\mathbf{C}_m)\longrightarrow0$ and the definition of d it follows that for every $n'\in\mathbf{N}$, $\mathsf{u}_{n'}^m\longrightarrow\mathsf{u}_{n'}$ and $\mathbf{C}_m\mathsf{u}_{n'}^m\longrightarrow\mathbf{C}\mathsf{u}_{n'}$ strongly in $H^{-1}(\Omega;\mathbf{R}^r)$ as $m\longrightarrow\infty$. Due to (12) and the upper boundedness of $\mathcal{M}_r(\alpha,\beta;\Omega)$ it follows that (for fixed n') the sequences $(\mathsf{u}_{n'}^m)_m$ and $(\mathbf{C}_m\mathsf{u}_{n'}^m)_m$ are bounded in $L^2(\Omega;\mathbf{R}^r)$, and therefore converge weakly on a subsequence. Therefore, from their strong convergence in $H^{-1}(\Omega;\mathbf{R}^r)$, we conclude the convergence of the whole sequences:

$$\begin{aligned}\mathsf{u}_{n'}^m&\rightharpoonup\mathsf{u}_{n'}\quad\text{in }L^2(\Omega;\mathbf{R}^r)\quad\text{as }m\longrightarrow\infty\,,\\\mathbf{C}_m\mathsf{u}_{n'}^m&\rightharpoonup\mathbf{C}\mathsf{u}_{n'}\quad\text{in }L^2(\Omega;\mathbf{R}^r)\quad\text{as }m\longrightarrow\infty\,.\end{aligned}\tag{13}$$

Subtracting the equations for u and $\mathsf{u}_{n'}$, and similarly for u^m and $\mathsf{u}_{n'}^m$, we get

$$\begin{aligned}(\mathcal{L}_0+\mathbf{C})(\mathsf{u}-\mathsf{u}_{n'})&=\mathsf{f}-\mathsf{f}_{n'}\,,\\(\mathcal{L}_0+\mathbf{C}_m)(\mathsf{u}^m-\mathsf{u}_{n'}^m)&=\mathsf{f}-\mathsf{f}_{n'}\,.\end{aligned}$$

Similarly as before, from the convergence of the sequence $(\mathsf{f}_{n'})$ to f, the upper boundedness of $\mathcal{M}_r(\alpha,\beta;\Omega)$ and (12) we conclude $\mathsf{u}_{n'} \longrightarrow \mathsf{u}$ and $\mathbf{C}\mathsf{u}_{n'} \longrightarrow \mathbf{C}\mathsf{u}$ in $\mathrm{L}^2(\Omega;\mathbf{R}^r)$, as well as $\mathsf{u}^m_{n'} \longrightarrow \mathsf{u}^m$ and $\mathbf{C}_m\mathsf{u}^m_{n'} \longrightarrow \mathbf{C}_m\mathsf{u}^m$ in $\mathrm{L}^2(\Omega;\mathbf{R}^r)$ uniformly in m as $n' \longrightarrow \infty$. This, together with (13) implies

$$\begin{aligned} \mathsf{u}^m &\rightharpoonup \mathsf{u} \quad \text{in } \mathrm{L}^2(\Omega;\mathbf{R}^r)\,, \\ \mathbf{C}_m\mathsf{u}^m &\rightharpoonup \mathbf{C}\mathsf{u} \quad \text{in } \mathrm{L}^2(\Omega;\mathbf{R}^r)\,, \end{aligned}$$

and finally, by the arbitrariness of $\mathsf{f} \in \mathrm{L}^2(\Omega;\mathbf{R}^r)$, $\mathbf{C}_m \xrightharpoonup{H} \mathbf{C}$.

Q.E.D.

We proceed further with the compactness result which, in particular, shows that the parameters α' and β' in the definition of H-convergence are equal to α and β, respectively. However, as Friedrichs theory applies to various differential equations, we cannot expect a compactness result in this full generality. For example, the transport equation can be written as a Friedrichs systems, and it is well known that some nonlocal effects occur in the homogenisation of this equation, where the limiting equation is some integro-differential equation (see [28] for a simple model example of such nonlocal effects). Therefore, we need some additional assumptions that distinguish cases as this one, and which would be easy to verify for equations of interest. The majority of our proofs in this paper follow some ideas from classical homogenisation theory of particular equations, such as the stationary diffusion equation or the heat equation. However, in homogenisation of these equations, due to their specific structure, one can use some nice compactness properties, such as the Div-rot lemma. As our setting is very general, one does not have such properties for granted.

Here are the assumptions that we shall use throughout the rest of this article: for fixed $\mathcal{L}_0$ and V satisfying (V), we say that family $\mathcal{M}_r(\alpha,\beta;\Omega)$ (respectively, family $\mathcal{F}(\alpha,\beta;\Omega)$) has property:

(K1) if for every sequence $\mathbf{C}_n \in \mathcal{M}_r(\alpha,\beta;\Omega)$ (respectively, $\mathcal{C}_n \in \mathcal{F}(\alpha,\beta;\Omega)$), and every $\mathsf{f} \in L$, the sequence $\mathsf{u}_n \in V$ defined by $\mathsf{u}_n := (\mathcal{L}_0 + \mathbf{C}_n)^{-1}\mathsf{f}$ (respectively $\mathsf{u}_n := (\mathcal{L}_0 + \mathcal{C}_n)^{-1}\mathsf{f}$) satisfies the following: if (u_n) weakly converges to u in W, then also

$$_{W'}\langle D\mathsf{u}_n, \mathsf{u}_n \rangle_W \longrightarrow {}_{W'}\langle D\mathsf{u}, \mathsf{u} \rangle_W;$$

(K2) if for every sequence $\mathbf{C}_n \in \mathcal{M}_r(\alpha,\beta;\Omega)$ (respectively, $\mathcal{C}_n \in \mathcal{F}(\alpha,\beta;\Omega)$), and every $\mathsf{f} \in L$, the sequence $\mathsf{u}_n \in V$ defined by $\mathsf{u}_n := (\mathcal{L}_0 + \mathbf{C}_n)^{-1}\mathsf{f}$ (respectively $\mathsf{u}_n := (\mathcal{L}_0 + \mathcal{C}_n)^{-1}\mathsf{f}$) satisfies the following: if (u_n) weakly converges to u in W, then also

$$(\forall \varphi \in \mathrm{C}^\infty_c(\Omega)) \quad \langle \mathcal{L}_0\mathsf{u}_n \mid \varphi\mathsf{u}_n \rangle_L \longrightarrow \langle \mathcal{L}_0\mathsf{u} \mid \varphi\mathsf{u} \rangle_L\,.$$

Remark. From the definition of the boundary operator D it follows that for every $\mathsf{w} \in W$ one has

$$_{W'}\langle D\mathsf{w}, \mathsf{w} \rangle_W = 2\langle \mathcal{L}_0\mathsf{w} \mid \mathsf{w} \rangle_L\,,$$

which implies that the convergence required in (K1) is equivalent to

$$\langle \mathcal{L}_0\mathsf{u}_n \mid \mathsf{u}_n \rangle_L \longrightarrow \langle \mathcal{L}_0\mathsf{u} \mid \mathsf{u} \rangle_L\,.$$

∎

Theorem 5. *For fixed $\mathcal{L}_0$ and V, if family $\mathcal{F}(\alpha,\beta;\Omega)$ satisfies (K1), then for any sequence $(\mathcal{C}_n)$ in $\mathcal{F}(\alpha,\beta;\Omega)$ there exists a subsequence of $\mathcal{L}_n := \mathcal{L}_0 + \mathcal{C}_n$ which G-converges to $\mathcal{L} := \mathcal{L}_0 + \mathcal{C}$ with $\mathcal{C} \in \mathcal{F}(\alpha,\beta;\Omega)$.*

Dem. The proof resembles the original proof of Spagnolo in the case of parabolic G-convergence.

By (11), for given $\mathsf{f} \in L$, the sequence $\mathsf{u}_n := \mathcal{L}_n^{-1}\mathsf{f}$ is bounded in W:

$$\|\mathsf{u}_n\|_{\mathcal{L}_0} \leqslant c\,\|\mathsf{f}\|_L\,,$$

(with the constant c depending on α and β) so it possesses a weakly converging subsequence. Due to the separability of L, using the Cantor diagonal procedure one can pass to a subsequence (for which we keep the same notation) such that $\mathcal{L}_n^{-1}\mathsf{f}$ weakly converges in W, for any $\mathsf{f} \in L$. The limit will be denoted by $\mathsf{u} = \mathcal{B}\mathsf{f} \in V$ (V is closed in W [18], and therefore weakly closed, as well), where $\mathcal{B} \in \mathcal{L}(L; W)$.

Let us define $\mathcal{K} \in \mathcal{L}(L)$ by $\mathcal{K}\mathsf{f} := \mathsf{f} - \mathcal{L}_0\mathsf{u} = \mathsf{f} - \mathcal{L}_0\mathcal{B}\mathsf{f}$. Then

$$\mathcal{L}_0\mathsf{u}_n + \mathcal{C}_n\mathsf{u}_n = \mathsf{f} = \mathcal{L}_0\mathcal{B}\mathsf{f} + \mathcal{K}\mathsf{f}\,, \tag{14}$$

and as $\mathcal{L}_0\mathsf{u}_n \rightharpoonup \mathcal{L}_0\mathcal{B}\mathsf{f}$ in L, it follows

$$\mathcal{C}_n\mathsf{u}_n \rightharpoonup \mathcal{K}\mathsf{f} \quad \text{in } L\,. \tag{15}$$

Multiplying the left equality in (14) by u_n, and the right by $\mathsf{u} = \mathcal{B}\mathsf{f}$, we get

$$\langle\, \mathcal{L}_0\mathsf{u}_n \mid \mathsf{u}_n \,\rangle_L + \langle\, \mathcal{C}_n\mathsf{u}_n \mid \mathsf{u}_n \,\rangle_L = \langle\, \mathsf{f} \mid \mathsf{u}_n \,\rangle_L\,, \tag{16}$$

$$\langle\, \mathcal{L}_0\mathsf{u} \mid \mathsf{u} \,\rangle_L + \langle\, \mathcal{K}\mathsf{f} \mid \mathsf{u} \,\rangle_L = \langle\, \mathsf{f} \mid \mathsf{u} \,\rangle_L\,. \tag{17}$$

The right-hand side in (16) converges to $\langle\, \mathsf{f} \mid \mathsf{u} \,\rangle_L$, while the first term on the left-hand side, by (K1), converges to $\langle\, \mathcal{L}_0\mathsf{u} \mid \mathsf{u} \,\rangle_L$. Therefore, by using (17) we obtain the convergence

$$\langle\, \mathcal{C}_n\mathsf{u}_n \mid \mathsf{u}_n \,\rangle_L \longrightarrow \langle\, \mathcal{K}\mathsf{f} \mid \mathsf{u} \,\rangle_L\,. \tag{18}$$

Let us show that $\mathcal{B}$ is injective: if $\mathcal{B}\mathsf{f} = \mathsf{0}$ for some $\mathsf{f} \in L$, then (18) reads $\langle\, \mathcal{C}_n\mathsf{u}_n \mid \mathsf{u}_n \,\rangle_L \longrightarrow 0$ which, together with the second inequality in the definition of $\mathcal{F}(\alpha,\beta;\Omega)$, implies the strong convergence $\mathcal{C}_n\mathsf{u}_n \longrightarrow \mathsf{0}$, and finally from (15) we are able to conclude that $\mathsf{0} = \mathcal{K}\mathsf{f} = \mathsf{f} - \mathcal{L}_0\mathcal{B}\mathsf{f} = \mathsf{f}$.

The injectivity of $\mathcal{B}$ enables us to define the linear operator $\mathcal{C} : \mathcal{B}(L) \to L$ by $\mathcal{C}(\mathcal{B}\mathsf{f}) := \mathcal{K}\mathsf{f}$. Using (15), the second inequality in the definition of $\mathcal{F}(\alpha,\beta;\Omega)$, and (18), respectively, we have

$$\frac{1}{\beta}\|\mathcal{C}\mathsf{u}\|_L^2 \leq \liminf_n \frac{1}{\beta}\|\mathcal{C}_n\mathsf{u}_n\|_L^2 \leq \liminf_n \langle\, \mathcal{C}_n\mathsf{u}_n \mid \mathsf{u}_n \,\rangle_L = \langle\, \mathcal{C}\mathsf{u} \mid \mathsf{u} \,\rangle_L \leq \|\mathcal{C}\mathsf{u}\|_L \|\mathsf{u}\|_L\,,$$

which shows the continuity of $\mathcal{C}$ (in the pair of L norms).

Let us now prove that $\mathcal{B}(L)$ is dense in L, and consequently that $\mathcal{C}$ can be uniquely extended by continuity to the whole L: arguing by contradiction, let us assume that there exists a non-zero $\mathsf{f} \in L$, orthogonal to $\mathcal{B}(L)$. In particular, the equality $\langle\, \mathsf{f} \mid \mathcal{B}\mathsf{f} \,\rangle_L = 0$ holds, which together with (17) implies

$$\langle\, \mathcal{L}_0\mathsf{u} \mid \mathsf{u} \,\rangle_L + \langle\, \mathcal{K}\mathsf{f} \mid \mathsf{u} \,\rangle_L = 0\,.$$

Since $\langle\, \mathcal{L}_0\mathsf{u} \mid \mathsf{u} \,\rangle_L = \frac{1}{2}{}_{W'}\langle\, D\mathsf{u}, \mathsf{u} \,\rangle_W \geq 0$, we have $\langle\, \mathcal{K}\mathsf{f} \mid \mathsf{u} \,\rangle_L \leq 0$. However, from the first inequality in the definition of $\mathcal{F}(\alpha,\beta;\Omega)$ and (18) it follows $\langle\, \mathcal{K}\mathsf{f} \mid \mathsf{u} \,\rangle_L \geq 0$ and therefore (18) reduces to $\langle\, \mathcal{C}_n\mathsf{u}_n \mid \mathsf{u}_n \,\rangle_L \longrightarrow 0$. Now, one proceeds similarly as before to conclude $\mathsf{f} = \mathsf{0}$, which gives a contradiction.

To sum up, the operator $\mathcal{C} : L \to L$ satisfies the same bounds as operators $\mathsf{u} \mapsto \mathcal{C}_n\mathsf{u}$:

$$\begin{aligned} &\langle\, \mathcal{C}\mathsf{u} \mid \mathsf{u} \,\rangle_L \geq \alpha\|\mathsf{u}\|_L^2 \\ &\langle\, \mathcal{C}\mathsf{u} \mid \mathsf{u} \,\rangle_L \geq \frac{1}{\beta}\|\mathcal{C}\mathsf{u}\|_L^2 \end{aligned}\,, \quad \mathsf{u} \in L\,. \tag{19}$$

Here, the second inequality has already been shown, while the first one follows analogously: in $\langle\, \mathcal{C}\mathsf{u}_n \mid \mathsf{u}_n \,\rangle_L \geq \alpha\|\mathsf{u}_n\|_L^2$ the left-hand side converges to $\langle\, \mathcal{C}\mathsf{u} \mid \mathsf{u} \,\rangle_L$ by (18), while for the right-hand side it is enough to use the inequality $\liminf_n \|\mathsf{u}_n\|_L^2 \geq \|\mathsf{u}\|_L^2$.

Q.E.D.

Theorem 6. *For fixed $\mathcal{L}_0$ and V, if the family $\mathcal{M}_r(\alpha,\beta;\Omega)$ satisfies (K1) and (K2), then it is compact with respect to H-convergence, i.e. from any sequence $(\mathbf{C}_n)$ in $\mathcal{M}_r(\alpha,\beta;\Omega)$ one can extract a H-converging subsequence whose limit belongs to $\mathcal{M}_r(\alpha,\beta;\Omega)$.*

Dem. As the first step, we apply the last theorem to the operators $\mathcal{C}_n \in \mathcal{F}(\alpha,\beta;\Omega)$ defined by $(\mathcal{C}_n\mathsf{u})(\mathbf{x}) = \mathbf{C}_n(\mathbf{x})\mathsf{u}(\mathbf{x})$, for a.e. $\mathbf{x}\in\Omega$ and $\mathsf{u}\in L$, obtaining the operator $\mathcal{C}\in\mathcal{F}(\alpha,\beta;\Omega)$. It remains to prove that $\mathcal{C}$ is of the form $(\mathcal{C}\mathsf{u})(\mathbf{x}) = \mathbf{C}(\mathbf{x})\mathsf{u}(\mathbf{x})$ for almost every $\mathbf{x}\in\Omega$, with $\mathbf{C}\in\mathrm{L}^\infty(\Omega;\mathrm{M}_r(\mathbf{R}))$; the bounds on $\mathbf{C}$ will then follow from (19), thus proving $\mathbf{C}\in\mathcal{M}_r(\alpha,\beta;\Omega)$.

For given $\mathsf{u}\in V$ let us define $\mathsf{f} := \mathcal{L}\mathsf{u}\in L$ and the sequence $\mathsf{u}_n := \mathcal{L}_n^{-1}\mathsf{f}\in V$, so that $\mathsf{u}_n \rightharpoonup \mathsf{u}$ in W and $\mathbf{C}_n\mathsf{u}_n \rightharpoonup \mathcal{C}\mathsf{u}$ in L.

Testing the equalities $\mathsf{f} = \mathcal{L}\mathsf{u} = \mathcal{L}_n\mathsf{u}_n$ on $\varphi\mathsf{u}_n$, with $\varphi\in\mathrm{C}^\infty_c(\Omega)$ we obtain:

$$\langle\, \mathcal{L}_0\mathsf{u} \mid \varphi\mathsf{u}_n \,\rangle_L + \langle\, \mathcal{C}\mathsf{u} \mid \varphi\mathsf{u}_n \,\rangle_L = \langle\, \mathcal{L}_0\mathsf{u}_n \mid \varphi\mathsf{u}_n \,\rangle_L + \langle\, \mathbf{C}_n\mathsf{u}_n \mid \varphi\mathsf{u}_n \,\rangle_L \,. \tag{20}$$

The left-hand side converges to $\langle\, \mathcal{L}_0\mathsf{u} \mid \varphi\mathsf{u} \,\rangle_L + \langle\, \mathcal{C}\mathsf{u} \mid \varphi\mathsf{u} \,\rangle_L$ as n tends to ∞, so by (K2) we conclude

$$\langle\, \mathbf{C}_n\mathsf{u}_n \mid \varphi\mathsf{u}_n \,\rangle_L \longrightarrow \langle\, \mathcal{C}\mathsf{u} \mid \varphi\mathsf{u} \,\rangle_L \,.$$

In the sequel we consider only positive $\varphi\in\mathrm{C}^\infty_c(\Omega)$. As $\sqrt{\varphi}\mathbf{C}_n\mathsf{u}_n$ weakly converges to $\sqrt{\varphi}\mathcal{C}\mathsf{u}$ in L, we have $\liminf_n \|\sqrt{\varphi}\mathbf{C}_n\mathsf{u}_n\|_L \ge \|\sqrt{\varphi}\mathcal{C}\mathsf{u}\|_L$, so the inequalities $\varphi\mathbf{C}_n\mathsf{u}_n\cdot\mathsf{u}_n \ge \frac{1}{\beta}\varphi|\mathbf{C}_n\mathsf{u}_n|^2$, $n\in\mathbf{N}$, almost everywhere on Ω imply

$$\int_\Omega \varphi\mathcal{C}\mathsf{u}\cdot\mathsf{u}\,d\mathbf{x} = \lim_n \int_\Omega \varphi\mathbf{C}_n\mathsf{u}_n\cdot\mathsf{u}_n\,d\mathbf{x} \ge \frac{1}{\beta}\int_\Omega \varphi|\mathcal{C}\mathsf{u}|^2\,d\mathbf{x}\,.$$

Now one concludes

$$\mathcal{C}\mathsf{u}\cdot\mathsf{u} \ge \frac{1}{\beta}|\mathcal{C}\mathsf{u}|^2\,, \qquad \text{a.e. on } \Omega\,,$$

and, in particular, for any $\mathsf{u}\in V$

$$|\mathcal{C}\mathsf{u}| \le \beta|\mathsf{u}|\,, \qquad \text{a.e. on } \Omega\,. \tag{21}$$

Let $\omega_1\subseteq\omega_2\subseteq\cdots$ be an increasing sequence of compact sets in Ω such that $\bigcup_n \omega_n = \Omega$. The inequality (21) enables one to define a matrix-valued function $\mathbf{C}\in\mathrm{L}^\infty(\Omega;\mathrm{M}_r(\mathbf{R}))$ by defining it on each ω_n: if $\varphi_n\in\mathrm{C}^\infty_c(\Omega)$ equals 1 on ω_n, for any $\boldsymbol{\xi}\in\mathbf{R}^r$ we take

$$\mathbf{C}(\mathbf{x})\boldsymbol{\xi} := \mathcal{C}(\boldsymbol{\xi}\varphi_n)(\mathbf{x})\,,\ \text{a.e. } \mathbf{x}\in\omega_n\,.$$

This definition is correct, since if $m>n$ then on ω_n in this way we obtain the same value:

$$|\mathcal{C}(\boldsymbol{\xi}\varphi_n) - \mathcal{C}(\boldsymbol{\xi}\varphi_m)| \le \beta|\boldsymbol{\xi}(\varphi_n-\varphi_m)| = 0\,.$$

In particular, due to the linearity of $\mathcal{C}$, this definition implies the equality $\mathcal{C}\mathsf{v} = \mathbf{C}\mathsf{v}$ almost everywhere on Ω, for any step function v on Ω. As step functions are dense in L, another application of (21) implies that the same holds on the whole L.

Q.E.D.

Let us set up the notation for the next theorem: if $\mathcal{L}$ is a Friedrichs operator of the form $\mathcal{L} = \mathcal{L}_0 + \mathcal{C}$, with $\mathcal{L}_0 = \sum_{k=1}^d \mathbf{A}_k\partial_k$ and $\mathcal{C}\in\mathcal{F}(\alpha,\beta;\Omega)$, as before, then its formal adjoint is $\tilde{\mathcal{L}} = -\mathcal{L}_0 + \mathcal{C}^*$, where $\mathcal{C}^*\in\mathcal{L}(L)$ is the adjoint operator of the operator $\mathcal{C}$ (in the sense of Hilbert spaces), belonging also to $\mathcal{F}(\alpha,\beta;\Omega)$. In particular, if $\mathcal{C}$ stands for the multiplication by a matrix-valued function $\mathbf{C}\in\mathcal{M}_r(\alpha,\beta;\Omega)$, then $\mathcal{C}^*$ is of the same kind, with the transpose matrix $\mathbf{C}^\top\in\mathcal{M}_r(\alpha,\beta;\Omega)$ instead of $\mathbf{C}$. By Theorem 1, $\tilde{\mathcal{L}}$ is an isomorphism from $\widetilde{V}$ to L, and the definition of G-convergence is analogously written in terms of its inverse.

Theorem 7. *If a sequence of Friedrichs operators of the form $\mathcal{L}_n = \mathcal{L}_0+\mathcal{C}_n$, for $\mathcal{C}_n \in \mathcal{F}(\alpha,\beta;\Omega)$, G-converges to $\mathcal{L} = \mathcal{L}_0+\mathcal{C}$, then the sequence of its formal adjoints $\tilde{\mathcal{L}}_n = -\mathcal{L}_0+\mathcal{C}_n^*$ G-converges to $\tilde{\mathcal{L}} = -\mathcal{L}_0+\mathcal{C}^*$. Moreover, if a sequence $(\mathbf{C}_n)$ in $\mathcal{M}_r(\alpha,\beta;\Omega)$ H-converges to $\mathbf{C}$ with respect to $\mathcal{L}_0$ and V, then $(\mathbf{C}_n^\top)$ H-converges to $\mathbf{C}^\top$ with respect to $-\mathcal{L}_0$ and $\widetilde{V}$.*

Dem. For given $f, g \in L$ we define $\mathsf{u}_n = \mathcal{L}_n^{-1}\mathsf{f} \in V$ and $\mathsf{v}_n := \tilde{\mathcal{L}}_n^{-1}\mathsf{g} \in \widetilde{V}$. Since $\mathcal{L}_n \xrightharpoonup{G} \mathcal{L}$, the convergence $\mathsf{u}_n \xrightharpoonup{W} \mathsf{u}$ holds, with $\mathsf{u} = \mathcal{L}^{-1}\mathsf{f} \in V$. Due to the uniform bound on $\tilde{\mathcal{L}}_n^{-1}$, we can pass to a subsequence of (v_n) which converges to $\mathsf{v} \in \widetilde{V}$ weakly in W. After testing $\mathcal{L}_n\mathsf{u}_n = \mathsf{f}$ and $\tilde{\mathcal{L}}_n\mathsf{v}_n = \mathsf{g}$ to v_n and u_n, respectively, and then subtracting these equalities, we get

$$\langle\, \mathcal{L}_n\mathsf{u}_n \mid \mathsf{v}_n \,\rangle_L - \langle\, \tilde{\mathcal{L}}_n\mathsf{v}_n \mid \mathsf{u}_n \,\rangle_L = \langle\, \mathsf{f} \mid \mathsf{v}_n \,\rangle_L - \langle\, \mathsf{g} \mid \mathsf{u}_n \,\rangle_L \,, \tag{22}$$

The left-hand side of (22) equals ${}_{W'}\langle\, \mathcal{D}\mathsf{u}_n, \mathsf{v}_n \,\rangle_W$ which is zero by (V2), as u_n belongs to V, and v_n belongs to $\widetilde{V}$. Therefore, passing to the limit in (22) we obtain

$$\langle\, \mathsf{f} \mid \mathsf{v} \,\rangle_L = \lim_n \langle\, \mathsf{f} \mid \mathsf{v}_n \,\rangle_L = \lim_n \langle\, \mathsf{g} \mid \mathsf{u}_n \,\rangle_L = \langle\, \mathsf{g} \mid \mathsf{u} \,\rangle_L \,.$$

Finally, as $\mathsf{u} \in V$ and $\mathsf{v} \in \widetilde{V}$ we have ${}_{W'}\langle\, \mathcal{D}\mathsf{u}, \mathsf{v} \,\rangle_W = 0$, or equivalently

$$\langle\, \mathsf{u} \mid \tilde{\mathcal{L}}\mathsf{v} \,\rangle_L = \langle\, \mathcal{L}\mathsf{u} \mid \mathsf{v} \,\rangle_L = \langle\, \mathsf{f} \mid \mathsf{v} \,\rangle_L = \langle\, \mathsf{g} \mid \mathsf{u} \,\rangle_L \,.$$

Since $\mathsf{u} \in V$ is arbitrary and $\mathrm{C}_c^\infty(\Omega;\mathbf{R}^r) \subseteq W_0 \subseteq V$ are dense in L (see e.g. [18, Lemma 3.1]), it follows that $\mathsf{v} = \tilde{\mathcal{L}}^{-1}\mathsf{g}$. Since the accumulation point v of the sequence (v_n) is uniquely determined, the whole sequence converges to v, and arbitrariness of g completes the proof of the first claim.

Additionally, let us assume that for any $n \in \mathbf{N}$ the operator $\mathcal{C}_n : L \to L$ is a multiplication operator by a matrix-valued function $\mathbf{C}_n \in \mathcal{M}_r(\alpha,\beta;\Omega)$ and $\mathbf{C}_n \xrightharpoonup{H} \mathbf{C}$ with respect to $\mathcal{L}_0$ and V, which also implies that $\mathcal{C}$ is the multiplication operator by $\mathbf{C}$. For an arbitrary $\mathsf{g} \in L$, let us define $\mathsf{v}_n = \tilde{\mathcal{L}}_n^{-1}\mathsf{g}$. By the first part of the proof we know that $\mathsf{v}_n \rightharpoonup \mathsf{v}$ in W, where $\mathsf{v} = \tilde{\mathcal{L}}^{-1}\mathsf{g} \in \widetilde{V}$, implying also the weak convergence $-\mathcal{L}_0\mathsf{v}_n \rightharpoonup -\mathcal{L}_0\mathsf{v}$ in L. Then $-\mathcal{L}_0\mathsf{v}_n + \mathcal{C}_n^*\mathsf{v}_n = \mathsf{g} = -\mathcal{L}_0\mathsf{v} + \mathbf{C}^\top\mathsf{v}$ implies that $\mathbf{C}_n^\top\mathsf{v}_n = \mathcal{C}_n^*\mathsf{v}_n \rightharpoonup \mathbf{C}^\top\mathsf{v}$ weakly in L, which implies $\mathbf{C}_n^\top \xrightharpoonup{H} \mathbf{C}^\top$ with respect to $-\mathcal{L}_0$ and $\widetilde{V}$, by the definition.

Q.E.D.

In Theorem 6 we have proved that, under the additional assumptions (K1) and (K2), for a sequence of classical Friedrichs systems satisfying some uniform bounds and sharing the same $\mathcal{L}_0$ part, we get a system of the same type in the limit. Since we want to reinterpret the homogenisation results of some classical equations in the setting of Friedrichs systems, we would like to conclude that, for example for the system corresponding to the stationary diffusion equation, we get in the limit the stationary diffusion equation in the form of Friedrichs system again. However, Theorem 6 does not provide that information – it only claims that the limiting equation is the Friedrichs system (with the same $\mathcal{L}_0$ part and satisfying the same bounds), and not necessarily the system that corresponds to the stationary diffusion equation. Clearly, this is the question of the *conservation of the same structure* of the coefficient matrix $\mathbf{C}$ through the homogenisation process, studied in the following theorem.

Theorem 8. *Let a sequence $(\mathbf{C}_n)$ in $\mathcal{M}_r(\alpha,\beta;\Omega)$ H-converge towards $\mathbf{C}$ with respect to $\mathcal{L}_0$ and V. If each $\mathbf{C}_n$ has the block-diagonal structure, with one block not depending on n:*

$$\mathbf{C}_n = \begin{bmatrix} \mathbf{C}_n^{\boldsymbol{\sigma}} & \mathbf{0} \\ \mathbf{0} & \mathbf{C}^u \end{bmatrix} ,$$

then $\mathbf{C}$ has the same block-diagonal structure, i. e.

$$\mathbf{C} = \begin{bmatrix} \mathbf{C}^{\boldsymbol{\sigma}} & \mathbf{0} \\ \mathbf{0} & \mathbf{C}^u \end{bmatrix} ,$$

for some bounded matrix field $\mathbf{C}^{\boldsymbol{\sigma}}$ on Ω.

Dem. Let
$$\mathbf{C} = \begin{bmatrix} \mathbf{C}^{\boldsymbol\sigma} & \mathbf{D} \\ \mathbf{E} & \mathbf{F} \end{bmatrix},$$
and let us prove that $\mathbf{E} = \mathbf{D}^\top = \mathbf{0}$ and $\mathbf{F} = \mathbf{C}^u$. For an arbitrary $\mathsf{u} = (\mathsf{u}^{\boldsymbol\sigma}, \mathsf{u}^u)^\top \in V$ let us define $\mathsf{f} := (\mathcal{L}_0 + \mathbf{C})\mathsf{u}$ and $\mathsf{u}_n := (\mathcal{L}_0 + \mathbf{C}_n)^{-1}\mathsf{f} = (\mathsf{u}^{\boldsymbol\sigma}_n, \mathsf{u}^u_n)^\top$, where the $\boldsymbol\sigma$-component of these functions has the size of the block $\mathbf{C}^\sigma$, and similarly for the u-component. Then the convergence $\mathbf{C}_n\mathsf{u}_n \rightharpoonup \mathbf{C}\mathsf{u}$ in L implies
$$\begin{aligned} \mathbf{C}^{\boldsymbol\sigma}_n\mathsf{u}^{\boldsymbol\sigma}_n &\rightharpoonup \mathbf{C}^{\boldsymbol\sigma}\mathsf{u}^{\boldsymbol\sigma} + \mathbf{D}\mathsf{u}^u \qquad \text{in } L\,, \\ \mathbf{C}^u\mathsf{u}^u_n &\rightharpoonup \mathbf{E}\mathsf{u}^{\boldsymbol\sigma} + \mathbf{F}\mathsf{u}^u \qquad \text{in } L\,. \end{aligned}$$
However, the convergence $\mathsf{u}_n \rightharpoonup \mathsf{u}$ in L, implies $\mathbf{C}^u\mathsf{u}^u_n \rightharpoonup \mathbf{C}^u\mathsf{u}^u$ and therefore $\mathbf{E}\mathsf{u}^{\boldsymbol\sigma}+\mathbf{F}\mathsf{u}^u = \mathbf{C}^u\mathsf{u}^u$. Choosing u such that $\mathsf{u}^u = 0$ and $\mathsf{u}^{\boldsymbol\sigma}$ being an arbitrary smooth function with a compact support in Ω, we conclude $\mathbf{E} = \mathbf{0}$. Similarly, taking $\mathsf{u}^{\boldsymbol\sigma} = 0$ and an arbitrary smooth u^u with a compact support, we have $\mathbf{F} = \mathbf{C}^u$.

Using Theorem 7 and the same argumentation as above for the transpose operators $\mathbf{C}_n^\top$ and $\mathbf{C}^\top$ (and H-convergence with respect to $-\mathcal{L}_0$ and $\widetilde{V}$), we get $\mathbf{D} = \mathbf{0}$, which concludes the proof.

Q.E.D.

4. Examples

In this section we apply our homogenisation results to the stationary diffusion equation and the heat equation. For these examples, we are able to obtain the complete characterisation of the H-limit, in terms of the classical H-limit for the original equations. The last example considers the opposite case: the simplest equation for which the memory effects occur.

As mentioned before, in order to verify that compactness conditions are fulfilled for the first two examples, we shall use the Quadratic theorem of compensated compactness [29], which we state here.

Theorem 9. (**Quadratic theorem**) *Let $\Omega \subseteq \mathbf{R}^d$ be open, $\mathbf{D}^k \in \mathrm{M}_{q,p}(\mathbf{R})$, $k \in 1..d$, and*
$$\Lambda := \Big\{ \boldsymbol\lambda \in \mathbf{R}^p \,:\, (\exists\, \boldsymbol\xi \in \mathbf{R}^d \setminus \{0\}) \quad \sum_{k=1}^d \xi_k \mathbf{D}^k \boldsymbol\lambda = 0 \Big\}.$$
Moreover, for given $\mathbf{Q} \in \mathrm{M}_p(\mathbf{R})$, let the quadratic form $Q(\boldsymbol\lambda) := \mathbf{Q}\boldsymbol\lambda \cdot \boldsymbol\lambda$, $\boldsymbol\lambda \in \mathbf{R}^p$ satisfy
$$(\forall\, \boldsymbol\lambda \in \Lambda) \quad Q(\boldsymbol\lambda) = 0\,.$$
Then every sequence of functions (w^n), with properties

$(P1)$ $$\mathsf{w}^n \rightharpoonup \mathsf{w} \quad \textit{weakly in} \quad \mathrm{L}^2(\Omega; \mathbf{R}^p)\,,$$

$(P2)$ $$\Big(\sum_{k=1}^d \mathbf{D}^k \partial_k \mathsf{w}^n\Big) \quad \textit{is relatively compact in} \quad \mathrm{H}^{-1}(\Omega; \mathbf{R}^q)\,,$$

satisfies
$$Q \circ \mathsf{w}^n \rightharpoonup Q \circ \mathsf{w} \quad \textit{in} \quad \mathcal{D}'(\Omega)\,.$$

∎

Stationary diffusion equation

In the introductory section we have seen how stationary diffusion equation can be written as a Friedrichs system. We are now interested in the homogenisation of that Friedrichs system. More precisely, we shall show that in this particular case the family $\mathcal{M}_r(\alpha, \beta; \Omega)$ satisfies (K1) and (K2), for the various choices of boundary conditions (subspaces V), and make the comparison with the classical notion of H-convergence for the stationary diffusion equation.

Let us assume that we have given a sequence of equations

$$-\mathsf{div}\,(\mathbf{A}^n \nabla u_n) + c_n u_n = f \tag{23}$$

in an open and bounded set $\Omega \subseteq \mathbf{R}^d$ with the Lipschitz boundary Γ, where $f \in \mathrm{L}^2(\Omega)$, $\mathbf{A}^n \in \mathcal{M}_d(\alpha', \beta'; \Omega)$ and $c_n \in \mathrm{L}^\infty(\Omega)$ with $\frac{1}{\beta'} \leqslant c_n \leqslant \frac{1}{\alpha'}$, for some $\beta' \geq \alpha' > 0$. We have already seen how this type of equation can be written as the Friedrichs system, with coefficient matrices $\mathbf{A}_k = \mathbf{e}_k \otimes \mathbf{e}_{d+1} + \mathbf{e}_{d+1} \otimes \mathbf{e}_k \in \mathrm{M}_{d+1}(\mathbf{R})$, for $k \in 1..d$ and the block-diagonal matrix-valued function

$$\mathbf{C}_n = \begin{bmatrix} (\mathbf{A}^n)^{-1} & \mathsf{0} \\ \mathsf{0}^\top & c_n \end{bmatrix} \in \mathrm{L}^\infty(\Omega; \mathrm{M}_{d+1}(\mathbf{R}))\,. \tag{24}$$

Clearly, the assumptions on c_n and $\mathbf{A}^n$ are equivalent to $\mathbf{C}_n \in \mathcal{M}_{d+1}(\alpha, \beta; \Omega)$, for $\alpha = \frac{1}{\beta'}$ and $\beta = \frac{1}{\alpha'}$.

Let us recall that graph space is $W = \mathrm{L}^2_{\mathrm{div}}(\Omega) \times \mathrm{H}^1(\Omega)$ and

$$\mathcal{L}_0 \begin{bmatrix} \mathsf{u}^{\boldsymbol{\sigma}} \\ u^u \end{bmatrix} = \begin{bmatrix} \nabla u^u \\ \mathsf{div}\, \mathsf{u}^{\boldsymbol{\sigma}} \end{bmatrix}, \tag{25}$$

where $\mathsf{u}^{\boldsymbol{\sigma}}$ contains the first d components of u, and u^u is the last component. Dirichlet, Neumann and Robin boundary conditions are imposed by the following choice of V and $\widetilde{V}$, respectively (we put indices to distinguish them):

$$\begin{aligned} V_D = \widetilde{V}_D :=& \mathrm{L}^2_{\mathrm{div}}(\Omega) \times \mathrm{H}^1_0(\Omega)\,, \\ V_N = \widetilde{V}_N :=& \{(\mathsf{u}^{\boldsymbol{\sigma}}, u^u)^\top \in W : \boldsymbol{\nu} \cdot \mathsf{u}^{\boldsymbol{\sigma}} = 0\}\,, \\ V_R :=& \{(\mathsf{u}^{\boldsymbol{\sigma}}, u^u)^\top \in W : \boldsymbol{\nu} \cdot \mathsf{u}^{\boldsymbol{\sigma}} = a u^u_{|\Gamma}\}\,, \\ \widetilde{V}_R :=& \{(\mathsf{u}^{\boldsymbol{\sigma}}, u^u)^\top \in W : \boldsymbol{\nu} \cdot \mathsf{u}^{\boldsymbol{\sigma}} = -a u^u_{|\Gamma}\}\,. \end{aligned}$$

Thus, in order to apply the theorems presented here, it remains to prove the compactness assumptions (K1) and (K2) for $\mathcal{M}_{d+1}(\alpha, \beta; \Omega)$:

Lemma 1. *For $\mathcal{L}_0$ as in (25) and any $V \in \{V_D, V_N, V_R\}$, the family $\mathcal{M}_{d+1}(\alpha, \beta; \Omega)$ satisfies assumptions (K1) and (K2). The same is true for the operator $-\mathcal{L}_0$ and the subspace $\widetilde{V}$, instead of $\mathcal{L}_0$ and V, respectively.*

Dem. For the proof of (K1) we shall use the following representation of D [9]:

$${}_{W'}\langle\, D(\mathsf{u}^{\boldsymbol{\sigma}}, u^u)^\top, (\mathsf{v}^{\boldsymbol{\sigma}}, v^u)^\top \,\rangle_W = {}_{\mathrm{H}^{-\frac{1}{2}}}\langle\, \boldsymbol{\nu} \cdot \mathsf{u}^{\boldsymbol{\sigma}}, v^u \,\rangle_{\mathrm{H}^{\frac{1}{2}}} + {}_{\mathrm{H}^{-\frac{1}{2}}}\langle\, \boldsymbol{\nu} \cdot \mathsf{v}^{\boldsymbol{\sigma}}, u^u \,\rangle_{\mathrm{H}^{\frac{1}{2}}}\,,$$

for $\mathsf{u} = (\mathsf{u}^{\boldsymbol{\sigma}}, u^u)^\top, \mathsf{v} = (\mathsf{v}^{\boldsymbol{\sigma}}, v^u)^\top \in W$. It is now clear that ${}_{W'}\langle\, D\mathsf{v}, \mathsf{v} \,\rangle_W = 0$ if v belongs to any of the sets V_D, $\widetilde{V}_D$, V_N, or $\widetilde{V}_N$, so the condition (K1) is trivially satisfied. In the case of Robin boundary conditions, for $\mathsf{v} \in V_R$, we have

$${}_{W'}\langle\, D\mathsf{v}, \mathsf{v} \,\rangle_W = 2a \|v^u\|^2_{\mathrm{L}^2(\Gamma)}\,. \tag{26}$$

If a sequence (u_n) in V_R weakly converges to u in W, then the sequence of the last components (u^u_n) weakly converges to u^u in $\mathrm{H}^1(\Omega)$. Now, using the weak continuity of trace operator $\mathrm{H}^1(\Omega) \longrightarrow \mathrm{H}^{\frac{1}{2}}(\Gamma)$, and compact imbedding of $\mathrm{H}^{\frac{1}{2}}(\Gamma)$ in $\mathrm{L}^2(\Gamma)$, we conclude $\|u^u_n\|_{\mathrm{L}^2(\Gamma)} \longrightarrow \|u^u\|_{\mathrm{L}^2(\Gamma)}$, which together with (26) proves (K1) for V_R. Similarly for $\widetilde{V}_R$.

As we mentioned above, in order to check the assumption (K2) we shall use Quadratic theorem of compensated compactness: let u_n be a sequence which weakly converges to u in W and $\varphi \in \mathrm{C}_c^\infty(\Omega)$ arbitrary. Using integration by parts, from

$$\langle\, \mathcal{L}_0 \mathsf{u}_n \mid \varphi \mathsf{u}_n \,\rangle_L = \int_\Omega \sum_{k=1}^d \mathbf{A}_k \partial_k \mathsf{u}_n \cdot \varphi \mathsf{u}_n \, d\mathbf{x}\,,$$

we easily get

$$\langle\, \mathcal{L}_0 \mathsf{u}_n \mid \varphi \mathsf{u}_n \,\rangle_L = -\frac{1}{2} \int_\Omega \partial_k \varphi \sum_{k=1}^d \mathbf{A}_k \mathsf{u}_n \cdot \mathsf{u}_n \, d\mathbf{x}\,,$$

and a similar formula for u instead u_n. Therefore, in order to prove (K2) it is enough to prove that $\mathbf{A}_i \mathsf{u}_n \cdot \mathsf{u}_n \rightharpoonup \mathbf{A}_i \mathsf{u} \cdot \mathsf{u}$ in $\mathcal{D}'(\Omega)$, for every $i \in 1..d$. Now we apply Quadratic theorem with $p = q = d+1$, $\mathbf{D}^k = \mathbf{A}_k$ and $\mathbf{Q} = \mathbf{A}_i$, for fixed $i \in 1..d$. Note that the conditions (P1) and (P2) of Quadratic theorem are satisfied for our sequence because $\mathsf{u}_n \rightharpoonup \mathsf{u}$ in W. Now it is easy to calculate Λ:

$$\Lambda = \{\boldsymbol{\lambda} \in \mathbf{R}^{d+1} : \lambda_{d+1} = 0\}\,,$$

and since $\mathbf{A}_i \boldsymbol{\lambda} \cdot \boldsymbol{\lambda} = 2\lambda_i \lambda_{d+1}$ it follows that our quadratic form is zero on it. Therefore, the conditions of Quadratic theorem are satisfied, which proves (K2).

Q.E.D.

The above lemma implies that our general homogenisation results can be applied to this example. Let us now compare the notion of H-convergence of this Friedrichs system with the classical H-convergence in case of stationary diffusion equation. More precisely, consider a sequence of matrices $\mathbf{C}_n$ of the form (24) H-converging to $\mathbf{C}$ with respect to $\mathcal{L}_0$, given by (25), and some $V \in \{V_D, V_N, V_R\}$.

For the moment let us consider the simplest case: let (c_n) be a stationary sequence. Then by Theorem 8 the H-limit $\mathbf{C}$ has the same block-diagonal structure:

$$\mathbf{C} = \begin{bmatrix} \mathbf{B} & \mathsf{0} \\ \mathsf{0}^\top & c \end{bmatrix}.$$

If the right-hand side f vanishes, except at the last component f, the limiting system turns to the stationary diffusion equation for the last component of u:

$$-\mathsf{div}\,(\mathbf{B}^{-1} \nabla u) + cu = f\,.$$

On the other hand, as mentioned in the Introduction, the sequence $(\mathbf{A}^n)$ has a subsequence that H-converges to $\mathbf{A}$ in the original sense for the stationary diffusion problem (we shall address this H-convergence as the *classical H-convergence*). Now, the natural question appears: does $\mathbf{A} = \mathbf{B}^{-1}$ hold?

We shall obtain this in Theorem 11, without the stationarity assumption on the sequence (c_n), but we first cite the following theorem, which shows that the classical H-convergence is independent of boundary conditions [22]. Actually, the possibility of an improvement of the result of Theorem 8 for this example lies in the special structure of matrices $\mathbf{A}_k$, as well as the strong convergence in L of the sequence of the solutions u_n^u.

Theorem 10. *If a sequence $(\mathbf{A}^n)$ classically H-converges to $\mathbf{A}$ and $u_n \rightharpoonup u$ in $\mathrm{H}^1(\Omega)$ with $\mathsf{div}\,(\mathbf{A}^n \nabla u_n)$ belonging to a compact set of $\mathrm{H}^{-1}(\Omega)$ strong, then $\mathbf{A}^n \nabla u_n \rightharpoonup \mathbf{A} \nabla u$ in $\mathrm{L}^2(\Omega; \mathbf{R}^d)$.*

∎

Theorem 11. *For the Friedrichs system corresponding to the stationary diffusion equation, a sequence $(\mathbf{C}_n)$ in $\mathcal{M}_{d+1}(\alpha, \beta; \Omega)$ of the form (24) H-converges with respect to $\mathcal{L}_0$ and V_D if and only if $(\mathbf{A}^n)$ classically H-converges to some $\mathbf{A}$ and (c_n) L^∞ weakly $*$ converges to some c. In that case, the H-limit is the matrix-valued function*

$$\mathbf{C} = \begin{bmatrix} \mathbf{A}^{-1} & \mathsf{0} \\ \mathsf{0}^\top & c \end{bmatrix}, \tag{27}$$

Dem. For the proof of the necessity part let us denote the H-limit by

$$\mathbf{C} = \begin{bmatrix} \mathbf{B} & \mathsf{D} \\ \mathsf{E} & F \end{bmatrix},$$

where, as before, the size of the matrix-valued function $\mathbf{B}$ is $d \times d$, and since $\mathbf{C}$ is uniformly positive definite, so is $\mathbf{B}$. Due to the compactness theorem for classical H-convergence, and the boundedness of the sequence (c_n) in L^∞, we can pass to a subsequence such that $(\mathbf{A}^n)$ classically H-converges to $\mathbf{A}$, and c_n weakly $*$ converges to c.

The operator $\mathcal{A}u := \mathsf{div}\,\mathbf{A}\nabla u + cu$ is an isomorphism from $\mathrm{H}^1_0(\Omega)$ to $\mathrm{H}^{-1}(\Omega)$, and since $\mathrm{L}^2(\Omega)$ is dense in $\mathrm{H}^{-1}(\Omega)$, so is $D(\mathcal{A}) := \mathcal{A}^{-1}(\mathrm{L}^2(\Omega))$ dense in $\mathrm{H}^1_0(\Omega)$. Let us take an arbitrary $u^u \in D(\mathcal{A})$ and define $\mathsf{u} = (\mathsf{u}^{\boldsymbol\sigma}, u^u)^\top$ with $\mathsf{u}^{\boldsymbol\sigma} = -\mathbf{A}\nabla u^u \in \mathrm{L}^2_{\mathrm{div}}(\Omega)$, $f := -\mathsf{div}\,(\mathbf{A}\nabla u^u) + cu^u \in L$, and $\mathsf{f} = (0, \dots, 0, f)^\top$.

Furthermore, we take $\mathsf{u}_n = (\mathsf{u}^{\boldsymbol\sigma}_n, u^u_n)^\top := \mathcal{L}_n^{-1}\mathsf{f}$, with $\mathcal{L}_n = \mathcal{L}_0 + \mathbf{C}_n$. By the a priori bound (12), the sequence of the solutions (u_n) is bounded in W so we can take another subsequence such that $\mathsf{u}_n \rightharpoonup \mathsf{v}$ in W. Since first d components of f are equal to zero, the Friedrichs system for u_n reads: $\mathsf{u}^{\boldsymbol\sigma}_n = -\mathbf{A}^n\nabla u^u_n$ and $u^u_n \in \mathrm{H}^1_0(\Omega)$ (which weakly converges to v^u in $\mathrm{H}^1_0(\Omega)$) solves the equation $-\mathsf{div}\,(\mathbf{A}^n\nabla u^u_n) = f - c_n u^u_n$. By the Rellich compactness theorem, the weak convergence of u^u_n implies $u^u_n \longrightarrow v^u$ in $\mathrm{L}^2(\Omega)$, which together with weak $*$ convergence of c_n implies $c_n u^u_n \rightharpoonup cv^u$ in $\mathrm{L}^2(\Omega)$. Using again the Rellich theorem, we obtain that the right-hand side in the equation for u^u_n strongly converges to $f - cv^u$ in $\mathrm{H}^{-1}(\Omega)$, and therefore $\mathbf{A}^n\nabla u^u_n \rightharpoonup \mathbf{A}\nabla v^u$ in $\mathrm{L}^2(\Omega; \mathbf{R}^d)$, by Theorem 10. Therefore, passing to the limit in the equation for u^u_n we obtain $-\mathsf{div}\,(\mathbf{A}\nabla v^u) = f - cv^u$, and the uniqueness of the solution implies $v^u = u^u$. Moreover, we have $\mathsf{u}^{\boldsymbol\sigma}_n = -\mathbf{A}^n\nabla u^u_n \rightharpoonup -\mathbf{A}\nabla v^u = -\mathbf{A}\nabla u^u = \mathsf{u}^{\boldsymbol\sigma}$ in $\mathrm{L}^2(\Omega; \mathbf{R}^d)$, showing the equality $\mathsf{v} = \mathsf{u}$.

On the other side, by the H-convergence $\mathbf{C}_n \xrightarrow{H} \mathbf{C}$, we know that $\mathsf{u}_n = \mathcal{L}_n^{-1}\mathsf{f} \rightharpoonup \mathcal{L}^{-1}\mathsf{f}$ in W, with $\mathcal{L} = \mathcal{L}_0 + \mathbf{C}$, so the uniqueness of the limit implies $\mathsf{u} = \mathcal{L}^{-1}\mathsf{f}$.

The convergence $\mathbf{C}_n\mathsf{u}_n \rightharpoonup \mathbf{C}\mathsf{u}$ means

$$\begin{aligned} -\nabla u^u_n &\rightharpoonup \mathbf{B}\mathsf{u}^{\boldsymbol\sigma} + \mathsf{D}u^u && \text{in } L\,, \\ c_n u^u_n &\rightharpoonup \mathsf{E}\mathsf{u}^{\boldsymbol\sigma} + Fu^u && \text{in } L\,. \end{aligned}$$

Since we already obtained $\nabla u^u_n \rightharpoonup \nabla u^u$ in L and $c_n u^u_n \rightharpoonup cu^u$ (due to the strong convergence of the sequence (u^u_n) in L) we have

$$\begin{aligned} -\nabla u^u &= \mathbf{B}\mathsf{u}^{\boldsymbol\sigma} + \mathsf{D}u^u\,, \\ cu^u &= \mathsf{E}\mathsf{u}^{\boldsymbol\sigma} + Fu^u\,. \end{aligned} \tag{28}$$

These two equalities hold for an arbitrary $u^u \in D(\mathcal{A})$ and $\mathsf{u}^{\boldsymbol\sigma} = -\mathbf{A}\nabla u^u$, so they also hold for any $u^u \in \mathrm{H}^1_0(\Omega)$ and $\mathsf{u}^{\boldsymbol\sigma} = -\mathbf{A}\nabla u^u$. Indeed, using density of $D(\mathcal{A})$ in $\mathrm{H}^1_0(\Omega)$, for given $u^u \in \mathrm{H}^1_0(\Omega)$ let us take a sequence $v^u_n \in D(\mathcal{A})$ converging to u^u strongly in $\mathrm{H}^1(\Omega)$, implying also $\mathsf{v}^{\boldsymbol\sigma}_n := -\mathbf{A}\nabla v^u_n \longrightarrow -\mathbf{A}\nabla u^u$ in $\mathrm{L}^2(\Omega)$. Now, taking the limit in equalities (28) with v^u_n instead of u^u and $\mathsf{v}^{\boldsymbol\sigma}_n$ instead of $\mathsf{u}^{\boldsymbol\sigma}$, as n tends to ∞, we obtain (28).

As $u^u \in \mathrm{C}^\infty_c(\Omega)$ can be chosen arbitrarily, and $\mathsf{u}^{\boldsymbol\sigma} = -\mathbf{A}\nabla u^u$, let us take $u^u \equiv 1$ on some set ω compactly included in Ω. Substituting this in (28), we obtain $\mathsf{D} = \mathsf{0}$ and $F = c$ on ω, and therefore, on the whole Ω. Now, from the above equalities one easily concludes $\mathsf{E} = \mathsf{0}$ and $\mathbf{A} = \mathbf{B}^{-1}$ almost everywhere on Ω. The uniqueness of the limit of all converging subsequences of $(\mathbf{A}_n)$ and (c_n), implies the convergence of the whole sequences.

Conversely, if the sequence $(\mathbf{A}_n)$ classically H-converges to $\mathbf{A}$, and c_n weakly $*$ converges to c, let us pass to a subsequence of $(\mathbf{C}_n)$ which H-converges to $\mathbf{C}$ with respect to $\mathcal{L}_0$ and V_D. However, from the first part of this proof, one obtains that $\mathbf{C}$ is given by (27), implying that the whole sequence converges to the same limit, as this is its only cluster point.

Q.E.D.

Thanks to the characterisation given by the last theorem, it is possible to conclude that the notion of H-convergence for Friedrichs system given by the operator $\mathcal{L}_0$ from (26) is independent of the prescribed boundary conditions, i.e. independent of the choice of the subspace: V_D, V_N or V_R.

Corollary 1. *The notion of H-convergence with respect to $\mathcal{L}_0$ given by (26) does not depend on the particular choice of the subspace V_D,V_N or V_R.*

Dem. The claim of the last theorem still holds if we change the subspace V_D by V_R or V_N, with actually the same proof. The operator $\mathcal{A} : \mathrm{H}^1(\Omega) \to \mathrm{H}^1(\Omega)'$, from the beginning of that proof, defined classically via corresponding bilinear form, is isomorphic by Lax-Milgram lemma. But $\mathcal{A}$ could also be restricted to an isomorphism from $D(\mathcal{A}) := \mathcal{A}^{-1}(\mathrm{L}^2(\Omega))$ (with respect to the graph norm) onto $\mathrm{L}^2(\Omega)$. Now the same proof applies because $D(\mathcal{A})$ is dense in $\mathrm{H}^1(\Omega)$ (for details see [13, VI.3.2 and VII.1.3]).

Q.E.D.

Heat equation

Let, as before, $\Omega \subseteq \mathbf{R}^d$ be an open and bounded set with the Lipschitz boundary Γ, $T > 0$ and $\Omega_T := \Omega \times \langle 0, T\rangle$. We consider a sequence of heat equations in the following form

$$\partial_t u_n - \mathsf{div}_{\mathbf{x}}(\mathbf{A}^n \nabla_{\mathbf{x}} u_n) + c_n u_n = f \qquad \text{in } \Omega_T ,$$

where $f \in \mathrm{L}^2(\Omega_T)$, $c_n \in \mathrm{L}^\infty(\Omega_T)$, $\frac{1}{\beta'} \leqslant c_n \leqslant \frac{1}{\alpha'}$ and $\mathbf{A}^n \in \mathcal{M}_d(\alpha', \beta'; \Omega_T)$, for some $\beta' \geq \alpha' > 0$ (note that the coefficients depend both on $\mathbf{x}$ and t).

Similarly as it is the case for the stationary diffusion equation, this equation can be rewritten as the Friedrichs system $(\mathcal{L}_0 + \mathbf{C}_n)\mathsf{u}_n = \mathsf{f}$ for the vector function (see [6] for details)

$$\mathsf{u}_n = \begin{bmatrix} \mathsf{u}_n^{\boldsymbol{\sigma}} \\ u_n^u \end{bmatrix} = \begin{bmatrix} -\mathbf{A}^n \nabla_{\mathbf{x}} u_n \\ u_n \end{bmatrix} ,$$

and the right-hand side $\mathsf{f} = (0, \dots, 0, f)^\top \in \mathrm{L}^2(\Omega_T; \mathbf{R}^{d+1})$. The operator $\mathcal{L}_0$ is given by

$$\mathcal{L}_0 \begin{bmatrix} \mathsf{u}^{\boldsymbol{\sigma}} \\ u^u \end{bmatrix} = \begin{bmatrix} \nabla_{\mathbf{x}} u^u \\ \partial_t u^u + \mathsf{div}_{\mathbf{x}} \mathsf{u}^{\boldsymbol{\sigma}} \end{bmatrix} , \tag{29}$$

while the corresponding graph space is

$$W = \Big\{ \mathsf{u} \in \mathrm{L}^2_{\mathrm{div}}(\Omega_T) : u^u \in \mathrm{L}^2(0, T; \mathrm{H}^1(\Omega)) \Big\} .$$

Therefore, the matrices $\mathbf{A}_k = \mathbf{e}_k \otimes \mathbf{e}_{d+1} + \mathbf{e}_{d+1} \otimes \mathbf{e}_k \in \mathrm{M}_{d+1}(\mathbf{R})$, for $k \in 1..d$ and the block matrix-valued function

$$\mathbf{C}_n = \begin{bmatrix} (\mathbf{A}^n)^{-1} & \mathsf{0} \\ \mathsf{0}^\top & c_n \end{bmatrix}$$

coincides with the one used for the stationary diffusion equation, while $\mathbf{A}_{d+1} = \mathbf{e}_{d+1} \otimes \mathbf{e}_{d+1} \in \mathrm{M}_{d+1}(\mathbf{R})$. Here we again have that each $\mathbf{A}_k$ is a constant symmetric matrix, while $\mathbf{C}_n \in \mathcal{M}_{d+1}(\alpha, \beta; \Omega_T)$, for $\alpha = \frac{1}{\beta'}$ and $\beta = \frac{1}{\alpha'}$. Note that our domain is now Ω_T, and that the first d variables are the space variables of the original equation, while $(d + 1)$-st variable is the time variable.

The subspaces V and $\widetilde{V}$ are chosen with respect to the homogenous Dirichlet boundary condition on $\Gamma \times \langle 0, T\rangle$ and the zero initial condition:

$$V = \Big\{ \mathsf{u} \in W : u_{d+1} \in \mathrm{L}^2(0, T; \mathrm{H}^1_0(\Omega)), \quad u_{d+1}(\cdot, 0) = 0 \text{ a.e. on } \Omega \Big\} , \tag{30}$$

$$\widetilde{V} = \Big\{ \mathsf{v} \in W : v_{d+1} \in \mathrm{L}^2(0, T; \mathrm{H}^1_0(\Omega)), \quad v_{d+1}(\cdot, T) = 0 \text{ a.e. on } \Omega \Big\} .$$

In [6] it was shown that they satisfy conditions (V1) and (V2) providing the well-posedness result. Therefore, in order to apply our homogenisation result for Friedrichs systems we need to check conditions (K1) and (K2).

Lemma 2. *For $\mathcal{L}_0$ as in (29) and V as in (30), the family $\mathcal{M}_{d+1}(\alpha, \beta; \Omega)$ satisfies the assumptions (K1) and (K2). The same is true for the operator $-\mathcal{L}_0$ and the subspace $\widetilde{V}$, instead of $\mathcal{L}_0$ and V, respectively.*

Dem. Using results from [6] (namely, Lemma 4 and the formula (7)) we easily get that for every $\mathsf{u} = (\mathsf{u}^{\boldsymbol{\sigma}}, u^u)^\top, \mathsf{v} = (\mathsf{v}^{\boldsymbol{\sigma}}, v^u)^\top \in V$ we have

$$
{}_{W'}\langle\, D(\mathsf{u}^{\boldsymbol{\sigma}}, u^u)^\top, (\mathsf{v}^{\boldsymbol{\sigma}}, v^u)^\top \,\rangle_W = \int_\Omega u^u(\cdot, T) v^u(\cdot, T)\, d\mathbf{x} \;,
$$

and in particular

$$
{}_{W'}\langle\, D\mathsf{v}, \mathsf{v} \,\rangle_W = \| v^u(\cdot, T)\|^2_{\mathrm{L}^2(\Omega)} \,. \tag{31}
$$

Similarly as in [32, Lemma 3] one can prove that for the sequence $\mathsf{u}_n := (\mathcal{L}_0 + \mathcal{C}_n)^{-1}\mathsf{f} \in V$, for arbitrary $\mathsf{f} \in \mathrm{L}^2(\Omega_T; \mathbf{R}^{d+1})$ and $\mathcal{C}_n \in \mathcal{F}(\alpha, \beta; \Omega)$, the corresponding sequence of the last components (u^u_n) is compact in $\mathrm{C}([0, T]; \mathrm{L}^2(\Omega))$, which together with (31) proves (K1) for V.

In order to prove (K2) we use the Quadratic theorem of compensated compactness similarly as in the proof of Lemma 1. Actually, the only differences arise from the fact that we now have $d+1$ instead of d variables. The set $\Lambda = \{\boldsymbol{\lambda} \in \mathbf{R}^{d+1} : \lambda_{d+1} = 0\}$ remains the same, as well as the quadratic forms $\mathbf{A}_i\boldsymbol{\lambda} \cdot \boldsymbol{\lambda} = 2\lambda_i\lambda_{d+1}$, for $k \in 1..d$, which are then, clearly, zero on Λ. In the case of the heat equation we have one additional quadratic form: $Q(\lambda) = \mathbf{A}_{d+1}\boldsymbol{\lambda} \cdot \boldsymbol{\lambda} = \lambda^2_{d+1}$ which is also zero on Λ and then we can apply Theorem 9 in order to conclude (K2).

The statements can be similarly proved for $-\mathcal{L}_0$ and $\widetilde{V}$.

Q.E.D.

The next theorem gives the comparison of H-convergence for the Friedrichs system corresponding to the heat equation with the parabolic H-convergence.

Theorem 12. *Let the operator $\mathcal{L}_0$ and the subspace V be defined by (29) and (30), respectively, and $(\mathbf{C}_n)$ be a sequence in $\mathcal{M}_{d+1}(\alpha, \beta; \Omega_T)$ of the form*

$$
\mathbf{C}_n = \begin{bmatrix} (\mathbf{A}^n)^{-1} & \mathsf{0} \\ \mathsf{0}^\top & c_n \end{bmatrix} .
$$

Then $\mathbf{C}_n$ H-converges to some $\mathbf{C} \in \mathcal{M}_{d+1}(\alpha, \beta; \Omega_T)$ with respect to $\mathcal{L}_0$ and V if and only if the sequence $\mathbf{A}^n \in \mathcal{M}_d(\alpha, \beta; \Omega_T)$ parabolically H-converges to some $\mathbf{A}$ and c_n converge to some c in L^∞ weakly $*$. In that case the limit is given by

$$
\mathbf{C} = \begin{bmatrix} (\mathbf{A})^{-1} & \mathsf{0} \\ \mathsf{0}^\top & c \end{bmatrix} .
$$

∎

The proof is done in a similar manner as the proof of Theorem 11. Here, instead of Theorem 10, the parallel result for the heat equation is used [26, Theorem 3].

Equation with memory effects in homogenisation

In the first two examples we were dealing with the equations whose homogenised limit is the equation of the same type. Of course, this is not always the case, as a number of equations are known that under the process of homogenisation exhibit so called memory effects, where the limiting equation contains some nonlocal term. We shall present here the simplest model problem where such nonlocal effects occur [28]:

$$
\begin{cases} \partial_t u(x,t) + c(x) u(x,t) = f(x,t)\,, & (x,t) \in \Omega_T = \Omega \times \langle 0, T\rangle \\ u(x, 0) = 0\,, \quad x \in \Omega\,, \end{cases} \tag{32}
$$

represent it as Friedrichs system, and then try to apply our general homogenisation setting for Friedrichs systems. Since our setting does not detect memory effects, we expect to find that some of our assumptions are not satisfied. Here, for simplicity, we take $\Omega = \langle 0, 1\rangle$, and, as before, $T > 0$, $f \in \mathrm{L}^2(\Omega_T)$, $\beta \geqslant c \geqslant \alpha > 0$ a.e. on Ω.

Actually, this single equation is already in a form of Friedrichs system with our matrix-valued functions now being scalars: $\mathbf{A}_1 = 0, \mathbf{A}_2 = 1, \mathbf{C} = c$. The graph space is given by

$$\begin{aligned} W &= \{u \in \mathrm{L}^2(\Omega_T) : \partial_t u \in \mathrm{L}^2(\Omega_T)\} \\ &= \{u \in \mathrm{L}^2(0, T; \mathrm{L}^2(\Omega)) : \partial_t u \in \mathrm{L}^2(0, T; \mathrm{L}^2(\Omega))\}\,, \end{aligned}$$

and it is continuously imbedded in $\mathrm{C}(0, T; \mathrm{L}^2(\Omega))$ [21]. As $\mathbf{A}_{\boldsymbol{\nu}} = \nu_2$ the boundary operator D is given by

$$(33) \qquad {}_{W'}\langle\, Du, v\,\rangle_W = \int_0^1 u(\cdot, T)v(\cdot, T)\, dx - \int_0^1 u(\cdot, 0)v(\cdot, 0)\, dx\,,$$

for smooth functions u and v, but similarly as in [6] it can be shown that this representation formula is valid for arbitrary $u, v \in W$. The natural choice of subspaces V and $\widetilde{V}$ that corresponds to the initial and boundary conditions in (32) is given by

$$\begin{aligned} V &= \{u \in W : u(\cdot, 0) = 0\}\,, \\ \widetilde{V} &= \{v \in W : v(\cdot, T) = 0\}\,, \end{aligned}$$

and one can easily verify that they satisfy conditions (V).

Let us now turn our attention to the homogenisation problem: if properties (K1) and (K2) hold then we would have compactness theorem and, similarly as in the previous two examples, the limiting equation would be of the same type. However, we know that this is not the case for this particular example, as shown in [28]. Therefore, some of our assumptions will not be satisfied: we shall prove that property (K1) is not valid for this Friedrichs system.

Let (c_n) be a sequence of bounded functions on $\langle 0, 1\rangle$ with $\beta \geqslant c_n \geqslant \alpha > 0$ a.e. on $\langle 0, 1\rangle$, and let ν be its Young measure. Take $f(x, t) \equiv 1$, so that the solution of

$$\begin{cases} \partial_t u_n(x, t) + c_n(x)u_n(x, t) = 1\,, & (x, t) \in \Omega_T = \langle 0, 1\rangle \times \langle 0, T\rangle \\ u(x, 0) = 0\,, \quad x \in \langle 0, 1\rangle \end{cases}$$

is given by the explicit formula $u_n(x, t) = F(c_n(x), t)$, where

$$F(y, t) = \frac{1}{y}\left(1 - \mathrm{e}^{-yt}\right)\,.$$

One can easily see that this sequence of the solutions converges weakly in $\mathrm{L}^2(\Omega_T)$ to

$$u(x, t) = \int_\alpha^\beta F(y, t)\, d\nu_x(y)\,.$$

From (33) it follows

$$\begin{aligned} {}_{W'}\langle\, Du_n, u_n\,\rangle_W &= \int_0^1 F^2(c_n(x), T)\, dx\,, \\ {}_{W'}\langle\, Du, u\,\rangle_W &= \int_0^1 \left(\int_\alpha^\beta F(y, T)\, d\nu_x(y)\right)^2 dx\,, \end{aligned}$$

and since

$$F^2(c_n(\cdot), T) \longrightarrow \int_\alpha^\beta F^2(y, T)\, d\nu_\cdot(y) \qquad \text{weakly } * \text{ in } \mathrm{L}^\infty(\langle 0, 1\rangle)\,,$$

after testing the last convergence on the constant function $1 \in \mathrm{L}^1(\langle 0,1\rangle)$, we conclude

$$ {}_{W'}\langle\, Du_n, u_n \,\rangle_W \longrightarrow \int_0^1 \int_\alpha^\beta F^2(y,T)\, d\nu_x(y)\, dx\,. $$

Therefore, if (K1) is valid, we have

$$ \int_0^1 \int_\alpha^\beta F^2(y,T)\, d\nu_x(y)\, dx = \int_0^1 \left(\int_\alpha^\beta F(y,T)\, d\nu_x(y) \right)^2 dx\,, $$

which, in case when the sequence (c_n) is of the form $c_n(x) = \tilde{c}(nx)$, for some periodic function $\tilde{c} : \mathbf{R} \longrightarrow \mathbf{R}$, and therefore has homogenous Young measure (independent of x), becomes

$$ \int_\alpha^\beta F^2(y,T)\, d\nu(y) = \left(\int_\alpha^\beta F(y,T)\, d\nu(y) \right)^2 . $$

As this equality is clearly not true for every homogenous Young measure, it follows that (K1) is not valid.

5. Concluding remarks

It is known that the abstract theory of Friedrichs systems can be applied to the stationary diffusion equation [18, 4, 10], and more recently to the heat equation [6]. In this paper we have developed the general theory of homogenisation of Friedrichs systems, and applied it to these two equations. In this way we rediscover some homogenisation results, such as the compactness theorem for *classical* H-convergence of the stationary diffusion and the heat equation.

There is also a number of open problems that arise from this work, the obvious one being the possible applications to other equations of interest (as Friedrichs systems can be applied to a wide variety of equations and systems [18, 4, 10, 20]). Furthermore, some properties of H-convergence for Friedrichs systems should be investigated, e.g. locality property, irrelevance of the boundary conditions, homogenisation of periodic or stratified materials, question of correctors etc.

It would also be of interest to develop the homogenisation theory for the two-field Friedrichs systems with partial coercivity [17]. This would enable the treatment of the stationary diffusion equation and the heat equation with zero non-derivative term ($c = 0$).

References

[1] Gregoire Allaire: *Shape optimization by the homogenization method*, Springer-Verlag, 2002.
[2] Nenad Antonić, Krešimir Burazin: *Graph spaces of first-order linear partial differential operators*, *Math. Communications* **14**(1) (2009) 135–155.
[3] Nenad Antonić, Krešimir Burazin: *Intrinsic boundary conditions for Friedrichs systems*, *Comm. Partial Diff. Eq.* **35** (2010) 1690–1715.
[4] Nenad Antonić, Krešimir Burazin: *Boundary operator from matrix field formulation of boundary conditions for Friedrichs systems*, *J. Diff. Eq.* **250** (2011) 3630–3651.
[5] Nenad Antonić, Krešimir Burazin, Marko Vrdoljak: *Second-order equations as Friedrichs systems*, Nonlinear Analysis: RWA, doi: 10.1016/j.nonrwa.2011.08.031
[6] Nenad Antonić, Krešimir Burazin, Marko Vrdoljak: *Heat equation as a Friedrichs system*, submitted
[7] Nenad Antonić, Marko Vrdoljak: *Parabolic H-convergence and small-amplitude homogenization*, *Applicable Analysis* **88** (2009) 1493–1508.
[8] Alain Bensoussan, Jacques-Lous Lions, Georgios Papanicolaou *Asymptotic Analysis in Periodic Structures*, NorthHolland, Amsterdam, 1978.
[9] Krešimir Burazin: *Contributions to the theory of Friedrichs' and hyperbolic systems* (in Croatian), Ph.D. thesis, University of Zagreb, 2008, `http://www.mathos.hr/~kburazin/papers/teza.pdf`

[10] Tan Bui-Thanh, Leszek Demkowicz, Omar Ghattas: *A unified discontinuous Petrov-Galerkin method and its analysis for Friedrichs' systems*, ICES REPORT 11-34, The Institute for Computational Engineering and Sciences, The University of Texas at Austin, November 2011, http://www.ices.utexas.edu/media/reports/2011/1134.pdf

[11] Andrea Dall'Aglio, Francois Murat: *A corrector result for H-converging parabolic problems with time-dependent coefficients*, *Annali della Scuola Normale Superiore di Pisa* **25** (1997) 329–373.

[12] Daniele Antonio Di Pietro, Alexandre Ern: *Mathematical aspects of discontinuous Galerkin methods*, Springer, 2012.

[13] Robert Dautray, Jacques-Louis Lions: *Mathematical analysis and numerical methods for science and technology, Vol.II*, Springer, 1992.

[14] Alexandre Ern, Jean-Luc Guermond: *Theory and practice of finite elements*, Springer, 2004.

[15] Alexandre Ern, Jean-Luc Guermond: *Discontinuous Galerkin methods for Friedrichs' systems. I. General theory, SIAM J. Numer. Anal.* **44** (2006) 753–778.

[16] Alexandre Ern, Jean-Luc Guermond: *Discontinuous Galerkin methods for Friedrichs' systems. II. Second-order elliptic PDEs, SIAM J. Numer. Anal.* **44** (2006) 2363-2388.

[17] Alexandre Ern, Jean-Luc Guermond: *Discontinuous Galerkin methods for Friedrichs' systems. III. Multifield theories with partial coercivity, SIAM J. Numer. Anal.* **46** (2008) 776–804.

[18] Alexandre Ern, Jean-Luc Guermond, Gilbert Caplain: *An intrinsic criterion for the bijectivity of Hilbert operators related to Friedrichs' systems, Comm. Partial Diff. Eq.* **32** (2007) 317–341.

[19] Kurt O. Friedrichs: *Symmetric positive linear differential equations, Comm. Pure Appl. Math.* **11** (1958) 333–418.

[20] Max Jensen: *Discontinuous Galerkin methods for Friedrichs systems with irregular solutions*, Ph.D. thesis, University of Oxford, 2004, http://maths.dur.ac.uk/~dma0mpj/thesisjensen.pdf

[21] Jacques-Lous Lions, Enrico Magenes: *Non-homogeneous boundary value problems and applications, Vol. 1*, Springer, 1972.

[22] François Murat, Luc Tartar: *H-convergence*, Séminaire d'Analyse Fonctionnelle et Numérique de l'Université d'Alger, mimeographed notes, 1978. English translation in *Topics in the mathematical modelling of composite materials*, A. Cherkaev, R. Kohn (eds.), 21–43, Birkhäuser, 1997.

[23] Jeffrey Rauch: *Symmetric positive systems with boundary characteristic of constant multiplicity, Trans. Amer. Math. Soc.* **291** (1985) 167–187.

[24] Sergio Spagnolo: *Sul limite delle soluzioni di problemi di Cauchy relativi all'equazione del calore, Ann. Scuola Norm. Sup. Pisa* **21** (1967) 657–699.

[25] Sergio Spagnolo: *Sulla convergenza di soluzioni di equazioni paraboliche ed ellittiche, Ann. Scuola Norm. Sup. Pisa* **22** (1968) 571–597.

[26] Sergio Spagnolo: *Convergence of parabolic equations, Bolletino della Unione Matematica Italiana* **14**–B (1977) 547–568.

[27] Luc Tartar: *Problèmes de contrôle des coefficients dans des équations aux dérivées partielles*, Control Theory, Numerical Methods and Computer Systems Modelling, Lecture notes in Economy and Mathematical Systems **107**, 420-426, Springer Verlag, 1975.

[28] Luc Tartar: *Nonlocal effects induced by homogenisation*, Partial Differential Equations and the Calculus of Variation, Essays in Honor of Ennio de Giorgi, II, 925–938, Birkhäuser, Boston, 1989.

[29] Luc Tartar: *The general theory of homogenization: A personalized introduction*, Springer, 2009.

[30] Marcus Waurick: *On the theory of homogenization of evolutionary equations in Hilbert spaces*, arXiv:1204.3768v2 [math.AP].

[31] Vasili V. Žikov, Sergei M. Kozlov, Ol'ga A. Oleinik, Ha Tien Ngoan: *Homogenization and G-convergence of differential operators, Uspehi Mat. Nauk* **34** (1979) 65-133.

[32] Vasili V. Žikov, Sergei M. Kozlov, Ol'ga A. Oleinik: *On G-convergence of parabolic operators, Russ. Math. Surv.* **36**:1 (1981) 9–60.