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Boundary operator from matrix field formulation of boundary conditions for Friedrichs systems

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Abstract

Following the recent progress in understanding the abstract setting for Friedrichs symmetric positive systems by Ern, Guermond and Caplain (2007), as well as Antonić and Burazin (2010), an attempt is made to relate these results to the classical Friedrichs theory.

A comparison of two approaches, via the trace operator and the boundary operator, has been made, favouring the latter. Finally, a particular set of sufficient conditions for a boundary matrix field to define a boundary operator in that case is given, and the applicability of this procedure in realistic situations is shown by two examples.

Keywords: symmetric positive system, first-order system of partial differential equations, boundary condition, boundary operator

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1. Introduction

Several years after his successful treatment of symmetric hyperbolic systems, Friedrichs (1958) introduced a class of boundary value problems, named positive symmetric systems, encompassing also a variety of elliptic and parabolic problems. Even more, he provided a framework for a successful treatment of some equations of mixed type, like the Tricomi equation.

Such a diversity of equations treated in a unified framework requires the inclusion of various boundary conditions. Friedrichs introduced a clever technique to describe them, by using adequate boundary matrix fields. A number of open questions raised was investigated by his students [LP, M1, M2, PS], but many remained a challenge even today.

To be specific, let $d, r \in \mathbf{N}$ and let $\Omega \subseteq \mathbf{R}^d$ be an open and bounded set with Lipschitz boundary Γ (its closure we shall denote by $\mathsf{Cl}\,\Omega = \Omega \cup \Gamma$). If real (for simplicity we do not consider the complex case here, which could also be treated as in [FL]) matrix functions $\mathbf{A}_k \in \mathrm{W}^{1,\infty}(\Omega; \mathrm{M}_r(\mathbf{R}))$, $k \in 1..d$, and $\mathbf{C} \in \mathrm{L}^\infty(\Omega; \mathrm{M}_r(\mathbf{R}))$ satisfy:

(F1) $$\mathbf{A}_k \text{ is symmetric:} \quad \mathbf{A}_k = \mathbf{A}_k^\top\,,$$

(F2) $$(\exists\, \mu_0 > 0) \quad \mathbf{C} + \mathbf{C}^\top + \sum_{k=1}^d \partial_k \mathbf{A}_k \geqslant 2\mu_0 \mathbf{I} \qquad (\text{a.e. on } \Omega)\,,$$

then the first-order differential operator $\mathcal{L} : \mathrm{L}^2(\Omega; \mathbf{R}^r) \longrightarrow \mathcal{D}'(\Omega; \mathbf{R}^r)$ defined by

$$\mathcal{L}\mathsf{u} := \sum_{k=1}^d \partial_k(\mathbf{A}_k \mathsf{u}) + \mathbf{C}\mathsf{u}$$

is called *the Friedrichs operator* or *the symmetric positive operator*, while (for given $\mathsf{f} \in \mathrm{L}^2(\Omega; \mathbf{R}^r)$) the first-order system of partial differential equations $\mathcal{L}\mathsf{u} = \mathsf{f}$ is called *the Friedrichs system* or *the symmetric positive system*.

In describing the boundary conditions, following Friedrichs [F] we first define

$$\mathbf{A}_{\boldsymbol{\nu}} := \sum_{k=1}^d \nu_k \mathbf{A}_k\,,$$

where $\boldsymbol{\nu} = (\nu_1, \nu_2, \cdots, \nu_d)$ is the outward unit normal on Γ, which is, as well as $\mathbf{A}_{\boldsymbol{\nu}}$ of class L^∞ on Γ. For a given matrix field on the boundary $\mathbf{M} : \Gamma \longrightarrow \mathrm{M}_r(\mathbf{R})$, the boundary condition is prescribed by

$$(\mathbf{A}_{\boldsymbol{\nu}} - \mathbf{M})\mathsf{u}_{|_\Gamma} = \mathsf{0}\,,$$

and by varying $\mathbf{M}$ one can enforce different boundary conditions. Friedrichs required the following two conditions (for a.e. $\mathbf{x} \in \Gamma$) to hold:

(FM1) $$(\forall\, \boldsymbol{\xi} \in \mathbf{R}^r) \quad \mathbf{M}(\mathbf{x})\boldsymbol{\xi} \cdot \boldsymbol{\xi} \geqslant 0\,,$$

(FM2) $$\mathbf{R}^r = \ker\Big(\mathbf{A}_{\boldsymbol{\nu}}(\mathbf{x}) - \mathbf{M}(\mathbf{x})\Big) + \ker\Big(\mathbf{A}_{\boldsymbol{\nu}}(\mathbf{x}) + \mathbf{M}(\mathbf{x})\Big)\,;$$

and such $\mathbf{M}$ he called *an admissible boundary condition*.

The boundary value problem thus reads: for given $\mathsf{f} \in \mathrm{L}^2(\Omega; \mathbf{R}^r)$ find u such that

(1) $$\begin{cases} \mathcal{L}\mathsf{u} = \mathsf{f} \\ (\mathbf{A}_{\boldsymbol{\nu}} - \mathbf{M})\mathsf{u}_{|_\Gamma} = \mathsf{0} \end{cases}.$$

Of course, under such weak assumptions the existence of a classical solution (C^1 or $\mathrm{W}^{1,\infty}$) cannot be expected. It can be shown that, in general, the solution belongs only to the graph space of operator $\mathcal{L}$:

$$W = \Big\{ \mathsf{u} \in \mathrm{L}^2(\Omega;\mathbf{R}^r) : \mathcal{L}\mathsf{u} \in \mathrm{L}^2(\Omega;\mathbf{R}^r) \Big\} .$$

W is a separable Hilbert space (see e.g. [AB1]) with the inner product

$$\langle\, \mathsf{u} \mid \mathsf{v} \,\rangle_{\mathcal{L}} := \langle\, \mathsf{u} \mid \mathsf{v} \,\rangle_{\mathrm{L}^2(\Omega;\mathbf{R}^r)} + \langle\, \mathcal{L}\mathsf{u} \mid \mathcal{L}\mathsf{v} \,\rangle_{\mathrm{L}^2(\Omega;\mathbf{R}^r)} ,$$

in which the restrictions of functions from $\mathrm{C}^\infty_c(\mathbf{R}^d;\mathbf{R}^r)$ to Ω are dense. The corresponding norm will be denoted by

$$\|\mathsf{u}\|_{\mathcal{L}} = \sqrt{\|\mathsf{u}\|^2_{\mathrm{L}^2(\Omega;\mathbf{R}^r)} + \|\mathcal{L}\mathsf{u}\|^2_{\mathrm{L}^2(\Omega;\mathbf{R}^r)}} .$$

However, with such a weak notion of solution in a quite large space, the question arises how to interpret the boundary condition. It is not a priori clear what would be the meaning of $\mathsf{u}_{|_\Gamma}$ for functions u from the graph space. Recently (cf. [AB1, J]) it has been shown that $\mathsf{u}_{|_\Gamma}$ can be interpreted as an element of $\mathrm{H}^{-\frac{1}{2}}(\Gamma;\mathbf{R}^r)$, and the appropriate well-posedness results for the weak formulation of (1), under additional assumptions, have been proven [Ra, J].

More recently the Friedrichs theory has been rewritten in an abstract setting by Ern, Guermond and Caplain [EG, EGC], in terms of operators acting on Hilbert spaces, such that the traces on the boundary have not been explicitly used. Instead, the boundary conditions have been represented in an intrinsic way. In fact, the trace operator has been replaced by the *boundary operator* $D \in \mathcal{L}(W;W')$ defined by

$$_{W'}\langle\, D\mathsf{u}, \mathsf{v} \,\rangle_W := \langle\, \mathcal{L}\mathsf{u} \mid \mathsf{v} \,\rangle_{\mathrm{L}^2(\Omega;\mathbf{R}^r)} - \langle\, \mathsf{u} \mid \tilde{\mathcal{L}}\mathsf{v} \,\rangle_{\mathrm{L}^2(\Omega;\mathbf{R}^r)} , \qquad \mathsf{u},\mathsf{v} \in W ,$$

where $\tilde{\mathcal{L}} : \mathrm{L}^2(\Omega;\mathbf{R}^r) \longrightarrow \mathcal{D}'(\Omega;\mathbf{R}^r)$, the formally adjoint operator to $\mathcal{L}$, is defined by:

$$\tilde{\mathcal{L}}\mathsf{v} := -\sum_{k=1}^d \partial_k(\mathbf{A}_k^\top \mathsf{v}) + \Big(\mathbf{C}^\top + \sum_{k=1}^d \partial_k \mathbf{A}_k^\top\Big)\mathsf{v} .$$

Furthermore, it has been shown that operator D has got better properties than the trace operator.

Lemma 1. *Denote by W_0 the closure of the space $\mathrm{C}^\infty_c(\Omega;\mathbf{R}^r)$ in W. Then the kernel and image of operator D are given by*

$$\ker D = W_0 \qquad \textit{and} \qquad \mathrm{im}\, D = W_0^0 := \{ g \in W' : (\forall\, \mathsf{u} \in W_0) \quad {}_{W'}\langle\, g, \mathsf{u} \,\rangle_W = 0 \} .$$

In particular, $\mathrm{im}\, D$ *is closed in* W'. ∎

The fact that $\ker D = W_0$ clarifies the term *boundary operator* for D.

In [EGC] the following *weak well–posedness result* has been shown as well.

Theorem 1. *Let (F1)–(F2) be valid for matrix functions $\mathbf{A}_k \in \mathrm{W}^{1,\infty}(\Omega;\mathrm{M}_r(\mathbf{R}))$, $k \in 1..d$, and $\mathbf{C} \in \mathrm{L}^\infty(\Omega;\mathrm{M}_r(\mathbf{R}))$. Further assume that there exists an operator $M \in \mathcal{L}(W;W')$ satisfying*

(M1) $$(\forall\, \mathsf{u} \in W) \quad {}_{W'}\langle\, M\mathsf{u}, \mathsf{u} \,\rangle_W \geqslant 0 ,$$

and

(M2) $$W = \ker(D - M) + \ker(D + M) .$$

Then the restricted operators

$$\mathcal{L}_{|_{\ker(D-M)}} : \ker(D - M) \longrightarrow \mathrm{L}^2(\Omega;\mathbf{R}^r) \qquad \textit{and} \qquad \tilde{\mathcal{L}}_{|_{\ker(D+M^*)}} : \ker(D + M^*) \longrightarrow \mathrm{L}^2(\Omega;\mathbf{R}^r)$$

are isomorphisms. ∎

The operator M from the theorem is also called the boundary operator, as $\ker M = \ker D = W_0$. In the sequel we shall refer to both properties (M1) and (M2) as (M); similarly we shall use (F) and (FM).

In the abstract setting Ern, Guermond and Caplain [EGC] considered, besides (M), two additional forms of the boundary conditions and their mutual relationship, rising a number of open questions. In the papers [AB1, AB2, AB3] we closed the most important question by proving that those abstract conditions are, in fact, all equivalent. The new development was based on the fact that the theory can be expressed in terms of Kreĭn spaces (a particular kind of indefinite inner product spaces). This approach allowed us to simplify a number of earlier proofs as well.

The above simplification of abstract theory paved the way to new investigations of precise relationship between the classical Friedrichs theory and its abstract counterpart.

The analogy between the properties (M) for operator M and the conditions (FM) for matrix boundary condition $\mathbf{M}$ is apparent. A natural question to be investigated is the nature of the relationship between the matrix field $\mathbf{M}$ and the boundary operator M. More precisely, our goal is to find additional conditions on the matrix field $\mathbf{M}$ with properties (FM) which will guarantee the existence of a *suitable operator* $M \in \mathcal{L}(W; W')$ with properties (M).

For a given matrix field $\mathbf{M}$, which M will be a *suitable operator*? The condition is satisfied by such an operator M that the result of Theorem 1 really presents the *weak well–posedness result* for problem (1) in the following sense: if for given $\mathsf{f} \in \mathrm{L}^2(\Omega; \mathbf{R}^r)$, $\mathsf{u} \in \ker(D - M)$ is such that $\mathcal{L}\mathsf{u} = \mathsf{f}$, where we additionally have $\mathsf{u} \in \mathrm{C}^1(\Omega; \mathbf{R}^r) \cap \mathrm{C}(\mathrm{Cl}\,\Omega; \mathbf{R}^r)$, then u satisfies (1) in the classical sense.

In such a way established connection between $\mathbf{M}$ and the boundary operator M we take as a first step towards better understanding of the relation between the existence and uniqueness results for the Friedrichs systems as in [EGC, AB2] and the earlier *classical* results [F, J, Ra]. Our motivation stems from the need of better such results in order to apply H-measures [A, AL] to symmetric systems.

The paper is organised as follows: in the second section we discuss the definition of boundary operator M by the aid of boundary matrix field $\mathbf{M}$, showing by an example that (FM) is not sufficient to guarantee the boundedness of M. Theorem 2 in the following section provides a set of sufficient conditions. Next two sections are devoted to the investigation whether so defined M satisfies condition (M), by using two approaches, via the trace operator and via the boundary operator, respectively. The latter venue proves to be better, and it is shown that the assumptions of Theorem 2 are already sufficient for (M) to hold. Finally, in the last section we present three examples (related to the scalar elliptic equation, the Maxwell system in the diffusive régime, and AN ODE), demonstrating that the assumptions of Theorem 2 are applicable in some relevant situations.

2. Boundary operator defined by matrix field

Boundary operator D can be expressed [AB1, EGC] via matrix function $\mathbf{A}_{\boldsymbol{\nu}}$:

$$(2) \qquad (\forall\, \mathsf{u}, \mathsf{v} \in \mathrm{C}^\infty_c(\mathbf{R}^d; \mathbf{R}^r)) \qquad {}_{W'}\langle D\mathsf{u}, \mathsf{v} \rangle_W = \int_\Gamma \mathbf{A}_{\boldsymbol{\nu}}(\mathbf{x}) \mathsf{u}_{|_\Gamma}(\mathbf{x}) \cdot \mathsf{v}_{|_\Gamma}(\mathbf{x}) dS(\mathbf{x})\,.$$

In fact, the above can easily be extended to $\mathsf{u}, \mathsf{v} \in \mathrm{H}^1(\Omega; \mathbf{R}^r)$, providing that the restriction to Γ is replaced by the trace operator $\mathcal{T}_{\mathrm{H}^1} : \mathrm{H}^1(\Omega; \mathbf{R}^r) \longrightarrow \mathrm{H}^{\frac{1}{2}}(\Gamma; \mathbf{R}^r)$. Of course, for M we expect to be of the following form (see [EG])

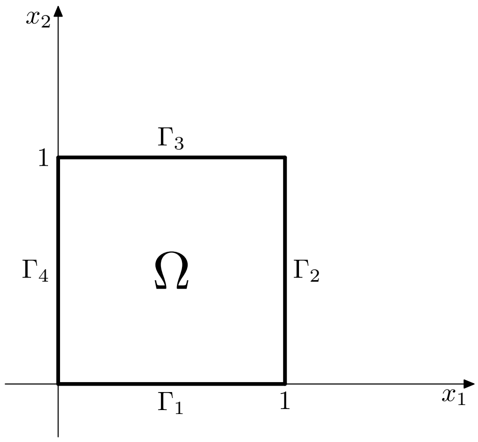
$$(3) \qquad (\forall\, \mathsf{u}, \mathsf{v} \in \mathrm{C}^\infty_c(\mathbf{R}^d; \mathbf{R}^r)) \qquad {}_{W'}\langle M\mathsf{u}, \mathsf{v} \rangle_W = \int_\Gamma \mathbf{M}(\mathbf{x}) \mathsf{u}_{|_\Gamma}(\mathbf{x}) \cdot \mathsf{v}_{|_\Gamma}(\mathbf{x}) dS(\mathbf{x})\,,$$

where we naturally assume that $\mathbf{M}$ is bounded, i.e. $\mathbf{M} \in \mathrm{L}^\infty(\Omega; \mathrm{M}_r(\mathbf{R}))$. For the above formula to define a unique bounded operator from $\mathcal{L}(W; W')$, it is necessary and sufficient that

$$(4) \qquad (\exists\, C > 0)(\forall\, \mathsf{u}, \mathsf{v} \in \mathrm{C}^\infty_c(\mathbf{R}^d; \mathbf{R}^r)) \qquad \Big| \int_\Gamma \mathbf{M}(\mathbf{x}) \mathsf{u}_{|_\Gamma}(\mathbf{x}) \cdot \mathsf{v}_{|_\Gamma}(\mathbf{x}) dS(\mathbf{x}) \Big| \leqslant C \|\mathsf{u}\|_{\mathcal{L}} \|\mathsf{v}\|_{\mathcal{L}}\,.$$

However, the properties (FM) do not guarantee that the preceding condition is satisfied, as it can be seen from the following example.

Example. By I denote the open unit interval $0 < x < 1$; let the open unit square $\Omega := I \times I \subseteq \mathbf{R}^2$ in the first quadrant be given, and let $\Gamma_1 := I \times \{0\}$, $\Gamma_2 := \{1\} \times I$, $\Gamma_3 := I \times \{1\}$ and $\Gamma_4 := \{0\} \times I$ denote its sides excluding the vertices.



Furthermore, let the operators $\mathcal{L}$ and $\tilde{\mathcal{L}}$ be defined by

$$\begin{aligned}\mathcal{L}\mathsf{u} &:= \partial_2(\mathbf{A}_2\mathsf{u}) + \mathbf{C}\mathsf{u}\,, \text{ and}\\ \tilde{\mathcal{L}}\mathsf{u} &:= -\partial_2(\mathbf{A}_2^\top\mathsf{u}) + (\mathbf{C}^\top + \partial_2\mathbf{A}_2^\top)\mathsf{u}\,,\end{aligned}$$

where

$$\begin{aligned}\mathbf{A}_2(x_1,x_2) &= -\frac{1}{2}\begin{bmatrix} e^{-\frac{2}{x_1}}(x_2-1) & -e^{-\frac{1}{x_1}}(x_2-1)\\ -e^{-\frac{1}{x_1}}(x_2-1) & 0\end{bmatrix} && \in \mathrm{W}^{1,\infty}(\Omega;\mathrm{M}_2(\mathbf{R}))\,,\\ \mathbf{C}(x_1,x_2) &= \frac{1}{4}\begin{bmatrix} e^{-\frac{2}{x_1}} + \varepsilon(x_1,x_2) & -e^{-\frac{1}{x_1}}\\ -e^{-\frac{1}{x_1}} & \varepsilon(x_1,x_2)\end{bmatrix} && \in \mathrm{L}^{\infty}(\Omega;\mathrm{M}_2(\mathbf{R}))\,,\end{aligned}$$

for some $\varepsilon \in \mathrm{L}^\infty(\Omega)$, such that $\varepsilon \geqslant 4\mu_0 > 0$ almost everywhere. The properties (F) can now be easily checked, so $\mathcal{L}$ is a Friedrichs operator.

If we define $\mathbf{M} \in \mathrm{L}^\infty(\Gamma;\mathrm{M}_2(\mathbf{R}))$ by the formula

$$\mathbf{M}(x_1,x_2) = \begin{cases} \frac{1}{2}\begin{bmatrix} e^{-\frac{2}{x_1}} & -e^{-\frac{1}{x_1}}\\ -e^{-\frac{1}{x_1}} & 2\end{bmatrix}, & \text{on } \Gamma_1\\ \mathbf{0}, & \text{on } \Gamma_2\cup\Gamma_3\cup\Gamma_4\end{cases},$$

the property (FM1) holds. As we have

$$\mathbf{A}_{\boldsymbol{\nu}}(x_1,x_2) = \begin{cases} -\frac{1}{2}\begin{bmatrix} e^{-\frac{2}{x_1}} & -e^{-\frac{1}{x_1}}\\ -e^{-\frac{1}{x_1}} & 0\end{bmatrix}, & \text{on } \Gamma_1\\ \mathbf{0}, & \text{on } \Gamma_2\cup\Gamma_3\cup\Gamma_4\end{cases},$$

on $\Gamma_2\cup\Gamma_3\cup\Gamma_4$ we get

$$\mathbf{A}_{\boldsymbol{\nu}} - \mathbf{M} = \mathbf{A}_{\boldsymbol{\nu}} + \mathbf{M} = \mathbf{0}\,,$$

therefore (FM2) is also clearly fulfilled here. Furthermore, on Γ_1 we have

$$\begin{aligned}\mathbf{A}_{\boldsymbol{\nu}} - \mathbf{M} &= -\begin{bmatrix} e^{-\frac{2}{x_1}} & -e^{-\frac{1}{x_1}}\\ -e^{-\frac{1}{x_1}} & 1\end{bmatrix},\\ \mathbf{A}_{\boldsymbol{\nu}} + \mathbf{M} &= -\begin{bmatrix} 0 & 0\\ 0 & -1\end{bmatrix},\end{aligned}$$

so it can easily be checked that for $(x_1, 0) \in \Gamma_1$

$$\ker(\mathbf{A}_{\boldsymbol{\nu}} - \mathbf{M})(x_1, 0) = \left\{(y_1, y_2)^\top \in \mathbf{R}^2 : y_2 = -e^{-\frac{1}{x_1}} y_1\right\},$$
$$\ker(\mathbf{A}_{\boldsymbol{\nu}} + \mathbf{M})(x_1, 0) = \left\{(y_1, y_2)^\top \in \mathbf{R}^2 : y_2 = 0\right\},$$

thus (FM2) is satisfied on Γ_1 as well.

Let us next show that the corresponding operator M is not continuous, i.e. that (4) does not hold. By choosing u and v such that $\mathsf{u} = \mathsf{v} = (0, u_2)^\top$, we get

$$\int_\Gamma \mathbf{M}(\mathbf{x})\mathsf{u}_{|_\Gamma}(\mathbf{x}) \cdot \mathsf{v}_{|_\Gamma}(\mathbf{x}) dS(\mathbf{x}) = \int_0^1 u_2(x_1, 0) dx_1 ,$$

and

$$\begin{aligned}
\|u\|^2_{\mathcal{L}} &= \int_\Omega u_2^2 + \frac{1}{16}\int_\Omega \Big(e^{-\frac{2}{x_1}} + \varepsilon(\mathbf{x})\Big) u_2^2(\mathbf{x}) d\mathbf{x} \\
&\quad + 4\int_\Omega e^{-\frac{2}{x_1}}(x_2-1)\partial_2 u_2(\mathbf{x})\Big(u_2(\mathbf{x}) + (x_2-1)\partial_2 u_2(\mathbf{x})\Big) d\mathbf{x} \\
&\leqslant C_1 \int_\Omega u_2^2 + \frac{1}{4}\int_\Omega e^{-\frac{2}{x_1}}(x_2-1)u_2(\mathbf{x})\partial_2 u_2(\mathbf{x}) d\mathbf{x} + \frac{1}{4}\int_\Omega e^{-\frac{2}{x_1}}(x_2-1)^2(\partial_2 u_2(\mathbf{x}))^2 d\mathbf{x},
\end{aligned}$$

for some $C_1 > 0$. The integrals appearing on the right hand side of the inequality we denote by I_1, I_2 and I_3, respectively. With a particular choice of $u_2(x_1, x_2) = (1-x_1)^m(1-x_2)^m$, $m \in \mathbf{N}$, we obtain

$$\int_\Gamma \mathbf{M}(\mathbf{x})\mathsf{u}_{|_\Gamma}(\mathbf{x}) \cdot \mathsf{u}_{|_\Gamma}(\mathbf{x}) dS(\mathbf{x}) = \frac{1}{2m+1},$$

while the above integrals take the following form:

$$\begin{aligned}
I_1 &= \frac{1}{(2m+1)^2}, \\
I_2 &= \frac{m}{2m+1}\int_0^1 e^{-\frac{2}{x_1}}(1-x_1)^{2m} dx_1 , \\
I_3 &= \frac{m^2}{2m+1}\int_0^1 e^{-\frac{2}{x_1}}(1-x_1)^{2m} dx_1 .
\end{aligned}$$

A simple calculation shows that for any $m \geqslant m_0$, for some $m_0 \in \mathbf{N}$, the integral appearing in I_2 and I_3 is bounded

$$\int_0^1 e^{-\frac{2}{x_1}}(1-x_1)^{2m} dx_1 \leqslant \frac{1}{m^3},$$

thus for some $C_2 > 0$

$$\|\mathsf{u}\|^2_{\mathcal{L}} \leqslant C_2 \frac{1}{2m+1}\Big(\frac{1}{2m+1} + \frac{1}{m} + \frac{1}{m^2}\Big) = C_2\Big(\frac{1}{2m+1} + \frac{1}{m} + \frac{1}{m^2}\Big)\int_\Gamma \mathbf{M}\mathsf{u} \cdot \mathsf{u}\, dS .$$

Therefore (4) is not valid and formula (3) does not define a bounded mapping from W to W'. ∎

3. Continuity of the boundary operator

In order to determine some additional conditions which will guarantee the continuity, we shall use the following characterisation of properties (FM) (cf. [F, J, B]). Let us first note that by a *pair of projections* we mean any two matrices $\mathbf{P}_1, \mathbf{P}_2 \in \mathrm{M}_r(\mathbf{R})$ satisfying

$$\mathbf{P}_1 + \mathbf{P}_2 = \mathbf{I} \qquad \text{and} \qquad \mathbf{P}_1\mathbf{P}_2 = \mathbf{P}_2\mathbf{P}_1 = \mathbf{0}.$$

Lemma 2. *Let matrix function* $\mathbf{M}$ *satisfy (FM1). Then the following statements are equivalent:*
a) $\mathbf{M}$ *satisfies (FM2);*
b) For almost every $\mathbf{x} \in \Gamma$ *there is a pair of projections* $\mathbf{P}_+(\mathbf{x})$*,* $\mathbf{P}_-(\mathbf{x})$*, such that*

$$(\mathbf{A}_{\boldsymbol{\nu}} + \mathbf{M})(\mathbf{x}) = 2\mathbf{A}_{\boldsymbol{\nu}}(\mathbf{x})\mathbf{P}_+(\mathbf{x}) \qquad \textit{and} \qquad (\mathbf{A}_{\boldsymbol{\nu}} - \mathbf{M})(\mathbf{x}) = 2\mathbf{A}_{\boldsymbol{\nu}}(\mathbf{x})\mathbf{P}_-(\mathbf{x})\,;$$

c) For almost every $\mathbf{x} \in \Gamma$ *there is a pair of projections* $\mathbf{S}_+(\mathbf{x})$*,* $\mathbf{S}_-(\mathbf{x})$*, such that*

$$(\mathbf{A}_{\boldsymbol{\nu}} + \mathbf{M})(\mathbf{x}) = 2\mathbf{S}_+^\top(\mathbf{x})\mathbf{A}_{\boldsymbol{\nu}}(\mathbf{x}) \qquad \textit{and} \qquad (\mathbf{A}_{\boldsymbol{\nu}} - \mathbf{M})(\mathbf{x}) = 2\mathbf{S}_-^\top(\mathbf{x})\mathbf{A}_{\boldsymbol{\nu}}(\mathbf{x})\,.$$

∎

For the boundedness of operator M defined by (3), we shall use the fact that $\mathbf{A}_{\boldsymbol{\nu}}$ by formula (2) defines a continuous operator D, and a representation of field $\mathbf{M}$ by $\mathbf{A}_{\boldsymbol{\nu}}$, which follows from the previous lemma. In the sequel, by $\mathcal{T}_{\mathrm{H}^1}$ we denote a surjective and continuous trace operator $\mathcal{T}_{\mathrm{H}^1} : \mathrm{H}^1(\Omega;\mathbf{R}^r) \longrightarrow \mathrm{H}^{\frac{1}{2}}(\Gamma;\mathbf{R}^r)$.

Theorem 2. *Let the matrix field* $\mathbf{M} \in \mathrm{L}^\infty(\Gamma;\mathrm{M}_r(\mathbf{R}))$ *satisfy (FM), and let* $\mathbf{S}_-$ *be as in Lemma 2. Additionally assume that* $\mathbf{S}_-$ *can be extended to a measurable matrix function* $\mathbf{S}_{-,p} : \mathsf{Cl}\,\Omega \longrightarrow \mathrm{M}_r(\mathbf{R})$ *satisfying*
(S1) The multiplication operator $\mathcal{S}_{-,p}$ *defined by* $\mathcal{S}_{-,p}(\mathsf{v}) := \mathbf{S}_{-,p}\mathsf{v}$ *for* $\mathsf{v} \in W$ *is in* $\mathcal{L}(W)$*.*
(S2) $(\forall \mathsf{v} \in \mathrm{H}^1(\Omega;\mathbf{R}^r)) \qquad \mathbf{S}_{-,p}\mathsf{v} \in \mathrm{H}^1(\Omega;\mathbf{R}^r) \quad \& \quad \mathcal{T}_{\mathrm{H}^1}(\mathbf{S}_{-,p}\mathsf{v}) = \mathbf{S}_-\mathcal{T}_{\mathrm{H}^1}\mathsf{v}$.
Then formula (3) defines a bounded operator $M \in \mathcal{L}(W;W')$*.*

Dem. From the second equality in Lemma 2(c) we get

$$\mathbf{M}(\mathbf{x}) = (\mathbf{I} - 2\mathbf{S}_-^\top(\mathbf{x}))\mathbf{A}_{\boldsymbol{\nu}}(\mathbf{x}) \qquad (\text{a.e. } \mathbf{x} \in \Gamma)\,,$$

so after multiplying by $\mathsf{u},\mathsf{v} \in \mathrm{C}_c^\infty(\mathbf{R}^d;\mathbf{R}^r)$ and integrating over Γ

$$\int_\Gamma \mathbf{M}\mathsf{u}_{|_\Gamma} \cdot \mathsf{v}_{|_\Gamma} dS = \int_\Gamma (\mathbf{I} - 2\mathbf{S}_-^\top)\mathbf{A}_{\boldsymbol{\nu}}\mathsf{u}_{|_\Gamma} \cdot \mathsf{v}_{|_\Gamma} dS = \int_\Gamma \mathbf{A}_{\boldsymbol{\nu}}\mathsf{u}_{|_\Gamma} \cdot (\mathbf{I} - 2\mathbf{S}_-)\mathsf{v}_{|_\Gamma} dS\,.$$

By (S2) it follows $(\mathbf{I} - 2\mathbf{S}_{-,p})\mathsf{v} \in \mathrm{H}^1(\Omega;\mathbf{R}^r)$ and $\mathcal{T}_{\mathrm{H}^1}\Big((\mathbf{I} - 2\mathbf{S}_{-,p})\mathsf{v}\Big) = (\mathbf{I} - 2\mathbf{S}_-)\mathsf{v}_{|_\Gamma}$, so from (2) we can conclude that

$$\begin{aligned} \int_\Gamma \mathbf{M}\mathsf{u}_{|_\Gamma} \cdot \mathsf{v}_{|_\Gamma} dS &= {}_{W'}\langle\, D\mathsf{u}, (\mathbf{I} - 2\mathbf{S}_{-,p})\mathsf{v}\,\rangle_W \\ &= {}_{W'}\langle\, D\mathsf{u}, (\mathcal{I}_W - 2\mathcal{S}_{-,p})\mathsf{v}\,\rangle_W\,, \end{aligned} \tag{5}$$

where $\mathcal{I}_W$ denotes the identity on W. Since all the operators appearing on the right hand side of the above equality are continuous, we conclude that

$$\Big|\int_\Gamma \mathbf{M}\mathsf{u}_{|_\Gamma} \cdot \mathsf{v}_{|_\Gamma} dS\Big| \leqslant \|D\|_{\mathcal{L}(W;W')} \cdot \|\mathcal{I}_W - 2\mathcal{S}_{-,p}\|_{\mathcal{L}(W)} \cdot \|\mathsf{u}\|_W \cdot \|\mathsf{v}\|_W\,,$$

and therefore M defined by (3) belongs to $\mathcal{L}(W;W')$.

Q.E.D.

Remark. Note that, under the assumptions of the above theorem, the operator M can be expressed by the operators D and $\mathcal{S}_{-,p}$. Indeed, if by $\mathcal{S}_{-,p}^* \in \mathcal{L}(W')$ we denote the adjoint operator to $\mathcal{S}_{-,p}$ defined (in the sense of Banach spaces) by

$${}_{W'}\langle\, \mathcal{S}_{-,p}^*\mathsf{g}, \mathsf{u}\,\rangle_W = {}_{W'}\langle\, \mathsf{g}, \mathcal{S}_{-,p}\mathsf{u}\,\rangle_W\,, \qquad \mathsf{g} \in W'\,, \quad \mathsf{u} \in W\,,$$

then from (5) it follows that

$$M = (\mathcal{I}_{W'} - 2\mathcal{S}_{-,p}^*)D = D - 2\mathcal{S}_{-,p}^*D\,, \tag{6}$$

where $\mathcal{I}_{W'}$ is the identity on W'.

∎

Remark. If $\mathbf{S}_-$ satisfies the assumptions of Theorem 2, then the same assumptions will be satisfied also by $\mathbf{S}_+ = \mathbf{I} - \mathbf{S}_-$. Indeed, if $\mathbf{S}_{-,p}$ is a measurable extension of the matrix function $\mathbf{S}_-$ to $\mathsf{Cl}\,\Omega$ satisfying (S1)–(S2), then by

$$\mathbf{S}_{+,p}(\mathbf{x}) := \mathbf{I} - \mathbf{S}_{-,p}(\mathbf{x})\,, \quad \mathbf{x} \in \mathsf{Cl}\,\Omega$$

a measurable extension of function $\mathbf{S}_+$ is given, for which it can easily be checked that satisfies analogous conditions to (S1)–(S2). ∎

At this point, it is natural to look for some sufficient conditions on $\mathbf{S}_-$, so that the assumptions of Theorem 2 will be fulfilled. In that direction the following result looks promising.

Lemma 3. *If $f : \Omega \longrightarrow \mathbf{R}$ is a Lipschitz function, then the multiplication $\mathsf{u} \mapsto f\mathsf{u}$ is a continuous linear operator on W.*

Dem. As $\mathrm{H}^1(\Omega;\mathbf{R}^r)$ is dense in W, it is enough to show that there is a constant $C > 0$, such that

$$(\forall\, \mathsf{u} \in \mathrm{H}^1(\Omega;\mathbf{R}^r)) \quad \|f\mathsf{u}\|_{\mathcal{L}} \leqslant C\|\mathsf{u}\|_{\mathcal{L}}\,.$$

For such u it can easily be seen that

$$\|f\mathsf{u}\|_{\mathrm{L}^2(\Omega;\mathbf{R}^r)} \leqslant \|f\|_{\mathrm{L}^\infty(\Omega)}\|\mathsf{u}\|_{\mathrm{L}^2(\Omega;\mathbf{R}^r)}\,,$$

so if we denote

$$A := \max_{k\in 1..d} \|\mathbf{A}_k\|_{\mathrm{L}^\infty(\Omega;\mathrm{M}_r(\mathbf{R}))}\,,$$

by the Leibniz formula for the derivative of product we get

$$\begin{aligned}\|\mathcal{L}(f\mathsf{u})\|_{\mathrm{L}^2(\Omega;\mathbf{R}^r)} &= \Big\|\sum_{k=1}^d (\partial_k f)\mathbf{A}_k\mathsf{u} + f\sum_{k=1}^d \partial_k(\mathbf{A}_k\mathsf{u}) + f\mathbf{C}\mathsf{u}\Big\|_{\mathrm{L}^2(\Omega;\mathbf{R}^r)}\\ &\leqslant C_1 A\|\nabla f\|_{\mathrm{L}^\infty(\Omega;\mathbf{R}^d)}\|\mathsf{u}\|_{\mathrm{L}^2(\Omega;\mathbf{R}^r)} + \|f\|_{\mathrm{L}^\infty(\Omega)}\|\mathcal{L}\mathsf{u}\|_{\mathrm{L}^2(\Omega;\mathbf{R}^r)}\\ &\leqslant C_2\|f\|_{\mathrm{W}^{1,\infty}(\Omega)}\|\mathsf{u}\|_{\mathcal{L}}\,,\end{aligned}$$

for some positive constants C_1 and C_2 which do not depend on u. Now we easily get

$$\|f\mathsf{u}\|_{\mathcal{L}} = \sqrt{\|f\mathsf{u}\|^2_{\mathrm{L}^2(\Omega;\mathbf{R}^r)} + \|\mathcal{L}(f\mathsf{u})\|^2_{\mathrm{L}^2(\Omega;\mathbf{R}^r)}} \leqslant C_3\|f\|_{\mathrm{W}^{1,\infty}(\Omega)}\|\mathsf{u}\|_{\mathcal{L}}\,,$$

for some constant $C_3 > 0$, thus obtaining the claim.

Q.E.D.

However, even though the multiplication by a scalar Lipschitz function is continuous on the graph space, the matrix multiplication need not be, as it can be seen from the following example.

Example. Let $\Omega \subseteq \mathbf{R}^2$ be an open bounded set, while

$$\mathbf{A}_1 = \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}, \qquad \mathbf{A}_2 = \mathbf{0} \qquad \text{and} \qquad \mathbf{C} = \mathbf{I}\,,$$

so that the operator $\mathcal{L}$ defined by

$$\mathcal{L}\mathsf{u} := \partial_1(\mathbf{A}_1\mathsf{u}) + \partial_2(\mathbf{A}_2\mathsf{u}) + \mathbf{C}\mathsf{u} = \begin{bmatrix} \partial_1 u_1 - \partial_1 u_2 + u_1 \\ -\partial_1 u_1 + \partial_1 u_2 + u_2 \end{bmatrix}, \qquad \text{for} \quad \mathsf{u} = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix},$$

is a Friedrichs operator (i.e. the conditions (F) hold).

If we take

$$\mathbf{S}_- \equiv \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix},$$

then

$$\mathbf{S}_{-,p} \equiv \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$$

is a natural Lipschitz extension of function $\mathbf{S}_-$ on $\mathsf{Cl}\,\Omega$, and we also have $\mathbf{S}_{-,p}^2 = \mathbf{S}_{-,p}$. Therefore

$$\mathcal{L}(\mathbf{S}_{-,p}\mathsf{u}) = \begin{bmatrix} \partial_1 u_1 + u_1 \\ -\partial_1 u_1 + u_2 \end{bmatrix}, \qquad \text{for} \quad \mathsf{u} = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}.$$

If we take $u_1 \in \mathrm{L}^2(\Omega)$ such that $\partial_1 u_1 \notin \mathrm{L}^2(\Omega)$ and $\mathsf{u} = (u_1, u_1)^\top$, then we have $\mathcal{L}\mathsf{u} = \mathsf{u} \in \mathrm{L}^2(\Omega; \mathbf{R}^2)$, and therefore $\mathsf{u} \in W$, while

$$\mathcal{L}(\mathbf{S}_{-,p}\mathsf{u}) = \begin{bmatrix} \partial_1 u_1 + u_1 \\ -\partial_1 u_1 + u_1 \end{bmatrix} \notin \mathrm{L}^2(\Omega; \mathbf{R}^2),$$

thus $\mathbf{S}_{-,p}\mathsf{u} \notin W$. The multiplication by a Lipschitz matrix function does not have to map the graph space into the graph space, so we cannot speak of continuity. Therefore the *smoothness* of the multiplying function does not guarantee the continuity in $W \longrightarrow W$. ∎

Remark. In the above example we have

$$\mathbf{A}_{\boldsymbol{\nu}} = \begin{bmatrix} \nu_1 & -\nu_1 \\ -\nu_1 & \nu_1 \end{bmatrix},$$

where $\boldsymbol{\nu} = (\nu_1, \nu_2)^\top$ is the unit outer normal on Γ, so

$$\mathbf{M} = (\mathbf{I} - 2\mathbf{S}_-)\mathbf{A}_{\boldsymbol{\nu}} = \begin{bmatrix} -\nu_1 & \nu_1 \\ -\nu_1 & \nu_1 \end{bmatrix}.$$

Thus $\mathbf{M}$ does not satisfy (FM1). At this point it is not clear whether the Lipschitz property of $\mathbf{S}_{-,p}$ together with (FM1) guarantees the continuity of multiplication on W. ∎

Remark. If $\mathbf{S}_- : \Gamma \longrightarrow \mathrm{M}_r(\mathbf{R})$ is a Lipschitz function, then it can be extended to a Lipschitz map defined on all of $\mathbf{R}^d$, by the Kirzbraun theorem [Fe, 2.10.43]. ∎

Remark. The Lipschitz property of $\mathbf{S}_{-,p} : \mathsf{Cl}\,\Omega \longrightarrow \mathrm{M}_r(\mathbf{R})$ implies (S2). Indeed, it can easily be seen that $\mathsf{u} \mapsto \mathbf{S}_{-,p}\mathsf{u}$ is continuous on $\mathrm{H}^1(\Omega; \mathbf{R}^r)$, while for $\mathsf{v} \in \mathrm{C}_c^\infty(\mathbf{R}^d; \mathbf{R}^r)$ we have

$$\mathcal{T}_{\mathrm{H}^1}(\mathbf{S}_{-,p}\mathsf{v}) = (\mathbf{S}_{-,p}\mathsf{v})_{|\Gamma} = \mathbf{S}_{-,p|_\Gamma}\mathsf{v}_{|\Gamma} = \mathbf{S}_-\mathsf{v}_{|\Gamma} = \mathbf{S}_-\mathcal{T}_{\mathrm{H}^1}\mathsf{v}.$$

Now, from the density of $\mathrm{C}_c^\infty(\mathbf{R}^d; \mathbf{R}^r)$ in $\mathrm{H}^1(\Omega; \mathbf{R}^r)$, the continuity of the trace operator $\mathcal{T}_{\mathrm{H}^1} : \mathrm{H}^1(\Omega; \mathbf{R}^r) \longrightarrow \mathrm{H}^{\frac{1}{2}}(\Gamma; \mathbf{R}^r)$ and the continuity of $\mathsf{z} \mapsto \mathbf{S}_-\mathsf{z}$ on $\mathrm{H}^{\frac{1}{2}}(\Gamma; \mathbf{R}^r)$ (cf. Lemma 4 below) we can easily get (S2). ∎

4. Approach via the trace operator

Our next goal is to determine some sufficient conditions for (M) in Theorem 1 to hold. The first idea is to use the trace operator on the graph space [AB1, J], as well as the well known results on continuity of the multiplication by a sufficiently smooth function on Sobolev spaces, namely the following lemma [T, p. 205]:

Lemma 4. *If $\mathbf{P} \in \mathrm{C}^{0,\frac{1}{2}}(\Gamma; \mathrm{M}_r(\mathbf{R}))$ (i.e. it is Hölder continuous of order $1/2$), then $\mathsf{z} \mapsto \mathbf{P}\mathsf{z}$ is a continuous linear operator on $\mathrm{H}^{\frac{1}{2}}(\Gamma; \mathbf{R}^r)$.* ∎

By $\mathcal{P} \in \mathcal{L}(\mathrm{H}^{\frac{1}{2}}(\Gamma; \mathbf{R}^r))$ denote the bounded linear operator from previous lemma, i.e.

$$\mathcal{P}(\mathsf{z}) := \mathbf{P}\mathsf{z}\,, \qquad \mathsf{z} \in \mathrm{H}^{\frac{1}{2}}(\Gamma; \mathbf{R}^r)\,, \tag{7}$$

while by $\mathcal{P}^* \in \mathcal{L}(\mathrm{H}^{-\frac{1}{2}}(\Gamma; \mathbf{R}^r))$ its adjoint operator defined by

$$_{\mathrm{H}^{-\frac{1}{2}}}\langle \mathcal{P}^* T, \mathsf{z} \rangle_{\mathrm{H}^{\frac{1}{2}}} := {}_{\mathrm{H}^{-\frac{1}{2}}}\langle T, \mathcal{P}\mathsf{z} \rangle_{\mathrm{H}^{\frac{1}{2}}}\,, \qquad T \in \mathrm{H}^{-\frac{1}{2}}(\Gamma; \mathbf{R}^r)\,, \, \mathsf{z} \in \mathrm{H}^{\frac{1}{2}}(\Gamma; \mathbf{R}^r)\,.$$

Lemma 5. *Let $\mathbf{P}_1, \mathbf{P}_2 \in \mathrm{C}^{0,\frac{1}{2}}(\Gamma; \mathrm{M}_r(\mathbf{R}))$ be such that $\mathbf{P}_1(\mathbf{x})$ and $\mathbf{P}_2(\mathbf{x})$ form a pair of projections (a.e. $\mathbf{x} \in \Gamma$). By $\mathcal{P}_1, \mathcal{P}_2 \in \mathcal{L}(\mathrm{H}^{\frac{1}{2}}(\Gamma; \mathbf{R}^r))$ denote the operators corresponding to matrix fields $\mathbf{P}_1$ and $\mathbf{P}_2$ as above, while by $\mathcal{P}_1^*, \mathcal{P}_2^*$ their adjoint operators. Then it holds:*

a) The operator $\mathcal{P}_1 + \mathcal{P}_2$ is an identity, while $\mathcal{P}_1 \circ \mathcal{P}_2 = \mathcal{P}_2 \circ \mathcal{P}_1$ is a nil-operator on $\mathrm{H}^{\frac{1}{2}}(\Gamma; \mathbf{R}^r)$;

b) The operator $\mathcal{P}_1^* + \mathcal{P}_2^*$ is an identity, while $\mathcal{P}_1^* \circ \mathcal{P}_2^* = \mathcal{P}_2^* \circ \mathcal{P}_1^*$ is a nil-operator on $\mathrm{H}^{-\frac{1}{2}}(\Gamma; \mathbf{R}^r)$.

Dem. (a) is a direct consequence of the fact that for almost every $\mathbf{x} \in \Gamma$ it holds

$$\mathbf{P}_1(\mathbf{x}) + \mathbf{P}_2(\mathbf{x}) = \mathbf{I} \quad \text{and} \quad \mathbf{P}_1(\mathbf{x})\mathbf{P}_2(\mathbf{x}) = \mathbf{P}_2(\mathbf{x})\mathbf{P}_1(\mathbf{x}) = \mathbf{0}\,,$$

while (b) follows from (a) and the definition of adjoint operator.

Q.E.D.

Let us note at the beginning that (M1) holds whenever M is continuous: namely, from (3) and (FM1) it follows that

$$(\forall \mathsf{u} \in \mathrm{C}_c^\infty(\mathbf{R}^d; \mathbf{R}^r)) \qquad {}_{W'}\langle M\mathsf{u}, \mathsf{u} \rangle_W \geqslant 0\,,$$

while the density of $\mathrm{C}_c^\infty(\mathbf{R}^d; \mathbf{R}^r)$ in W, together with the continuity of M, implies the validity of the above also for any $\mathsf{u} \in W$.

We shall use the properties of the trace operator $\mathcal{T}$ on the graph space [AB1]. Namely, on the graph space we can define operator $\mathcal{T} : W \longrightarrow \mathrm{H}^{-\frac{1}{2}}(\Gamma; \mathbf{R}^r)$, which for $\mathsf{u}, \mathsf{v} \in \mathrm{H}^1(\Omega; \mathbf{C}^r)$ satisfies

$$\begin{aligned} {}_{\mathrm{H}^{-\frac{1}{2}}(\Gamma;\mathbf{R}^r)}\langle \mathcal{T}\mathsf{u}, \mathcal{T}_{\mathrm{H}^1}\mathsf{v} \rangle_{\mathrm{H}^{\frac{1}{2}}(\Gamma;\mathbf{R}^r)} &= \langle \mathcal{L}\mathsf{u} \mid \mathsf{v} \rangle_{\mathrm{L}^2(\Omega;\mathbf{C}^r)} - \langle \mathsf{u} \mid \tilde{\mathcal{L}}\mathsf{v} \rangle_{\mathrm{L}^2(\Omega;\mathbf{C}^r)} \\ &= \langle \mathbf{A}_{\boldsymbol{\nu}} \mathcal{T}_{\mathrm{H}^1}\mathsf{u} \mid \mathcal{T}_{\mathrm{H}^1}\mathsf{v} \rangle_{\mathrm{L}^2(\Gamma;\mathbf{C}^r)}\,. \end{aligned} \tag{8}$$

In general, it is not an operator onto $\mathrm{H}^{-\frac{1}{2}}(\Gamma; \mathbf{R}^r)$, but still has a right inverse $\mathcal{E} : \operatorname{im}\mathcal{T} \longrightarrow W_0^\perp < W$, which satisfies

$$\mathcal{T}\mathcal{E}\mathsf{g} = \mathsf{g}\,, \quad \mathsf{g} \in \operatorname{im}\mathcal{T}\,.$$

As $\operatorname{im}\mathcal{T}$ is not necessarily closed in $\mathrm{H}^{-\frac{1}{2}}(\Gamma; \mathbf{R}^r)$, so neither $\mathcal{E}$ is necessarily continuous.

Theorem 3. *Assume that the matrix field $\mathbf{M} \in \mathrm{L}^\infty(\Gamma; \mathrm{M}_r(\mathbf{R}))$ satisfies (FM), and that by (3) is defined an operator $M \in \mathcal{L}(W; W')$. Then (M1) holds.*

Let the matrix function $\mathbf{S}_-$ from Lemma 2 additionally satisfies $\mathbf{S}_- \in \mathrm{C}^{0,\frac{1}{2}}(\Gamma; \mathrm{M}_r(\mathbf{R}))$. If by $\mathcal{S}_- \in \mathcal{L}(\mathrm{H}^{\frac{1}{2}}(\Gamma; \mathbf{R}^r))$ we denote the operator associated to the matrix field $\mathbf{S}_-$ as in (7), while by $\mathcal{S}_-^*$ we denote its adjoint operator, and by $\mathcal{T} : W \longrightarrow \mathrm{H}^{-\frac{1}{2}}(\Gamma; \mathbf{R}^r)$ the trace operator, then the condition $\mathcal{S}_-^*(\operatorname{im}\mathcal{T}) \subseteq \operatorname{im}\mathcal{T}$ implies (M2).

Dem. It only remains to show (M2). To this end it will be useful to express operators D and M through the trace operator $\mathcal{T}$. From the definition of D and (8) it follows that for $\mathsf{u}, \mathsf{v} \in \mathrm{H}^1(\Omega; \mathbf{R}^r)$ we have

$$\begin{aligned} {}_{W'}\langle D\mathsf{u}, \mathsf{v} \rangle_W &= \langle \mathcal{L}\mathsf{u} \mid \mathsf{v} \rangle_L - \langle \mathsf{u} \mid \tilde{\mathcal{L}}\mathsf{v} \rangle_L \\ &= {}_{\mathrm{H}^{-\frac{1}{2}}(\Gamma;\mathbf{R}^r)}\langle \mathcal{T}\mathsf{u}, \mathcal{T}_{\mathrm{H}^1}\mathsf{v} \rangle_{\mathrm{H}^{\frac{1}{2}}(\Gamma;\mathbf{R}^r)}\,. \end{aligned} \tag{9}$$

As $\mathrm{H}^1(\Omega; \mathbf{R}^r)$ is dense in W, while D and $\mathcal{T}$ are continuous, it can easily be seen that (9) remains valid also for $\mathsf{u} \in W$.

Following the same steps as in the proof of Theorem 2, after taking into the account (8), we get that for $\mathsf{u},\mathsf{v}\in \mathrm{C}^\infty_c(\mathbf{R}^d;\mathbf{R}^r)$ holds:

$$
\begin{aligned}
{}_{W'}\langle M\mathsf{u},\mathsf{v}\rangle_W &= \int_\Gamma \mathbf{A}_{\boldsymbol\nu}\mathsf{u}_{|_\Gamma}\cdot(\mathbf{I}-2\mathbf{S}_-)\mathsf{v}_{|_\Gamma}\,dS\\
&= {}_{\mathrm{H}^{-\frac12}(\Gamma;\mathbf{R}^r)}\langle \mathcal{T}\mathsf{u},(\mathcal{I}_{\mathrm{H}^{\frac12}}-2\mathcal{S}_-)\mathcal{T}_{\mathrm{H}^1}\mathsf{v}\rangle_{\mathrm{H}^{\frac12}(\Gamma;\mathbf{R}^r)}\\
&= {}_{\mathrm{H}^{-\frac12}(\Gamma;\mathbf{R}^r)}\langle (\mathcal{I}_{\mathrm{H}^{-\frac12}}-2\mathcal{S}_-^*)\mathcal{T}\mathsf{u},\mathcal{T}_{\mathrm{H}^1}\mathsf{v}\rangle_{\mathrm{H}^{\frac12}(\Gamma;\mathbf{R}^r)}\,,
\end{aligned}
\tag{10}
$$

where $\mathcal{I}_{\mathrm{H}^{\frac12}}:\mathrm{H}^{\frac12}(\Gamma;\mathbf{R}^r)\longrightarrow\mathrm{H}^{\frac12}(\Gamma;\mathbf{R}^r)$ and $\mathcal{I}_{\mathrm{H}^{-\frac12}}:\mathrm{H}^{-\frac12}(\Gamma;\mathbf{R}^r)\longrightarrow\mathrm{H}^{-\frac12}(\Gamma;\mathbf{R}^r)$ are identities. By the density of $\mathrm{C}^\infty_c(\mathbf{R}^d;\mathbf{R}^r)$ in W and the continuity of all operators appearing in (10), it easily follows that (10) remains valid for any $\mathsf{u}\in W$ and $\mathsf{v}\in\mathrm{H}^1(\Omega;\mathbf{R}^r)$.

By $\mathcal{S}_+\in\mathcal{L}(\mathrm{H}^{\frac12}(\Gamma;\mathbf{R}^r))$ denote the operator associated to the matrix field $\mathbf{S}_+$ as in (7), while by $\mathcal{S}_+^*$ its adjoint operator, and finally by $\mathcal{E}:\operatorname{im}\mathcal{T}\longrightarrow W$ the right inverse of the operator $\mathcal{T}$, as before. From $\mathcal{S}_-^*(\operatorname{im}\mathcal{T})\subseteq\operatorname{im}\mathcal{T}$ by Lemma 5 it follows that $\mathcal{S}_+^*(\operatorname{im}\mathcal{T})\subseteq\operatorname{im}\mathcal{T}$, so for given $\mathsf{w}\in W$ we have well defined

$$\mathsf{u}:=\mathcal{E}\mathcal{S}_+^*\mathcal{T}\mathsf{w}\quad\text{and}\quad \mathsf{v}:=\mathsf{w}-\mathsf{u}\,,$$

and obviously the decomposition $\mathsf{w}=\mathsf{u}+\mathsf{v}$.

Let us show that $\mathsf{u}\in\ker(D-M)$: for $\mathsf{z}\in\mathrm{H}^1(\Omega;\mathbf{R}^r)$ by (9), (10) and Lemma 5 we get

$$
\begin{aligned}
{}_{W'}\langle (D-M)\mathsf{u},\mathsf{z}\rangle_W &= {}_{\mathrm{H}^{-\frac12}(\Gamma;\mathbf{R}^r)}\langle 2\mathcal{S}_-^*\mathcal{T}\mathsf{u},\mathcal{T}_{\mathrm{H}^1}\mathsf{z}\rangle_{\mathrm{H}^{\frac12}(\Gamma;\mathbf{R}^r)}\\
&= {}_{\mathrm{H}^{-\frac12}(\Gamma;\mathbf{R}^r)}\langle 2\mathcal{S}_-^*\mathcal{T}\mathcal{E}\mathcal{S}_+^*\mathcal{T}\mathsf{w},\mathcal{T}_{\mathrm{H}^1}\mathsf{z}\rangle_{\mathrm{H}^{\frac12}(\Gamma;\mathbf{R}^r)}\\
&= {}_{\mathrm{H}^{-\frac12}(\Gamma;\mathbf{R}^r)}\langle 2\mathcal{S}_-^*\mathcal{S}_+^*\mathcal{T}\mathsf{w},\mathcal{T}_{\mathrm{H}^1}\mathsf{z}\rangle_{\mathrm{H}^{\frac12}(\Gamma;\mathbf{R}^r)}=0\,,
\end{aligned}
$$

thus $(D-M)\mathsf{u}=\mathsf{0}$, as $\mathcal{S}_-^*\mathcal{S}_+^*=0$.

It remains to show that $\mathsf{v}\in\ker(D+M)$: for $\mathsf{z}\in\mathrm{H}^1(\Omega;\mathbf{R}^r)$, similarly as above, it follows

$$
\begin{aligned}
{}_{W'}\langle (D+M)\mathsf{v},\mathsf{z}\rangle_W &= {}_{\mathrm{H}^{-\frac12}(\Gamma;\mathbf{R}^r)}\langle \mathcal{T}\mathsf{v}+(\mathcal{I}_{\mathrm{H}^{-\frac12}}-2\mathcal{S}_-^*)\mathcal{T}\mathsf{v},\mathcal{T}_{\mathrm{H}^1}\mathsf{z}\rangle_{\mathrm{H}^{\frac12}(\Gamma;\mathbf{R}^r)}\\
&= {}_{\mathrm{H}^{-\frac12}(\Gamma;\mathbf{R}^r)}\langle 2\mathcal{S}_+^*\mathcal{T}\mathsf{v},\mathcal{T}_{\mathrm{H}^1}\mathsf{z}\rangle_{\mathrm{H}^{\frac12}(\Gamma;\mathbf{R}^r)}\\
&= {}_{\mathrm{H}^{-\frac12}(\Gamma;\mathbf{R}^r)}\langle 2\mathcal{S}_+^*\mathcal{T}(\mathsf{w}-\mathcal{E}\mathcal{S}_+^*\mathcal{T}\mathsf{w}),\mathcal{T}_{\mathrm{H}^1}\mathsf{z}\rangle_{\mathrm{H}^{\frac12}(\Gamma;\mathbf{R}^r)}\\
&= {}_{\mathrm{H}^{-\frac12}(\Gamma;\mathbf{R}^r)}\langle 2\mathcal{S}_+^*(\mathcal{T}\mathsf{w}-\mathcal{T}\mathcal{E}\mathcal{S}_+^*\mathcal{T}\mathsf{w}),\mathcal{T}_{\mathrm{H}^1}\mathsf{z}\rangle_{\mathrm{H}^{\frac12}(\Gamma;\mathbf{R}^r)}\\
&= {}_{\mathrm{H}^{-\frac12}(\Gamma;\mathbf{R}^r)}\langle 2\mathcal{S}_+^*(\mathcal{T}\mathsf{w}-\mathcal{S}_+^*\mathcal{T}\mathsf{w}),\mathcal{T}_{\mathrm{H}^1}\mathsf{z}\rangle_{\mathrm{H}^{\frac12}(\Gamma;\mathbf{R}^r)}\\
&= {}_{\mathrm{H}^{-\frac12}(\Gamma;\mathbf{R}^r)}\langle 2\mathcal{S}_+^*(\mathcal{I}_{\mathrm{H}^{-\frac12}}-\mathcal{S}_+^*)\mathcal{T}\mathsf{w},\mathcal{T}_{\mathrm{H}^1}\mathsf{z}\rangle_{\mathrm{H}^{\frac12}(\Gamma;\mathbf{R}^r)}=0\,,
\end{aligned}
$$

as $\mathcal{S}_+^*(\mathcal{I}_{\mathrm{H}^{-\frac12}}-\mathcal{S}_+^*)=0$, thus $(D+M)\mathsf{v}=\mathsf{0}$ and we have the claim.

Q.E.D.

Theorems 2 and 3 provide us with sufficient conditions for operator $M:W\longrightarrow W'$, defined by (3), to be continuous and to satisfy (M). A natural question arises whether these conditions are *reasonable* and *usable*? A condition from Theorem 3, that $\mathbf{S}_-\in\mathrm{C}^{0,\frac12}(\Gamma;\mathrm{M}_r(\mathbf{R}))$, does not appear particularly restrictive, as it is expected that the conditions of Theorem 2 require even higher regularity on $\mathbf{S}_-$. However, another condition, requiring that the image of the trace operator is invariant under $\mathcal{S}_-^*$ appears somewhat *artificial* and *unnatural*. Therefore we should try yet another approach, by using the operator D instead of $\mathcal{T}$, as it was done in [EGC].

5. Approach via the boundary operator

It has already been said that the boundary operator D has *better* properties than the trace operator $\mathcal{T}$. A natural question to ask is whether these properties can be used to obtain *nicer* conditions than those in Theorem 3, which will still ensure the property (M2)?

By Lemma 1 $\ker D = W_0$, while $\operatorname{im} D = W_0^0$ is closed in W'. Therefore the restricted operator $D_{|_{W_0^\perp}} : W_0^\perp \longrightarrow W_0^0$ is a continuous linear bijection. As both $W_0^\perp$ and W_0^0 are closed (respectively in W and W'), by the Banach inverse mapping theorem its inverse $E : W_0^0 \longrightarrow W_0^\perp$ is also a continuous linear bijection. The operator E is clearly a right inverse of D:

$$DE\mathsf{g} = \mathsf{g}\,, \qquad \mathsf{g} \in W_0^0\,.$$

Lemma 6. *Under the assumptions of Theorem 2 we have that $\mathcal{S}_{-,p}(W_0) \subseteq W_0$ and $\mathcal{S}^*_{-,p}(W_0^0) \subseteq W_0^0$.*

Dem. For $\mathsf{u} \in \mathrm{C}_c^\infty(\Omega; \mathbf{R}^r)$ and $\mathsf{v} \in \mathrm{C}_c^\infty(\mathbf{R}^d; \mathbf{R}^r)$ we have

$$\begin{aligned}
{}_{W'}\langle\, D\mathcal{S}_{-,p}\mathsf{u}, \mathsf{v}\,\rangle_W &= {}_{W'}\langle\, D\mathbf{S}_{-,p}\mathsf{u}, \mathsf{v}\,\rangle_W \\
&= \int_\Gamma \mathbf{A}_{\boldsymbol\nu}\mathcal{T}_{\mathrm{H}^1}(\mathbf{S}_{-,p}\mathsf{u}) \cdot \mathsf{v}_{|_\Gamma}\, dS \\
&= \int_\Gamma \mathbf{A}_{\boldsymbol\nu}\mathbf{S}_-\mathsf{u}_{|_\Gamma} \cdot \mathsf{v}_{|_\Gamma}\, dS = 0\,,
\end{aligned}$$

as $\mathsf{u}_{|_\Gamma} = \mathsf{0}$. By the density of (the restrictions to Ω of the functions in) $\mathrm{C}_c^\infty(\mathbf{R}^d; \mathbf{R}^r)$ in W and the fact that v was taken to be arbitrary, we have that $D\mathcal{S}_{-,p}\mathsf{u} = 0$, or in other words $\mathcal{S}_{-,p}\mathsf{u} \in W_0$. Therefore $\mathcal{S}_{-,p}(\mathrm{C}_c^\infty(\Omega; \mathbf{R}^r)) \subseteq W_0$, while the density of $\mathrm{C}_c^\infty(\Omega; \mathbf{R}^r)$ in W_0 and the continuity of $\mathcal{S}_{-,p}$ implies $\mathcal{S}_{-,p}(W_0) \subseteq W_0$.

It remains to be shown that W_0^0 is invariant under $\mathcal{S}^*_{-,p}$: for arbitrary $\mathsf{g} \in W_0^0$ and $\mathsf{u} \in W_0$ (note that $\mathcal{S}_{-,p}\mathsf{u} \in W_0$) one has

$${}_{W'}\langle\, \mathcal{S}^*_{-,p}\mathsf{g}, \mathsf{u}\,\rangle_W = {}_{W'}\langle\, \mathsf{g}, \mathcal{S}_{-,p}\mathsf{u}\,\rangle_W = 0\,,$$

thus $\mathcal{S}^*_{-,p}\mathsf{g} \in W_0^0$, and we have the required invariance.

Q.E.D.

Theorem 4. *Under the assumptions of Theorem 2 we have that (3) defines operator $M \in \mathcal{L}(W; W')$ which satisfies (M).*

Dem. As it was already noted before Theorem 3, the continuity of M implies (M1), so it remains only to show (M2). Using the notation as in Theorem 2, let $\mathbf{S}_{+,p} := \mathbf{I} - \mathbf{S}_{-,p}$ be the extension of matrix function $\mathbf{S}_+$ on $\mathrm{Cl}\,\Omega$, let $\mathcal{S}_{+,p} \in \mathcal{L}(W)$ be the corresponding multiplication operator $\mathsf{u} \mapsto \mathbf{S}_{+,p}\mathsf{u}$, while by $\mathcal{S}^*_{+,p} \in \mathcal{L}(W')$ we denote its adjoint operator (in the Banach space sense). Clearly we have

$$\mathcal{S}_{-,p} + \mathcal{S}_{+,p} = \mathcal{I}_W\,, \quad \text{and} \quad \mathcal{S}^*_{-,p} + \mathcal{S}^*_{+,p} = \mathcal{I}_{W'}\,. \tag{11}$$

First we want to show that

$$\mathcal{S}^*_{-,p}\mathcal{S}^*_{+,p}D = \mathcal{S}^*_{+,p}\mathcal{S}^*_{-,p}D = 0\,. \tag{12}$$

Indeed, for $\mathsf{u}, \mathsf{v} \in \mathrm{C}_c^\infty(\mathbf{R}^d; \mathbf{R}^r)$ one has

$$\begin{aligned}
{}_{W'}\langle\, \mathcal{S}^*_{-,p}\mathcal{S}^*_{+,p}D\mathsf{u}, \mathsf{v}\,\rangle_W &= {}_{W'}\langle\, D\mathsf{u}, \mathcal{S}_{+,p}\mathcal{S}_{-,p}\mathsf{v}\,\rangle_W \\
&= {}_{W'}\langle\, D\mathsf{u}, \mathbf{S}_{+,p}\mathbf{S}_{-,p}\mathsf{v}\,\rangle_W\,,
\end{aligned}$$

which after taking into account condition (S2) of Theorem 2 (for $\mathbf{S}_{-,p}$ and $\mathbf{S}_{+,p}$), starting from the definition of boundary operator D (2), leads to

$$\begin{aligned}
{}_{W'}\langle\, \mathcal{S}^*_{-,p}\mathcal{S}^*_{+,p}D\mathsf{u}, \mathsf{v}\,\rangle_W &= \int_\Gamma \mathbf{A}_{\boldsymbol\nu}\mathsf{u}_{|_\Gamma} \cdot \mathcal{T}_{\mathrm{H}^1}(\mathbf{S}_{+,p}\mathbf{S}_{-,p}\mathsf{v})\, dS \\
&= \int_\Gamma \mathbf{A}_{\boldsymbol\nu}\mathsf{u}_{|_\Gamma} \cdot \mathbf{S}_+\mathcal{T}_{\mathrm{H}^1}(\mathbf{S}_{-,p}\mathsf{v})\, dS \\
&= \int_\Gamma \mathbf{A}_{\boldsymbol\nu}\mathsf{u}_{|_\Gamma} \cdot \mathbf{S}_+\mathbf{S}_-\mathcal{T}_{\mathrm{H}^1}(\mathsf{v})\, dS = 0\,,
\end{aligned}$$

where $\mathbf{S}_+\mathbf{S}_- = \mathbf{0}$, as $\mathbf{S}_+$ and $\mathbf{S}_-$ form a pair of projections, resulting in $\mathcal{S}^*_{-,p}\mathcal{S}^*_{+,p}D = 0$. Now from (11) we can easily get that $\mathcal{S}^*_{+,p}\mathcal{S}^*_{-,p}D = 0$.

Next we follow the same strategy of proof as in Theorem 3: for given $\mathsf{w} \in W$ we have well-defined

$$\mathsf{u} := E\mathcal{S}^*_{+,p}D\mathsf{w} \quad \text{and} \quad \mathsf{v} := \mathsf{w} - \mathsf{u}\,,$$

and it is obvious that $\mathsf{w} = \mathsf{u} + \mathsf{v}$.

Let us show that $\mathsf{u} \in \ker(D - M)$: by using the fact that E is a right inverse of D, after taking into account (6), (11) and (12) we get

$$\begin{aligned}(D - M)\mathsf{u} &= D\mathsf{u} - (D - 2\mathcal{S}^*_{-,p}D)\mathsf{u}\\ &= 2\mathcal{S}^*_{-,p}DE\mathcal{S}^*_{+,p}D\mathsf{w}\\ &= 2\mathcal{S}^*_{-,p}\mathcal{S}^*_{+,p}D\mathsf{w} = \mathsf{0}\,.\end{aligned}$$

It remains to show that $\mathsf{v} \in \ker(D + M)$: similarly as above

$$\begin{aligned}(D + M)\mathsf{v} &= (D + M)(\mathsf{w} - \mathsf{u})\\ &= (D + D - 2\mathcal{S}^*_{-,p}D)(\mathcal{I}_W - E\mathcal{S}^*_{+,p}D)\mathsf{w}\\ &= 2(\mathcal{I}_{W'} - \mathcal{S}^*_{-,p})D(\mathcal{I}_W - E\mathcal{S}^*_{+,p}D)\mathsf{w}\\ &= 2\mathcal{S}^*_{+,p}(D - \mathcal{S}^*_{+,p}D)\mathsf{w}\\ &= 2\mathcal{S}^*_{+,p}(\mathcal{I}_{W'} - \mathcal{S}^*_{+,p})D\mathsf{w}\\ &= 2\mathcal{S}^*_{+,p}\mathcal{S}^*_{-,p}D\mathsf{w} = \mathsf{0}\,,\end{aligned}$$

which gives the claim.

Q.E.D.

Note that the assumptions of Theorem 2, used to assure the continuity of operator M, are already sufficient for (M). Let us now check on several examples how reasonable the conditions of Theorem 2 really are.

6. Examples

Scalar elliptic equation

Let $\Omega \subseteq \mathbf{R}^d$ be an open and bounded set with the Lipschitz boundary Γ, and $\mu \in \mathrm{L}^\infty(\Omega)$ separated from zero in the sense that: $|\mu(\mathbf{x})| \geqslant \alpha_0 > 0$ (a.e. $\mathbf{x} \in \Omega$). Consider the following elliptic equation

$$-\triangle u + \mu u = f\,,$$

where $f \in \mathrm{L}^2(\Omega)$ is a given function. This equation can be rewritten as a first order system

$$\begin{cases}\mathsf{p} + \nabla u = \mathsf{0}\\ \mu u + \mathsf{div}\,\mathsf{p} = f\end{cases},$$

which turns out to be a Friedrichs system. Indeed, let

$$[\mathbf{A}_k]_{ij} = \begin{cases}1, & (i,j) \in \{(k, d+1), (d+1, k)\}\\ 0, & \text{otherwise}\end{cases},$$

$$[\mathbf{C}]_{ij} = \begin{cases}\mu(x), & i = j = d+1\\ 1, & i = j \neq d+1\\ 0, & \text{otherwise}\end{cases}.$$

Then $W = \mathrm{L}^2_{\mathrm{div}}(\Omega) \times \mathrm{H}^1(\Omega)$, where $\mathrm{L}^2_{\mathrm{div}}(\Omega) := \{\mathsf{u} \in \mathrm{L}^2(\Omega; \mathbf{R}^d) : \mathsf{div}\,\mathsf{u} \in \mathrm{L}^2(\Omega)\}$ is the Hilbert space. Let us also note that on $\mathrm{L}^2_{\mathrm{div}}(\Omega)$ a surjective *normal trace* $\mathcal{T}_{\mathrm{div}} : \mathrm{L}^2_{\mathrm{div}}(\Omega) \longrightarrow \mathrm{H}^{-\frac{1}{2}}(\Gamma)$ can be defined, which is for $\mathsf{u} \in \mathrm{H}^1(\Omega; \mathbf{R}^d)$ and $z \in \mathrm{H}^{\frac{1}{2}}(\Gamma)$ given by the formula

$${}_{\mathrm{H}^{-\frac{1}{2}}(\Gamma)}\langle\, \mathcal{T}_{\mathrm{div}}\mathsf{u}, z\,\rangle_{\mathrm{H}^{\frac{1}{2}}(\Gamma)} = {}_{\mathrm{H}^{-\frac{1}{2}}(\Gamma)}\langle\, \boldsymbol{\nu} \cdot \mathcal{T}_{\mathrm{H}^1}\mathsf{u}, z\,\rangle_{\mathrm{H}^{\frac{1}{2}}(\Gamma)}\,,$$

and then extended by continuity to a continuous linear operator on $\mathrm{L}^2_{\mathrm{div}}(\Omega)$.

Let us now describe the boundary operator $D : W \longrightarrow W'$. First we should mention that in the sequel by $\mathcal{T}_{\mathrm{H}^1}$ we shall denote any trace operator $\mathcal{T}_{\mathrm{H}^1} : \mathrm{H}^1(\Omega;\mathbf{R}^m) \longrightarrow \mathrm{H}^{\frac{1}{2}}(\Gamma;\mathbf{R}^m)$ (i.e. for any $m \in \mathbf{N}$ we shall use the same notation). Similarly, by ${}_{\mathrm{H}^{-\frac{1}{2}}}\langle\,\cdot,\cdot\,\rangle_{\mathrm{H}^{\frac{1}{2}}}$ we denote different duality products (both for functions taking values in $\mathbf{R}$ and $\mathbf{R}^{d+1}$). The meaning will be clear from the dimensions of the ranges.

As in this case

$$\mathbf{A}_{\boldsymbol{\nu}} = \begin{bmatrix} 0 & \cdots & 0 & \nu_1 \\ \vdots & & \vdots & \vdots \\ 0 & \cdots & 0 & \nu_d \\ \nu_1 & \cdots & \nu_d & 0 \end{bmatrix},$$

so from (2), after a short calculation, it follows that for any $(\mathsf{p},u)^\top, (\mathsf{r},v)^\top \in W$ we have

$$ {}_{W'}\langle\, D(\mathsf{p},u)^\top, (\mathsf{r},u)^\top \,\rangle_W = {}_{\mathrm{H}^{-\frac{1}{2}}}\langle\, \mathcal{T}_{\mathrm{div}}\mathsf{p}, \mathcal{T}_{\mathrm{H}^1}v \,\rangle_{\mathrm{H}^{\frac{1}{2}}} + {}_{\mathrm{H}^{-\frac{1}{2}}}\langle\, \mathcal{T}_{\mathrm{div}}\mathsf{r}, \mathcal{T}_{\mathrm{H}^1}u \,\rangle_{\mathrm{H}^{\frac{1}{2}}} . \tag{13}$$

The Dirichlet boundary condition $u_{|_\Gamma} = 0$ for the starting equation can be formulated by using different matrices $\mathbf{M}$; one possibility satisfying (FM) is

$$\mathbf{M} = \begin{bmatrix} 0 & \cdots & 0 & -\nu_1 \\ \vdots & & \vdots & \vdots \\ 0 & \cdots & 0 & -\nu_d \\ \nu_1 & \cdots & \nu_d & 0 \end{bmatrix}.$$

Since $\ker(\mathbf{A}_{\boldsymbol{\nu}} - \mathbf{M}^\top) \cap \ker(\mathbf{A}_{\boldsymbol{\nu}} + \mathbf{M}^\top) \neq \emptyset$, the choice of a pair of projections $\mathbf{S}_+$ and $\mathbf{S}_-$ is not unique. One possible choice consists in taking

$$\mathbf{S}_- = \mathbf{S}_-^\top = \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & 1 & 0 \\ 0 & \cdots & 0 & 0 & 0 \end{bmatrix}, \qquad \mathbf{S}_+ = \mathbf{S}_+^\top = \begin{bmatrix} 0 & \cdots & 0 & 0 \\ \vdots & & \vdots & \vdots \\ 0 & \cdots & 0 & 0 \\ 0 & \cdots & 0 & 1 \end{bmatrix}.$$

As the matrix function $\mathbf{S}_-$ is constant, it is only natural to extend it as the same constant, leading to

$$\mathbf{S}_{-,p} \begin{bmatrix} \mathsf{p} \\ u \end{bmatrix} = \begin{bmatrix} \mathsf{p} \\ 0 \end{bmatrix},$$

which obviously satisfies (S).

Therefore (3) defines an operator $M \in \mathcal{L}(W;W')$, satisfying (M). This operator can also be obtained from (5) and (13): for $(\mathsf{p},u)^\top \in W = \mathrm{L}^2_{\mathrm{div}}(\Omega) \times \mathrm{H}^1(\Omega)$ and $(\mathsf{r},v)^\top \in \mathrm{H}^1(\Omega;\mathbf{R}^d) \times \mathrm{H}^1(\Omega)$ one has

$$\begin{aligned} {}_{W'}\langle\, M(\mathsf{p},u)^\top, (\mathsf{r},v)^\top \,\rangle_W &= {}_{W'}\langle\, D(\mathsf{p},u)^\top, (\mathbf{I} - 2\mathbf{S}_{-,p})(\mathcal{T}_{\mathrm{H}^1}\mathsf{r}, \mathcal{T}_{\mathrm{H}^1}v)^\top \,\rangle_W \\ &= {}_{W'}\langle\, D(\mathsf{p},u)^\top, (-\mathcal{T}_{\mathrm{H}^1}\mathsf{r}, \mathcal{T}_{\mathrm{H}^1}v)^\top \,\rangle_W \\ &= {}_{\mathrm{H}^{-\frac{1}{2}}}\langle\, \mathcal{T}_{\mathrm{div}}\mathsf{p}, \mathcal{T}_{\mathrm{H}^1}v \,\rangle_{\mathrm{H}^{\frac{1}{2}}} - {}_{\mathrm{H}^{-\frac{1}{2}}}\langle\, \mathcal{T}_{\mathrm{div}}\mathsf{r}, \mathcal{T}_{\mathrm{H}^1}u \,\rangle_{\mathrm{H}^{\frac{1}{2}}} . \end{aligned}$$

As all the operators appearing in the above formula are continuous, while $\mathrm{H}^1(\Omega;\mathbf{R}^d) \times \mathrm{H}^1(\Omega)$ is dense in W, the formula remains valid for $(\mathsf{r},v)^\top \in W$ as well.

Now we can easily see that $\ker(D - M) = \mathrm{L}^2_{\mathrm{div}}(\Omega) \times \mathrm{H}^1_0(\Omega)$, which indeed corresponds to the Dirichlet boundary condition for the considered equation.

In general, the choice of matrix $\mathbf{M}$ defining the boundary condition is not unique. Namely, if we take

$$\mathbf{M} = \begin{bmatrix} 0 & \cdots & 0 & -\nu_1 \\ \vdots & & \vdots & \vdots \\ 0 & \cdots & 0 & -\nu_d \\ \nu_1 & \cdots & \nu_d & 2\alpha \end{bmatrix},$$

where $\alpha > 0$ is a constant, (FM) remains valid.

For matrix functions $\mathbf{S}_+$ and $\mathbf{S}_-$ in Lemma 2(c) we can take

$$\mathbf{S}_- = \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 & -\alpha\nu_1 \\ 0 & 1 & 0 & \cdots & 0 & -\alpha\nu_2 \\ \vdots & \ddots & \ddots & \ddots & \vdots & \vdots \\ 0 & \cdots & 0 & 1 & 0 & -\alpha\nu_{d-1} \\ 0 & \cdots & 0 & 0 & 1 & -\alpha\nu_d \\ 0 & \cdots & 0 & 0 & 0 & 0 \end{bmatrix}, \qquad \mathbf{S}_+ = \begin{bmatrix} 0 & \cdots & 0 & \alpha\nu_1 \\ \vdots & & \vdots & \vdots \\ 0 & \cdots & 0 & \alpha\nu_d \\ 0 & \cdots & 0 & 1 \end{bmatrix}.$$

In order to assure the conditions of Theorem 2, we shall require slightly higher regularity of the boundary, i.e. such that $\boldsymbol{\nu} : \Gamma \longrightarrow \mathbf{R}^d$ is a Lipschitz map. Then we can extend $\boldsymbol{\nu}$ to a Lipschitz map on the whole $\mathbf{R}^d$ (for simplicity, we maintain the same notation $\boldsymbol{\nu}$ for this extension) and define $\mathbf{S}_{-,p}$ by the same formula as $\mathbf{S}_-$. The condition (S2) will clearly be satisfied; on the other hand

$$\mathbf{S}_{-,p} \begin{bmatrix} \mathsf{p} \\ u \end{bmatrix} = \begin{bmatrix} \mathsf{p} - \alpha u\boldsymbol{\nu} \\ 0 \end{bmatrix} = \begin{bmatrix} \mathsf{p} \\ 0 \end{bmatrix} - \alpha \begin{bmatrix} u\boldsymbol{\nu} \\ 0 \end{bmatrix},$$

so (S1) follows immediately from the continuity of $u \mapsto u\boldsymbol{\nu}$ from $\mathrm{H}^1(\Omega)$ on $\mathrm{L}^2_{\mathrm{div}}(\Omega)$ (as this function is continuous from $\mathrm{H}^1(\Omega) \longrightarrow \mathrm{H}^1(\Omega;\mathbf{R}^d)$, it is also continuous when considered as a function $\mathrm{H}^1(\Omega) \longrightarrow \mathrm{L}^2_{\mathrm{div}}(\Omega)$).

As before, now easily follows that operator M, for $(\mathsf{p},u)^\top, (\mathsf{r},v)^\top \in W$, is given by

$$_{W'}\langle M(\mathsf{p},u)^\top, (\mathsf{r},v)^\top \rangle_W = {}_{\mathrm{H}^{-\frac12}}\langle \mathcal{T}_{\mathrm{div}}\mathsf{p}, \mathcal{T}_{\mathrm{H}^1}v \rangle_{\mathrm{H}^{\frac12}} - {}_{\mathrm{H}^{-\frac12}}\langle \mathcal{T}_{\mathrm{div}}\mathsf{r}, \mathcal{T}_{\mathrm{H}^1}u \rangle_{\mathrm{H}^{\frac12}} + 2\alpha \int_\Gamma \mathcal{T}_{\mathrm{H}^1}u\mathcal{T}_{\mathrm{H}^1}v\, dS.$$

The Robin boundary condition $(\boldsymbol{\nu}\cdot\nabla u + \alpha u)_{|_\Gamma} = 0$, for $\alpha > 0$, can be formulated by the choice of

$$\mathbf{M} = \begin{bmatrix} 0 & \cdots & 0 & \nu_1 \\ \vdots & & \vdots & \vdots \\ 0 & \cdots & 0 & \nu_d \\ -\nu_1 & \cdots & -\nu_d & 2\alpha \end{bmatrix},$$

with (FM) fulfilled.

For $\mathbf{S}_+$ and $\mathbf{S}_-$ in Lemma 2(c) we can take

$$\mathbf{S}_- = \begin{bmatrix} 0 & \cdots & 0 & -\alpha\nu_1 \\ \vdots & & \vdots & \vdots \\ 0 & \cdots & 0 & -\alpha\nu_d \\ 0 & \cdots & 0 & 1 \end{bmatrix}, \qquad \mathbf{S}_+ = \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 & \alpha\nu_1 \\ 0 & 1 & 0 & \cdots & 0 & \alpha\nu_2 \\ \vdots & \ddots & \ddots & \ddots & \vdots & \vdots \\ 0 & \cdots & 0 & 1 & 0 & \alpha\nu_{d-1} \\ 0 & \cdots & 0 & 0 & 1 & \alpha\nu_d \\ 0 & \cdots & 0 & 0 & 0 & 0 \end{bmatrix}.$$

As in the previous case, we can assure the condition in Theorem 2 by requiring $\boldsymbol{\nu} : \Gamma \longrightarrow \mathbf{R}^d$ to be Lipschitz, which can then be extended to a Lipschitz map on all of $\mathbf{R}^d$, while $\mathbf{S}_{-,p}$ we define by the same formula as $\mathbf{S}_-$.

M is now given by

$$_{W'}\langle M(\mathsf{p},u)^\top, (\mathsf{r},v)^\top \rangle_W = {}_{\mathrm{H}^{-\frac12}}\langle \mathcal{T}_{\mathrm{div}}\mathsf{r}, \mathcal{T}_{\mathrm{H}^1}u \rangle_{\mathrm{H}^{\frac12}} - {}_{\mathrm{H}^{-\frac12}}\langle \mathcal{T}_{\mathrm{div}}\mathsf{p}, \mathcal{T}_{\mathrm{H}^1}v \rangle_{\mathrm{H}^{\frac12}} + 2\alpha \int_\Gamma \mathcal{T}_{\mathrm{H}^1}u\mathcal{T}_{\mathrm{H}^1}v\, dS,$$

for all $(\mathsf{p},u)^\top, (\mathsf{r},v)^\top \in W$, and

$$\ker(D - M) = \{(\mathsf{p},u)^\top \in W : \mathcal{T}_{\mathrm{div}}\mathsf{p} = \alpha\mathcal{T}_{\mathrm{H}^1}u\},$$

which corresponds to the Robin boundary condition.

The Neumann boundary condition $(\boldsymbol{\nu}\cdot\nabla u)_{|\Gamma}=0$ can be formulated by

$$\mathbf{M}=\begin{bmatrix} 0 & \cdots & 0 & \nu_1 \\ \vdots & & \vdots & \vdots \\ 0 & \cdots & 0 & \nu_d \\ -\nu_1 & \cdots & -\nu_d & 0 \end{bmatrix},$$

for which (FM) holds.

For $\mathbf{S}_+$ and $\mathbf{S}_-$ we can take

$$\mathbf{S}_-=\begin{bmatrix} 0 & \cdots & 0 & 0 \\ \vdots & & \vdots & \vdots \\ 0 & \cdots & 0 & 0 \\ 0 & \cdots & 0 & 1 \end{bmatrix}, \qquad \mathbf{S}_+=\begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & 1 & 0 \\ 0 & \cdots & 0 & 0 & 0 \end{bmatrix},$$

so if $\mathbf{S}_{-,p}$ is defined by the same formula as $\mathbf{S}_-$, we can apply Theorem 2.

Now M will be given by

$${}_{W'}\langle\, M(\mathsf{p},u)^\top,(\mathsf{r},v)^\top\,\rangle_W=-{}_{\mathrm{H}^{-\frac12}}\langle\,\mathcal{T}_{\mathrm{div}}\mathsf{p},\mathcal{T}_{\mathrm{H}^1}v\,\rangle_{\mathrm{H}^{\frac12}}+{}_{\mathrm{H}^{-\frac12}}\langle\,\mathcal{T}_{\mathrm{div}}\mathsf{r},\mathcal{T}_{\mathrm{H}^1}u\,\rangle_{\mathrm{H}^{\frac12}}\,,$$

for all $(\mathsf{p},u)^\top,(\mathsf{r},v)^\top\in W$ and $\ker(D-M)=\{(\mathsf{p},u)^\top\in W:\mathcal{T}_{\mathrm{div}}\mathsf{p}=0\}$ corresponds to the Neumann boundary condition.

The Maxwell system in diffusive regime

Let $\Omega\subseteq\mathbf{R}^3$ be an open bounded set with a Lipschitz boundary, while $\mu,\sigma\in\mathrm{L}^\infty(\Omega)$ are separated from zero (in the sense defined in the previous example). For given $\mathsf{f},\mathsf{g}\in\mathrm{L}^2(\Omega;\mathbf{R}^3)$ we consider the system of equations

$$\begin{aligned}\mu\mathsf{H}+\operatorname{rot}\mathsf{E}&=\mathsf{f}\\ \sigma\mathsf{E}+\operatorname{rot}\mathsf{H}&=\mathsf{g}\end{aligned}\,.$$

This system can be rewritten in the Friedrichs form by taking

$$\mathbf{A}_1=\begin{bmatrix} \mathbf{0} & \begin{matrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{matrix} \\ \begin{matrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{matrix} & \mathbf{0} \end{bmatrix}, \qquad \mathbf{A}_2=\begin{bmatrix} \mathbf{0} & \begin{matrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{matrix} \\ \begin{matrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{matrix} & \mathbf{0} \end{bmatrix},$$

$$\mathbf{A}_3=\begin{bmatrix} \mathbf{0} & \begin{matrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{matrix} \\ \begin{matrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{matrix} & \mathbf{0} \end{bmatrix}, \qquad \text{and} \quad \mathbf{C}=\sigma\mathbf{I}\,.$$

Here we have $W=\mathrm{L}^2_{\mathrm{rot}}(\Omega)\times\mathrm{L}^2_{\mathrm{rot}}(\Omega)$, with $\mathrm{L}^2_{\mathrm{rot}}(\Omega):=\{\mathsf{u}\in\mathrm{L}^2(\Omega;\mathbf{R}^3):\operatorname{rot}\mathsf{u}\in\mathrm{L}^2(\Omega;\mathbf{R}^3)\}$ being the Hilbert space. On $\mathrm{L}^2_{\mathrm{rot}}(\Omega)$ we have well defined *tangential trace* $\mathcal{T}_{\mathsf{rot}}:\mathrm{L}^2_{\mathrm{rot}}(\Omega)\longrightarrow\mathrm{H}^{-\frac12}(\Gamma;\mathbf{R}^3)$, which is for $\mathsf{u}\in\mathrm{H}^1(\Omega;\mathbf{C}^3)$ and $\mathsf{z}\in\mathrm{H}^{\frac12}(\Gamma;\mathbf{C}^3)$ given by the formula

$${}_{\mathrm{H}^{-\frac12}(\Gamma;\mathbf{C}^3)}\langle\,\mathcal{T}_{\mathsf{rot}}\,\mathsf{u},\mathsf{z}\,\rangle_{\mathrm{H}^{\frac12}(\Gamma;\mathbf{C}^3)}={}_{\mathrm{H}^{-\frac12}(\Gamma;\mathbf{C}^3)}\langle\,\boldsymbol{\nu}\times\mathcal{T}_{\mathrm{H}^1}\mathsf{u},\mathsf{z}\,\rangle_{\mathrm{H}^{\frac12}(\Gamma;\mathbf{C}^3)}\,,$$

and then extended by density to a continuous linear operator on $\mathrm{L}^2_{\mathrm{rot}}(\Omega)$.

Matrix function $\mathbf{A}_{\boldsymbol{\nu}}$ can be written in the block form as

$$\mathbf{A}_{\boldsymbol{\nu}} = \begin{bmatrix} \mathbf{0} & \mathbf{A}_{\boldsymbol{\nu}}^{\mathrm{rot}} \\ -\mathbf{A}_{\boldsymbol{\nu}}^{\mathrm{rot}} & \mathbf{0} \end{bmatrix},$$

where

$$\mathbf{A}_{\boldsymbol{\nu}}^{\mathrm{rot}} = \begin{bmatrix} 0 & -\nu_3 & \nu_2 \\ \nu_3 & 0 & -\nu_1 \\ -\nu_2 & \nu_1 & 0 \end{bmatrix}.$$

As then

$$\mathbf{A}_{\boldsymbol{\nu}} \begin{bmatrix} \mathsf{H} \\ \mathsf{E} \end{bmatrix} = \boldsymbol{\nu} \times \mathsf{E} - \boldsymbol{\nu} \times \mathsf{H} \qquad \mathsf{H}, \mathsf{E} \in \mathbf{R}^3,$$

by following a similar procedure as in the case of the scalar elliptic equation we obtain that operator D is defined by

$$_{W'}\langle D(\mathsf{H},\mathsf{E})^\top, (\mathsf{r},\mathsf{v})^\top \rangle_W = {}_{\mathrm{H}^{-\frac12}}\langle \mathcal{T}_{\mathrm{rot}}\mathsf{E}, \mathcal{T}_{\mathrm{H}^1}\mathsf{r} \rangle_{\mathrm{H}^{\frac12}} - {}_{\mathrm{H}^{-\frac12}}\langle \mathcal{T}_{\mathrm{rot}}\mathsf{H}, \mathcal{T}_{\mathrm{H}^1}\mathsf{v} \rangle_{\mathrm{H}^{\frac12}},$$

for $(\mathsf{H},\mathsf{E}) \in W$ and $(\mathsf{r},\mathsf{v}) \in \mathrm{H}^1(\Omega;\mathbf{R}^3) \times \mathrm{H}^1(\Omega;\mathbf{R}^3)$.

It can easily be checked that the boundary condition $\boldsymbol{\nu} \times \mathsf{E}_{|\Gamma} = 0$ can be prescribed by a choice of

$$\mathbf{M} = \begin{bmatrix} \mathbf{0} & -\mathbf{A}_{\boldsymbol{\nu}}^{\mathrm{rot}} \\ -\mathbf{A}_{\boldsymbol{\nu}}^{\mathrm{rot}} & \mathbf{0} \end{bmatrix},$$

with (FM) satisfied.

A natural choice for matrices $\mathbf{S}_+$ and $\mathbf{S}_-$ is to take

$$\mathbf{S}_- = \begin{bmatrix} \mathbf{I} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix} \qquad \text{and} \qquad \mathbf{S}_+ = \begin{bmatrix} \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{I} \end{bmatrix},$$

where the blocks are of dimension 3×3, so if we again define $\mathbf{S}_{-,p}$ by the same formula as $\mathbf{S}_-$, the conditions of Theorem 2 hold.

The operator M in this example takes the following form: for $(\mathsf{H},\mathsf{E})^\top \in W = \mathrm{L}^2_{\mathrm{rot}}(\Omega) \times \mathrm{L}^2_{\mathrm{rot}}(\Omega)$ and $(\mathsf{r},\mathsf{v})^\top \in \mathrm{H}^1(\Omega;\mathbf{R}^3) \times \mathrm{H}^1(\Omega;\mathbf{R}^3)$ it holds

$$_{W'}\langle M(\mathsf{H},\mathsf{E})^\top, (\mathsf{r},\mathsf{v})^\top \rangle_W = -{}_{\mathrm{H}^{-\frac12}}\langle \mathcal{T}_{\mathrm{rot}}\mathsf{E}, \mathcal{T}_{\mathrm{H}^1}\mathsf{r} \rangle_{\mathrm{H}^{\frac12}} - {}_{\mathrm{H}^{-\frac12}}\langle \mathcal{T}_{\mathrm{rot}}\mathsf{H}, \mathcal{T}_{\mathrm{H}^1}\mathsf{v} \rangle_{\mathrm{H}^{\frac12}},$$

so $\ker(D - M) = \{(\mathsf{H},\mathsf{E}) \in W : \mathcal{T}_{\mathrm{rot}}\mathsf{E} = 0\}$.

Second order linear ODE

Let I be the open unit interval, and $p \in \mathrm{W}^{1,\infty}(I)$, $q \in \mathrm{L}^\infty(I)$, such that $p \geqslant \mu_0 > 0$ and $q \geqslant \mu_0 > 0$. Consider the following ordinary differential equation

$$-(p(x)u'(x))' + q(x)u(x) = f(x), \tag{14}$$

where $f \in \mathrm{L}^2(I)$ is a given function. In this simple example we will give complete clasifiction of boundary conditions that can be imposed by using Theorem 2 for two different representation of starting equation as a Friedrichs system.

By introducing $\mathsf{u} := (u, u')^\top$ this equation can easily be rewritten as a Friedrichs system

$$\mathcal{L}\mathsf{u} := (\mathbf{A}\mathsf{u})' + \mathbf{C}\mathsf{u} = \mathsf{f},$$

where

$$\mathbf{A} = \begin{bmatrix} 0 & -p \\ -p & 0 \end{bmatrix}, \qquad \mathbf{C} = \begin{bmatrix} q & 0 \\ p' & p \end{bmatrix}, \qquad \mathsf{f} = \begin{bmatrix} f \\ 0 \end{bmatrix}.$$

Since $W = \mathrm{H}^1(I) \times \mathrm{H}^1(I)$, it follows than any function $\mathbf{S}_{-,p} \in \mathrm{W}^{1,\infty}(I; \mathrm{M}_2(\mathbf{R}))$ stisfies (S1). One can easily prove that $\mathbf{S}_-$ cannot take values in $\{\mathbf{0}, \mathbf{I}\} \subseteq \mathrm{M}_2(\mathbf{R})$, as in these cases the condition (FM1) will not be satisfied. All other projectors on $\mathbf{R}^2$ take the form

$$\mathbf{S}_- = \begin{bmatrix} a & b \\ c & 1-a \end{bmatrix},$$

where $bc = a - a^2$. As

$$\mathbf{A}_\nu = \nu p \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix},$$

where $\nu(0) = -1$, $\nu(1) = 1$, for such $\mathbf{S}_-$ we have

$$\mathbf{A}_\nu - \mathbf{M} = 2\mathbf{S}_-^\top \mathbf{A}_\nu = -2\nu p \begin{bmatrix} c & a \\ 1-a & b \end{bmatrix}, \qquad \mathbf{M} = \nu p \begin{bmatrix} 2c & 2a-1 \\ 1-2a & 2b \end{bmatrix}.$$

Then (FM) ie equivalent to

$$c(0) \leqslant 0, \quad b(0) \leqslant 0, \quad c(1) \geqslant 0, \quad b(1) \geqslant 0. \tag{15}$$

Due to $bc = a - a^2$, the expresion $(\mathbf{A}_\nu - \mathbf{M})\mathsf{u} = \mathsf{0}$ impeses the following boundary condition for the starting equation:

$$c(0)u(0) + a(0)u'(0) = 0, \quad c(1)u(1) + a(1)u'(1) = 0. \tag{16}$$

Thus, by choosing different values for a and c on the boundary, one can propose different boundary conditions. For example the Dirichlet boundary condition $u(0) = 0$ can be enforced with $a(0) = 0$, $c(0) < 0$, the Neumann boundary condition $u'(0) = 0$ with $a(0) = 1$, $c(0) = 0$, while Robin boundary condition $\gamma u(0) + \alpha u'(0) = 0$ (for $\gamma < 0$, $\alpha \neq 0$) can be achieved with $a(0) = \alpha$, $c(0) = \gamma$. Note that proposed values for a and c do not contradict to conditions (15). The same thing can be done at $x = 1$

Now it can easily be proved that any combination of above mentioned boundary conditions at $x = 0$ and $x = 1$ can be proposed by appropriately choosen $a, b, c \in \mathrm{W}^{1,\infty}(\mathrm{Cl}\, I)$ that satisfy (15) and $b(x)c(x) = a(x) - a^2(x)$ for $x \in \{0, 1\}$. For such a, b, c, the function $\mathbf{S}_{-,p}$ defined with the same formula as $\mathbf{S}_-$ will clearly satisfy conditions of Theorem 2.

Note that (16) does not allow proposal of initial conditions $u(0) = 0$, $u'(0) = 0$ for the starting equation. Thus, in order to propose initial conditions we need to find a different representation of (14) as a Friedrichs system. By choosing $\mathsf{u} := (\mathrm{e}^{-\beta x}u', \mathrm{e}^{-\beta x}u)^\top$, for some $\beta \in \mathbf{R}$, the equation (14) can rewritten as

$$(\mathbf{A}\mathsf{u})' + \mathbf{C}\mathsf{u} = \mathsf{f},$$

with

$$\mathbf{A} = \begin{bmatrix} p & 0 \\ 0 & p \end{bmatrix}, \qquad \mathbf{C} = \begin{bmatrix} \beta p & -q \\ -p & \beta p - p' \end{bmatrix}, \qquad \mathsf{f} = \begin{bmatrix} \mathrm{e}^{-\beta x} f \\ 0 \end{bmatrix}.$$

This system is clearly symetric, and as

$$\mathbf{C} + \mathbf{C}^\top + \mathbf{A}' = \begin{bmatrix} 2\beta p + p' & -p-q \\ -p-q & 2\beta p - p' \end{bmatrix},$$

it is also positive for suficiently large β. Note that now q can be any bounded function.

Here again we have $W = \mathrm{H}^1(I) \times \mathrm{H}^1(I)$, and any function $\mathbf{S}_{-,p} \in \mathrm{W}^{1,\infty}(I; \mathrm{M}_2(\mathbf{R}^n))$ stisfies (S1).

One can easily prove that for this Friedrichs system there is only one choice of projector $\mathbf{S}_-$ that satisfy (FM): $\mathbf{S}_-(0) = \mathbf{I}$ and $\mathbf{S}_-(1) = \mathbf{0}$. In this case we have

$$\mathbf{M}(0) = -\mathbf{A}_\nu(0) = \begin{bmatrix} p(0) & 0 \\ 0 & p(0) \end{bmatrix}, \qquad \mathbf{M}(1) = \mathbf{A}_\nu(1) = \begin{bmatrix} p(0) & 0 \\ 0 & p(0) \end{bmatrix},$$

and thus the expresion $(\mathbf{A}_\nu - \mathbf{M})\mathsf{u} = \mathsf{0}$ impeses the initial boundary condition for the starting equation: $u(0) = 0$, $u'(0) = 0$. Clearly, the matrix function $\mathbf{S}_{-,p} \in \mathrm{W}^{1,\infty}(\mathsf{Cl}\, I; \mathrm{M}_2(\mathbf{R}))$ such that $\mathbf{S}_{-,p}(0) = \mathbf{I}$ and $\mathbf{S}_{-,p}(1) = \mathbf{0}$ can be found, and thus conditions of Theorem 2 are satisfied.

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