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Estimates on the weak solution of semilinear hyperbolic systems

Abstract

The Cauchy problem for a semilinear hyperbolic system of the type

$$\begin{cases} \partial_t \mathsf{u}(t,\mathbf{x}) + \sum_{k=1}^{d} \mathbf{A}^k(t,\mathbf{x})\partial_k \mathsf{u}(t,\mathbf{x}) = \mathsf{f}(t,\mathbf{x},\mathsf{u}(t,\mathbf{x})) \\ \mathsf{u}(0,\cdot) = \mathsf{v} \end{cases}$$

is considered, with each matrix function $\mathbf{A}^k$ being diagonal, bounded and locally Lipschitz in $\mathbf{x}$. Discrete models for the Boltzmann equation furnish examples of such systems. For bounded initial data, and right hand side that is locally Lipschitz and locally bounded in u, local existence and uniqueness results in L^∞ are well known, together with some estimates on weak solutions.

More precise estimates for weak solutions of the above Cauchy problem will be given, supplemented by estimates on the maximal time of existence for the solution, as well as the local existence and uniqueness in L^p setting $(1 < p < \infty)$.

Keywords: semilinear hyperbolic systems, discrete models for Boltzmann's equation, estimates on solution, time of existence

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1. Introduction

Let d and r be two positive integers, and $\Omega := \langle 0, T\rangle \times \mathbf{R}^d$ a strip in space-time. We consider the Cauchy problem for a semilinear hyperbolic system of the type

$$\begin{cases} \partial_t \mathsf{u}(t,\mathbf{x}) + \sum_{k=1}^{d} \mathbf{A}^k(t,\mathbf{x})\partial_k \mathsf{u}(t,\mathbf{x}) = \mathsf{f}(t,\mathbf{x},\mathsf{u}(t,\mathbf{x})) \\ \mathsf{u}(0,\cdot) = \mathsf{v} \end{cases} .$$

Here, the vector function $\mathsf{u} = (u_1, u_2, \cdots, u_r)$ is unknown, while the matrix coefficients $\mathbf{A}^k$, $k \in 1..d$, the right-hand side $\mathsf{f} = (f_1, f_2, \cdots, f_r)$ and the initial condition $\mathsf{v} = (v_1, v_2, \cdots, v_r)$ are known functions. We assume that each matrix function $\mathbf{A}^k$ is diagonal:

$$\mathbf{A}^k = \mathrm{diag}\{a_1^k, a_2^k, \cdots, a_r^k\} .$$

This assumption gives the above system a *decoupled* form. Namely, if we define vector functions a_i (for $i \in 1..r$) by

$$\mathsf{a}_i := (a_i^1, a_i^2, \cdots, a_i^d) ,$$

the above Cauchy problem can be written as

$$\text{(CP)} \qquad \begin{cases} \partial_t u_i(t,\mathbf{x}) + \mathsf{a}_i(t,\mathbf{x}) \cdot \nabla u_i(t,\mathbf{x}) = f_i(t,\mathbf{x},\mathsf{u}(t,\mathbf{x})) \\ u_i(0,\cdot) = v_i \end{cases} , \quad \text{for } i \in 1..r \ .$$

On the left-hand side of each equation in (CP) there is only one component of unknown vector function u.

The motivation for studying such systems can be found in kinetic theory of gases: Carleman, Broadwell and Maxwell systems (which are discrete models for the Boltzmann equation) are of this type. Also, it can be easily verified that any strictly hyperbolic system in one space variable can be reduced to a decoupled one (provided that coefficients are C^1 smooth). Of course, a single semilinear equation (case $r = 1$) is covered by (CP) as well.

We assume that each coefficient a_i (for $i \in 1..r$) is

(A1)
- bounded: $\mathsf{a}_i \in \mathrm{L}^\infty(\Omega; \mathbf{R}^d)$;
- Lipschitz in $\mathbf{x}$: $(\exists\, A > 0)(\forall\, \mathbf{x}, \mathbf{y} \in \mathbf{R}^d)$
$$|\mathsf{a}_i(t,\mathbf{x}) - \mathsf{a}_i(t,\mathbf{y})| \le A|\mathbf{x} - \mathbf{y}| \quad (\text{a.e. } t \in \langle 0, T\rangle) .$$

Here and in the rest of this paper (in order to avoid writing some constants) by $|\cdot|$ we denote the infinity vector norm (we use it also to denote the classical scalar norm), and by $\mathrm{K}_{\mathbf{R}^r}[0, r_1]$ we denote the closed unit ball $\{\mathbf{z} \in \mathbf{R}^r : |\mathbf{z}| \le r_1\}$ in $\mathbf{R}^r$. For the initial condition and the right-hand side we assume

(A2)
- $\mathsf{v} \in \mathrm{L}^\infty(\mathbf{R}^d; \mathbf{R}^r)$;
- $\mathsf{f} : \bar\Omega \times \mathbf{R}^r \longrightarrow \mathbf{R}^r$ is measurable ;
- f is locally Lipschitz in u:
$(\exists\, \Psi \in \mathrm{L}^\infty_{\mathrm{loc}}(\mathbf{R}))(\forall\, r_1 > 0)(\forall\, \mathbf{w}, \mathbf{z} \in \mathrm{K}_{\mathbf{R}^r}[0, r_1])$
$$|\mathsf{f}(t,\mathbf{x},\mathbf{w}) - \mathsf{f}(t,\mathbf{x},\mathbf{z})| \le \Psi(r_1)|\mathbf{w} - \mathbf{z}| \quad (\text{a.e. } (t,\mathbf{x}) \in \Omega) ;$$
- f is locally bounded in u:
$(\exists\, \Phi \in \mathrm{L}^\infty_{\mathrm{loc}}([0,T] \times \mathbf{R}))(\forall\, r_1 > 0)(\forall\, \mathbf{w} \in \mathrm{K}_{\mathbf{R}^r}[0, r_1])$
$$|\mathsf{f}(t,\mathbf{x},\mathbf{w})| \le \Phi(t, r_1) \quad (\text{a.e. } (t,\mathbf{x}) \in \Omega) .$$

Under such rather weak hypothesis on the coefficients, the initial condition and the right-hand side, one can only look for a weak solution.

Definition. *Function* $\mathsf{u} \in \mathrm{L}^1_{\mathrm{loc}}(\Omega; \mathbf{R}^r)$ *is a weak solution of problem (CP) on* Ω *(under assumptions (A1) and (A2)) if for each* $i \in 1..r$ *and for each* $\varphi \in \mathrm{C}^1_c([0,T\rangle \times \mathbf{R}^d)$ *it holds*

$$-\int_0^T\!\!\int_{\mathbf{R}^d} u_i[\partial_t\varphi + \mathsf{div}\,(\varphi \mathsf{a}_i)]d\mathbf{x}dt = \int_0^T\!\!\int_{\mathbf{R}^d} f_i(\cdot,\cdot,\mathsf{u}(\cdot,\cdot))\varphi d\mathbf{x}dt + \int_{\mathbf{R}^d} v_i\varphi(0,\cdot)d\mathbf{x}\,.$$

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The space $\mathrm{C}^1_c([0,T\rangle \times \mathbf{R}^d)$ consists of restrictions of functions from $\mathrm{C}^1_c(\langle-\infty,T\rangle \times \mathbf{R}^d)$ to $[0,T\rangle \times \mathbf{R}^d$.

The paper is organised as follows: in the second section the local existence and uniqueness result is proved. It is paired with an estimate on the solution of a certain type. This is a known result (see [T1]), which is here proved under slightly generalised assumptions that do not change the proof, but allow a more precise estimate on the solution. The estimates on the solution and the time of its existence is the main topic of section three, which is the central part of this paper. It is shown how to achieve the best possible estimate on the solution and its time of existence (the best among all estimates of a certain type—the type provided by the existence and uniqueness theorem). This section concludes with two examples. The first one compares the best possible estimate on the solution achieved under *old* assumptions, with the *new* one (given by generalised assumptions), showing that the *new* estimate is essentially better when the right-hand side depends on the time variable. The second example shows that the estimate of the type we are dealing with can be quite rough (for example, in some situations when the right-hand side depends *essentially* on the space variable). In section four the L^p version (for $p \in \langle 1,\infty\rangle$) of the existence and uniqueness theorem is briefly discussed. The paper concludes with an appendix containing some results on ordinary differential equations that were used in previous sections (for more information see [CL] and the references there), as well as the existence and uniqueness result for the linear transport equation (which can be found in [T1]) used in Theorem 1.

2. Existence and uniqueness theorem

The existence and uniqueness result for (CP) can be found in unpublished lecture notes by Luc Tartar [T1] (under assumption (A1) and slightly modified assumption (A2): function Φ defining the local bound on f is not allowed to depend on the time variable). We shall state a more precise version of the existence and uniqueness theorem together with its proof. The proof uses existence and uniqueness result for the linear transport equation which is stated in the Appendix.

Theorem 1. *Let assumptions (A1) and (A2) hold, and assume that* $u : [0,S\rangle \longrightarrow \mathbf{R}$ *is an absolutely continuous solution of*

$$(ODE\text{-}t) \qquad \begin{cases} u'(t) = \Phi(t,u(t)) \\ u(0) = \|v\|_{\mathrm{L}^\infty(\mathbf{R}^d)} \end{cases},$$

for some $S \in \langle 0,T]$. *Then there exists a unique function* $\mathsf{u} \in \mathrm{L}^\infty_{\mathrm{loc}}([0,S\rangle;\mathrm{L}^\infty(\mathbf{R}^d;\mathbf{R}^r))$, *which is a weak solution of problem (CP) on* $\langle 0,S\rangle \times \mathbf{R}^d$. *Additionally,* u *satisfies the estimate*

$$(E) \qquad \|\mathsf{u}(t,\cdot)\|_{\mathrm{L}^\infty(\mathbf{R}^d;\mathbf{R}^r)} \le u(t) \quad (\text{a.e. } t \in \langle 0,S\rangle)\,.$$

Dem. Let us prove the uniqueness first. Assume that $\mathsf{u},\mathsf{v} \in \mathrm{L}^\infty_{\mathrm{loc}}([0,S\rangle;\mathrm{L}^\infty(\mathbf{R}^d;\mathbf{R}^r))$ are two weak solutions, and denote $g(t) := \|\mathsf{u}(t,\cdot)-\mathsf{v}(t,\cdot)\|_{\mathrm{L}^\infty(\mathbf{R}^d;\mathbf{R}^r)}$. Then (by locally Lipschitz property of f)

$$(\forall\,\varepsilon>0)(\exists\,D_\varepsilon>0)(\forall\,i\in 1..r)$$
$$\|f_i(t,\cdot,\mathsf{u}(t,\cdot)) - f_i(t,\cdot,\mathsf{v}(t,\cdot))\|_{\mathrm{L}^\infty(\mathbf{R}^d)} \le D_\varepsilon g(t) \quad (\text{a.e. } t\in\langle 0,S-\varepsilon\rangle)\,.$$

If we subtract equations for u and v (better said, their weak formulations), and use the estimate from Theorem 8, we get (for $i \in 1..r$ and a.e. $t \in \langle 0,S-\varepsilon\rangle$)

$$\|u_i(t,\cdot)-v_i(t,\cdot)\| \le \int_0^t \|f_i(s,\cdot,\mathsf{u}(s,\cdot)) - f_i(s,\cdot,\mathsf{v}(s,\cdot))\|_{\mathrm{L}^\infty(\mathbf{R}^d)}ds\,.$$

After taking maximum in i we have

$$g(t) \leq \int_0^t D_\varepsilon g(s)ds\,, \qquad (\text{a.e. } t \in \langle 0, S-\varepsilon\rangle)\,,$$

and by Gronwall's inequality it follows that $g = 0$ on $\langle 0, S-\varepsilon\rangle$. Now the arbitrariness of ε implies $\mathsf{u} = \mathsf{v}$ almost everywhere on $\langle 0, S\rangle$, and the uniqueness is proved.

To prove the existence we first inductively define u^n, $n \in \mathbf{N}$, as the weak solution of a linear problem

$$\begin{cases} \partial_t u_i^n(t,\mathbf{x}) + \mathsf{a}_i(t,\mathbf{x}) \cdot \nabla u_i^n(t,\mathbf{x}) = f_i(t,\mathbf{x},\mathsf{u}^{n-1}(t,\mathbf{x})) \\ u_i^n(0,\cdot) = v_i \end{cases}, \quad \text{for } i \in 1..r\,,$$

starting from a bounded function u^0 on $\langle 0, S\rangle \times \mathbf{R}^d$ that satisfies

$$\|\mathsf{u}^0(t,\cdot)\|_{\mathrm{L}^\infty(\mathbf{R}^d;\mathbf{R}^r)} \leq u(t) \quad (\text{a.e. } t \in \langle 0, S\rangle)\,.$$

By Theorem 8 each u_n is a well defined bounded function. Also, u^1 satisfies the estimate

$$\begin{aligned} \|\mathsf{u}^1(t,\cdot)\|_{\mathrm{L}^\infty(\mathbf{R}^d;\mathbf{R}^r)} &\leq \|\mathsf{v}\|_{\mathrm{L}^\infty(\mathbf{R}^d)} + \int_0^t \|\mathsf{f}(s,\cdot,\mathsf{u}^0(s,\cdot))\|ds \\ &\leq \|\mathsf{v}\|_{\mathrm{L}^\infty(\mathbf{R}^d)} + \int_0^t \Phi(s,u(s))ds = u(t) \quad (\text{a.e. } t \in \langle 0, S\rangle)\,. \end{aligned}$$

Inductively, the same estimate can be proved for each u^n.

We now distingnish two cases. The first one is when u does not have a blow up in S (and therefore is bounded). In this case, there exists a constant P such that $|\Psi(u(\cdot))| \leq P$. If we subtract equations for u^{n+1} and u^n, then use the estimate from Theorem 9 and locally Lipschitz property of f, we get (for a.e. $t \in \langle 0, S-\varepsilon\rangle$)

$$\begin{aligned} \|\mathsf{u}^{n+1}(t,\cdot) - \mathsf{u}^n(t,\cdot)\|_{\mathrm{L}^\infty(\mathbf{R}^d;\mathbf{R}^r)} &\leq \int_0^t \|\mathsf{f}(s,\cdot,\mathsf{u}^n(s,\cdot)) - \mathsf{f}(s,\cdot,\mathsf{u}^{n-1}(s,\cdot))\|_{\mathrm{L}^\infty(\mathbf{R}^d;\mathbf{R}^r)}ds \\ &\leq P\int_0^t \|\mathsf{u}^n(s,\cdot) - \mathsf{u}^{n-1}(s,\cdot)\|_{\mathrm{L}^\infty(\mathbf{R}^d;\mathbf{R}^r)}ds\,. \end{aligned}$$

Now one can easily prove by induction that (for a.e. $t \in \langle 0, S-\varepsilon\rangle$)

$$\|\mathsf{u}^{n+1}(t,\cdot) - \mathsf{u}^n(t,\cdot)\|_{\mathrm{L}^\infty(\mathbf{R}^d;\mathbf{R}^r)} \leq \frac{RP^n t^n}{n!} \leq \frac{RP^n S^n}{n!}\,,$$

where $R := \|\mathsf{u}^1 - \mathsf{u}^0\|_{\mathrm{L}^\infty(\langle 0,S\rangle\times\mathbf{R}^d;\mathbf{R}^r)}$. As

$$\sum_{j=0}^n \|\mathsf{u}^{j+1} - \mathsf{u}^j\|_{\mathrm{L}^\infty(\langle 0,S\rangle\times\mathbf{R}^d;\mathbf{R}^r)} \leq \sum_{j=0}^n \frac{RP^jS^j}{j!} \leq R\mathrm{e}^{PS}\,,$$

we conclude that series $\sum_{j=0}^n(\mathsf{u}^{j+1} - \mathsf{u}^j) = \mathsf{u}^{n+1} - \mathsf{u}^0$ converges absolutely in $\mathrm{L}^\infty(\langle 0, S\rangle \times \mathbf{R}^d; \mathbf{R}^r)$, which implies that u^n converges and we denote the limit of u^n by u. By locally Lipschitz property of function f it follows

$$\mathsf{f}(\cdot,\cdot,\mathsf{u}^n(\cdot,\cdot)) \longrightarrow \mathsf{f}(\cdot,\cdot,\mathsf{u}(\cdot,\cdot)) \qquad \text{in} \qquad \mathrm{L}^\infty(\langle 0, S\rangle \times \mathbf{R}^d; \mathbf{R}^r)\,.$$

After passing to the limit in weak formulation for each u^n one can easily verify that u satisfies the weak formulation of (CP) and the estimate (E), which proves the existence in this case.

If u has blow up in S we repeat the argument of the previous case, thus getting a weak solution u^{S_1} on $\langle 0, S_1\rangle \times \mathbf{R}^d$ for each $S_1 \in \langle 0, S\rangle$. One can also easily check that $\mathsf{u}^{S_1} = \mathsf{u}^{S_2}$ on $\langle 0, S_1\rangle \times \mathbf{R}^d$, for $0 < S_1 < S_2 < S$, which ensures the existence of a function u on $\langle 0, S\rangle \times \mathbf{R}^d$ with the property $\mathsf{u}^{S_1} = \mathsf{u}$ on $\langle 0, S_1\rangle \times \mathbf{R}^d$, for each $S_1 < S$. It is easy to see that u is a weak solution of (CP) on $\langle 0, S\rangle \times \mathbf{R}^d$, and that estimate (E) holds.

Q.E.D.

Remark. The only assumption on function Φ is that it belongs to $\mathrm{L}^\infty_{\mathrm{loc}}([0,T]\times\mathbf{R})$, which is not enough to guarantee the existence of solution for (ODE-t). However, if we take some function Φ_1 that is larger than Φ, then this function will also be a local bound for f. Therefore we can look for a solution of (ODE-t) with Φ_1 on the right-hand side instead of Φ. If we choose Φ_1 as below (the one that does not depend on time variable and it is continuous), this will ensure the existence of a C^1 solution for (ODE-t).

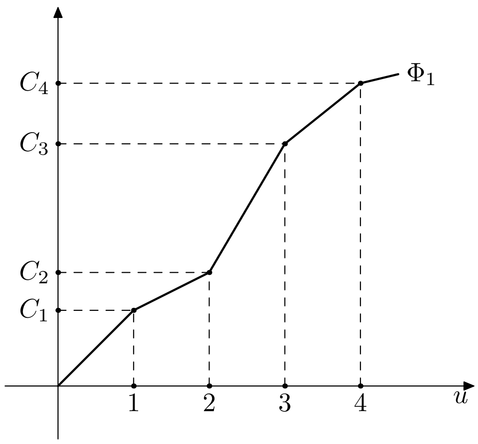
If we denote

$$C_n := \|\Phi\|_{\mathrm{L}^\infty([0,T]\times[0,n])}\,, \quad n\in\mathbf{N}\,,$$

and define function Φ_1 by

$$\Phi_1(u) := C_{n+1} + (C_{n+2}-C_{n+1})(u-n)\,, \quad u\in[n,n+1]\,, \quad n\in\mathbf{N}_0\,,$$

then the graph of Φ_1 is given on the below figure.



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3. Estimates on solution

The bound on the solution u from Theorem 1 obviously depends on the solution u to (ODE-t), and therefore on the choice of function Φ. From the proof of Theorem 1 it is clear that the time of existence of solution u also depends on the time of existence of solution u to (ODE-t): roughly speaking, if solution of (ODE-t) exists until some time S, then the solution of (CP) will also exist at least until time S. As we have a large set to choose Φ from (as mentioned before, if one function is a local bound for f, then any larger function is also a local bound for f), questions that naturally arise are *which Φ will give the best estimate on solution of (CP)* and *which Φ will give the largest time of existence for solution to (CP)*. The next theorem partially gives the answer to the first question.

Theorem 2. *Assume that $\Phi_1,\Phi_2 : [0,T]\times\mathbf{R}\longrightarrow\mathbf{R}$ are two measurable functions such that $\Phi_1\le\Phi_2$ (a.e.) and Φ_1 is locally Lipschitz in second variable:*

$$(\exists\,\Psi\in\mathrm{L}^\infty_{\mathrm{loc}}(\mathbf{R}))(\forall\,u,v\in\mathbf{R})$$
$$|u|\ge|v| \quad\Longrightarrow\quad |\Phi_1(t,u)-\Phi_1(t,v)|\le\Psi(|u|)|u-v| \quad (\text{a.e. } t\in\langle 0,T\rangle)\,.$$

Furthermore, let (for $i=1,2$ and $T_i\le T$) $u_i:[0,T_i\rangle\longrightarrow\mathbf{R}$, be a solution of

$$\begin{cases} u_i'(t)=\Phi_i(t,u_i(t)) & (\text{a.e. } t) \\ u_i(0)=u_0 \end{cases},$$

and $u_1\in\mathrm{W}^{1,\infty}_{\mathrm{loc}}([0,T_1\rangle)$, $u_2\in\mathrm{W}^{1,1}_{\mathrm{loc}}([0,T_2\rangle)$. Then $u_1\le u_2$ on the intersection of their intervals of existence.

Dem. Suppose that there is $c > 0$ such that $u_1(c) > u_2(c)$. As u_1 and u_2 are continuous, inequality $u_1 > u_2$ holds on some open interval that contains c. Let $\langle a, b\rangle$ be the union of all such intervals. Obviously $u_1(a) = u_2(a) =: u_a$, and for $t \in \langle a, b\rangle$ the formulae

$$u_i(t) = u_a + \int_a^t \Phi_i(s, u_i(s))ds\,, \qquad i = 1, 2$$

hold. For $\varepsilon > 0$ small let $K := \Psi\Big(\|u_1\|_{\mathrm{L}^\infty([a,b-\varepsilon])}\Big)$; then for $t \in [a, b-\varepsilon]$ we have

$$\begin{aligned}|u_1(t) - u_2(t)| = u_1(t) - u_2(t) &= \int_a^t \Phi_1(s, u_1(s)) \overbrace{-\Phi_2(s, u_2(s)) + \Phi_1(s, u_2(s))}^{\leq 0} -\Phi_1(s, u_2(s))ds \\ &\leq \int_a^t \Phi_1(s, u_1(s)) - \Phi_1(s, u_2(s))ds \\ &\leq \int_a^t |\Phi_1(s, u_1(s)) - \Phi_1(s, u_2(s))|ds \leq \int_a^t K|u_1(s) - u_2(s)|ds\,.\end{aligned}$$

Finally, Gronwall's inequality implies $u_1 = u_2$ on $[a, b-\varepsilon]$, which contradicts $u_1 > u_2$ on $\langle a, b\rangle$.

Q.E.D.

The above theorem suggests that the best possible estimate of type (E) will be given by the smallest possible function Φ. Let us investigate the properties of such Φ.

Theorem 3. *The function*

$$h(t, u) := \max_{|\mathsf{u}|\leq u} \operatorname*{vrai\,sup}_{\mathbf{x}\in\mathbf{R}^d} |\mathsf{f}(t, x, \mathsf{u})|\,, \quad t \in [0, T]\,,\ u \in \mathbf{R}_0^+\,,$$

which is the smallest local bound for f *has the following properties:*

- $h \in \mathrm{L}^\infty_{\mathrm{loc}}([0, T] \times \mathbf{R}_0^+)$;
- $h \geq 0$ *and* $h(t, \cdot)$ *is nondecresing,* $t \in [0, T]$;
- h *is locally Lipschitz in* u *(with the same* Ψ *as in (A2))*:
  $(\forall\, u, v \in \mathbf{R}_0^+) \quad u \geq v \quad \Longrightarrow \quad |h(t, u) - h(t, v)| \leq \Psi(u)|u - v| \quad$ (a.e. $t \in [0, T]$).

Dem. The first two properties are obvious, and it only remains to show the last one. In order to do that let us first show that function $g : [0, T] \times \mathbf{R}^r \longrightarrow \mathbf{R}$ defined by

$$g(t, \mathsf{u}) := \operatorname*{vrai\,sup}_{\mathbf{x}\in\mathbf{R}^d} |\mathsf{f}(t, x, \mathsf{u})|$$

is locally Lipschitz in u:

(1) $\qquad (\forall\, r_1 > 0)(\forall\, \mathbf{w}, \mathbf{z} \in \mathrm{K}_{\mathbf{R}^r}[0, r_1]) \quad |g(t, \mathbf{w}) - g(t, \mathbf{z})| \leq \Psi(r_1)|\mathbf{w} - \mathbf{z}| \quad$ (a.e. $t \in [0, t]$).

Indeed, if (for $r_1 > 0$) $\mathbf{w}$ and $\mathbf{z}$ from $\mathrm{K}_{\mathbf{R}^r}[0, r_1]$ are fixed, then the locally Lipschitz property of function f (from (A2)) implies

(2) $$|\mathsf{f}(t, \mathbf{x}, \mathbf{w})| \leq |\mathsf{f}(t, \mathbf{x}, \mathbf{z})| + \Psi(r_1)|\mathbf{w} - \mathbf{z}| \quad \text{(a.e. } (t, \mathbf{x}) \in \Omega)\,.$$

By definition of essential supremum for each $t \in [0, t]$ and for each $\varepsilon > 0$ there exists a set of positive measure $E_\varepsilon^{t,\mathbf{w}} \subseteq \mathbf{R}^d$, such that

$$(\forall\, \mathbf{x} \in E_\varepsilon^{t,\mathbf{w}}) \quad g(t, \mathbf{w}) \leq \varepsilon + |\mathsf{f}(t, \mathbf{x}, \mathbf{w})|\,.$$

Combining this with (2) we get

$$g(t,\mathbf{w}) \le \varepsilon + |\mathsf{f}(t,\mathbf{x},\mathbf{z})| + \Psi(r_1)|\mathbf{w}-\mathbf{z}| \quad (\text{a.e. } t\in[0,T]\,,\ \mathbf{x}\in E_\varepsilon^{t,\mathbf{w}})\,,$$

and then easily

$$g(t,\mathbf{w}) \le \varepsilon + g(t,\mathbf{z}) + \Psi(r_1)|\mathbf{w}-\mathbf{z}| \quad (\text{a.e. } t\in[0,T])\,.$$

By arbitrariness of $\varepsilon > 0$ we derive

$$g(t,\mathbf{w}) - g(t,\mathbf{z}) \le \Psi(r_1)|\mathbf{w}-\mathbf{z}| \quad (\text{a.e. } t\in[0,T])\,.$$

In a similar way one can prove

$$g(t,\mathbf{z}) - g(t,\mathbf{w}) \le \Psi(r_1)|\mathbf{w}-\mathbf{z}| \quad (\text{a.e. } t\in[0,T])\,,$$

which completes the proof of statement (1).

In order to prove that h is locally Lipschitz in the last variable, choose $u \ge v \ge 0$, and for $t\in[0,T]$ denote by $\mathbf{w}_t$ some point from $\mathrm{K}_{\mathbf{R}^r}[\mathsf{0},u]$, such that $h(t,u) = g(t,\mathbf{w}_t)$. If $\mathbf{w}_t \in \mathrm{K}_{\mathbf{R}^r}[\mathsf{0},v]$, then $h(t,u) = h(t,v)$, and the inequality in locally Lipschitz condition for h is trivially satisfied. If $\mathbf{w}_t \notin \mathrm{K}_{\mathbf{R}^r}[\mathsf{0},v]$, then by $\mathbf{z}_t$ denote the intersection of $\mathrm{K}_{\mathbf{R}^r}[\mathsf{0},v]$ with segment line connecting $\mathbf{w}$ and $\mathsf{0}$ in $\mathbf{R}^r$. It holds

$$g(t,\mathbf{w}_t) - g(t,\mathbf{z}_t) \le \Psi(u)|\mathbf{w}_t-\mathbf{z}_t| \le \Psi(u)|u-v| \quad (\text{a.e. } t\in[0,T])\,,$$

and therefore

$$h(t,u) \le g(t,\mathbf{z}_t) + \Psi(u)|u-v| \le h(t,v) + \Psi(u)|u-v| \quad (\text{a.e. } t\in[0,T])\,.$$

This implies

$$|h(t,u)-h(t,v)| = h(t,u)-h(t,v) \le \Psi(u)|u-v| \quad (\text{a.e. } t\in[0,T])\,,$$

as claimed.

Q.E.D.

Remark. If we extend h to $[-T,T]\times\mathbf{R}$ by

$$\begin{aligned} h(-t,u) &:= h(t,u)\,, \quad (t,u)\in[0,T]\times\mathbf{R}_0^+\,;\\ h(t,-u) &:= h(t,u)\,, \quad (t,u)\in[-T,T]\times\mathbf{R}_0^+\,; \end{aligned}$$

it will still be a locally bounded function (now from $\mathrm{L}^\infty_{\mathrm{loc}}([-T,T]\times\mathbf{R})$) and locally Lipschitz in the second variable (with the same Ψ as before).

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Corollary 1. *Let h be as in previous theorem, $u_0 \ge 0$, and let $u_h \in \mathrm{W}^{1,\infty}_{\mathrm{loc}}(\langle\alpha,\beta\rangle)$ be the maximal solution of*

$$\begin{cases} u_h'(t) = h(t,u_h(t)) & (\text{a.e. } t)\\ u(0) = u_0 \end{cases}$$

(such u_h exists by Theorem 6 from the Appendix). Assume that $v\in\mathrm{W}^{1,1}_{\mathrm{loc}}([0,T'\rangle)$ is a solution of

$$\begin{cases} v'(t) = \Phi(t,v(t)) & (\text{a.e. } t\in\langle 0,T'\rangle)\\ v(0) = u_0 \end{cases},$$

for some measurable function $\Phi \ge h$. Then it is necessary that $T' \le \beta \le T$ and $u \le v$ on $[0,T'\rangle$.

Therefore, u_h is the best estimate on the solution u to (CP) of type (E), and u exists at least until time β (which is the greatest time of existence that Theorem 1 can ensure).

Dem. By Theorem 7 from Appendix u has blow up in β, and we already know by Theorem 2 that $u \le v$. Therefore $T' \le \beta$.

Q.E.D.

Let us now look at two (academic) examples. The first one shows that, in order to get the best possible estimate, it is crucial to allow Φ (local bound for f) to depend on time variable (of course, this is the case when the right-hand side actually depends on t).

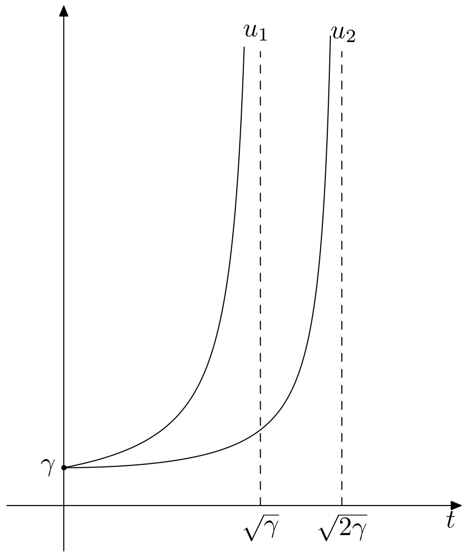
Example. Consider the Cauchy problem for a single equation in one space dimension

$$\begin{cases} \partial_t U(t,x) + \partial_x U(t,x) = f(t,x,U) := tU^2 \quad \text{in} \quad \langle 0,T\rangle \times \mathbf{R} \\ U(0,\cdot) = \dfrac{1}{\gamma} \end{cases},$$

where $\gamma > 0$ is a constant. The unique solution of this problem is given by

$$U(t,x) = \frac{1}{\gamma - t^2/2},$$

and it exists until time $t_c = \min\{T, \sqrt{2\gamma}\}$. If we do not allow the local bound for f to depend on time variable, the best one (the smallest one) we get is $\Phi(u) = Tu^2$, and the solution of corresponding (ODE-t) problem $u_1(t) = \dfrac{1}{\gamma - Tt}$ exists until time $t_1 = \min\{T, \dfrac{\gamma}{T}\}$. However, if we allow the local bound for f to depend on time variable, we get $h(t,u) = tu^2$. Then the solution of (ODE-h) problem $u_2(t) = \dfrac{1}{\gamma - t^2/2} = U(t,x)$ exists until time $t_2 = t_c = \min\{T, \sqrt{2\gamma}\}$. It is clear that $u_1(t) > u_2(t) = \|U(t,\cdot)\|_{\mathrm{L}^\infty(\mathbf{R})}$ and $t_1 \le t_2 = t_c$ as it is illustrated on picture below (if we were solving the above Cauchy problem on the whole half-space, we would get $t_1 = \sqrt{\gamma} < \sqrt{2\gamma} = t_2 = t_c$).



∎

The second example illustrates that an estimate of type (E) is not the best possible.

Example. Let

$$g(x) = \begin{cases} \mathrm{e}^{-x}, & x \ge 0 \\ \mathrm{e}^{x}, & x \le 0 \end{cases}.$$

One can easily check that the unique solution of problem

$$\begin{cases} \partial_t U + \partial_x U = U^2(g + tg') \quad \text{a.e. in} \quad \langle 0,\infty\rangle \times \mathbf{R} \\ U(0,\cdot) = \dfrac{1}{\gamma} > 0 \end{cases}$$

exists until time $t_c = \gamma$, as it is given by

$$U(t,x) = \frac{1}{\gamma - tg(x)}.$$

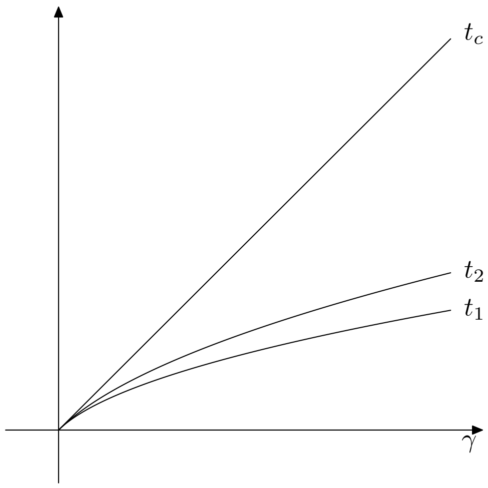
Here the best (the smallest) local bound for above right-hand side that does not depend on t is $\Phi(u) = (1+T)u^2$ (on strip $\langle 0, T\rangle \times \mathbf{R}$), while $h(t,u) = (1+t)u^2$. The solution u_1 of (ODE-t) and the solution u_2 of (ODE-h) satisfy (for $t > 0$)

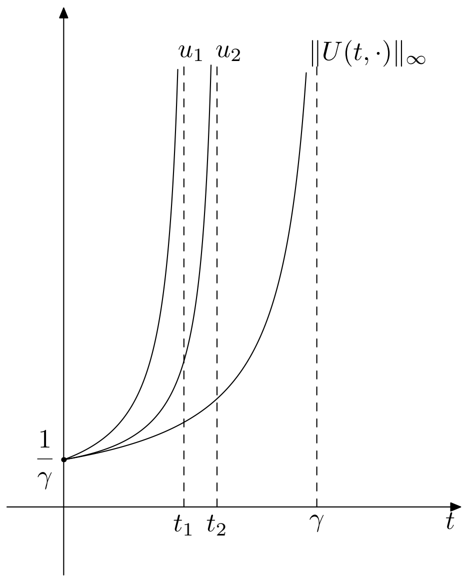
$$\|U(t,\cdot)\|_{\mathrm{L}^\infty(\mathbf{R})} = \frac{1}{\gamma - t} < u_2(t) = \frac{1}{\gamma - t - \frac{1}{2}t^2} < u_1(t) = \frac{1}{\gamma - (1+T)t}\,,$$

and for the (best possible) corresponding times of existence we have

$$T = t_1 = \frac{-1+\sqrt{1+4\gamma}}{2} < t_2 = -1 + \sqrt{1+2\gamma} < t_c = \gamma\,.$$

This is represented in the figures below.





∎

4. L^p case

Let $p \in \langle 1, +\infty\rangle$, and suppose that the initial condition and the right-hand side satisfy the following assumptions:

$$\text{(A3)}\qquad \begin{aligned}
&\bullet\ \mathsf{v} \in \mathrm{L}^p(\mathbf{R}^d;\mathbf{R}^r)\,;\\
&\bullet\ \mathsf{f} : \Omega \times \mathbf{R}^r \longrightarrow \mathbf{R}^r \text{ is mesurable}\,;\\
&\bullet\ (\exists\,\Psi \in \mathrm{L}^\infty_{\mathrm{loc}}(\mathbf{R}))(\forall\, r_1 > 0)(\forall\,\mathbf{w},\mathbf{z} \in \mathrm{K}_{\mathrm{L}^p(\mathbf{R}^d;\mathbf{R}^r)}[0,r_1])\\
&\qquad\qquad \|\mathsf{f}(t,\cdot,\mathbf{w}(\cdot)) - \mathsf{f}(t,\cdot,\mathbf{z}(\cdot))\|_{\mathrm{L}^p} \le \Psi(r_1)\|\mathbf{w}-\mathbf{z}\|_{\mathrm{L}^p} \quad (\text{a.e. } t \in [0,T])\,;\\
&\bullet\ (\exists\,\Phi \in \mathrm{L}^\infty_{\mathrm{loc}}([0,T]\times\mathbf{R}))(\forall\, r_1 > 0)(\forall\,\mathbf{w} \in \mathrm{K}_{\mathrm{L}^p(\mathbf{R}^d;\mathbf{R}^r)}[0,r_1])\\
&\qquad\qquad\qquad \|\mathsf{f}(t,\cdot,\mathbf{w}(\cdot))\|_{\mathrm{L}^p} \le \Phi(t,r_1) \quad (\text{a.e. } t \in [0,T])\,.
\end{aligned}$$

Theorem 4. *Let assumptions (A1) and (A3) hold, and let C_p be the constant from Theorem 8. Assume that $u : [0,S\rangle \longrightarrow \mathbf{R}$ is an absolutely continuous solution of*

$$\begin{cases} u'(t) = C_p\Phi(t,u(t)) \\ u(0) = C_p\|v\|_{\mathrm{L}^\infty(\mathbf{R}^d)} \end{cases},$$

for some $S \in \langle 0,T]$. Then there exists a unique function $\mathsf{u} \in \mathrm{L}^\infty_{\mathrm{loc}}([0,S\rangle;\mathrm{L}^p(\mathbf{R}^d;\mathbf{R}^r))$, which is a weak solution of problem (CP) on $\langle 0,S\rangle \times \mathbf{R}^d$ (note that the definition of weak solution is meaningful under assumptions (A1) and (A3)). Additionally, u satisfies the estimate

$$\|\mathsf{u}(t,\cdot)\|_{\mathrm{L}^p(\mathbf{R}^d;\mathbf{R}^r)} \le u(t) \quad (\text{a.e. } t \in \langle 0,S\rangle)\,.$$

∎

We do not present the proof of this theorem, as it is analogous to the proof of Theorem 1.

Remark. Assumptions (A3), when compared to assumptions (A2), are more difficult to verify. They also appear to be relatively restrictive on function f, as they impose a sublinear growth in variable $\mathbf{w}$ of function f, which is well known from the theory of Nemitski operators. ∎

Appendix

Let $h : [-T,T]\times\mathbf{R} \longrightarrow \mathbf{R}$ be a function from $\mathrm{L}^\infty_{\mathrm{loc}}([-T,T]\times\mathbf{R})$, which is also locally Lispchitz in the second variable:

$$(\exists\,\Psi \in \mathrm{L}^\infty_{\mathrm{loc}}(\mathbf{R}))(\forall\, u,v \in \mathbf{R}) \quad |u| \ge |v| \Longrightarrow |h(t,u) - h(t,v)| \le \Psi(|u|)|u-v| \quad (\text{a.e. } t \in [-T,T]).$$

We are interested in finding a solution u of the Cauchy problem

$$\text{(ODE-h)}\qquad \begin{cases} u'(t) = h(t,u(t)) \qquad (\text{a.e. } t) \\ u(0) = u_0 \end{cases},$$

for some $u_0 \in \mathbf{R}$. Using classical Picard's iterative process one can construct a sequence that converges to a solution of (ODE-h) (in an appropriate class), thus proving the following theorem (uniqueness can be proved by Gronwal's lemma).

Theorem 5. *Under the above assumption on h, there is $S > 0$ and unique function u from $\mathrm{W}^{1,\infty}_{\mathrm{loc}}(\langle -S,S\rangle)$ that is s solution of (ODE-h) on $\langle -S,S\rangle$.* ∎

In the same fashion as in the classical (smooth) case one can prove the existence of maximal solution.

Theorem 6. *There is a maximal (open) interval $I \ni 0$ and unique maximal solution $u \in \mathrm{W}^{1,\infty}_{\mathrm{loc}}(I)$ of problem (ODE-h). More precisely, if $J \subseteq \langle -T,T\rangle$ is an open interval, and $v \in \mathrm{W}^{1,\infty}_{\mathrm{loc}}(J)$ some solution of (ODE-h), then we have $J \subseteq I$ and $v = u_{|J}$.* ∎

Lemma 1. *If $u \in \mathrm{W}^{1,\infty}_{\mathrm{loc}}(\langle S_1, S_2\rangle)$ is some solution of (ODE-h), and*

$$\lim_{t\to S_2^-} u(t) \in \mathbf{R}\,,$$

then $u \in \mathrm{W}^{1,\infty}_{\mathrm{loc}}(\langle S_1, S_2])$.

Dem. It is obvious that $u \in \mathrm{L}^{\infty}_{\mathrm{loc}}(\langle S_1, S_2])$. Therefore, for each $\varepsilon > 0$ we have $u \in \mathrm{L}^{\infty}([S_1+\varepsilon, S_2])$ and, as h is locally bounded, for $s, t \in [S_1+\varepsilon, S_2]$ it holds

$$|u(t)-u(s)| = \Big|\int_s^t h(\sigma, u(\sigma))d\sigma\Big| \le \int_s^t |h(\sigma, u(\sigma))|d\sigma \le C_\varepsilon|t-s|\,,$$

for some constant $C_\varepsilon > 0$. It follows that u is a Lipschitz function on $[S_1+\varepsilon, S_2]$, for each $\varepsilon > 0$, which implies $u \in \mathrm{W}^{1,\infty}_{\mathrm{loc}}(\langle S_1, S_2])$.

Q.E.D.

Theorem 7. *Additionally suppose that $h \ge 0$ and $u_0 \ge 0$. If $u \in \mathrm{W}^{1,\infty}_{\mathrm{loc}}(\langle \alpha, \beta\rangle)$ is the maximal solution of (ODE-h), and $\beta < T$, then*

$$\lim_{t\to\beta^-} u(t) = +\infty\,.$$

Dem. First note that $h \ge 0$ and $u_0 \ge 0$ implies $u \ge 0$. Now suppose that $\lim_{t\to\beta^-} u(t) = u_\beta \in \mathbf{R}$ (this limit always exists as u is continuous). By Theorem 5 there exists $\varepsilon > 0$ and $\tilde u \in \mathrm{W}^{1,\infty}_{\mathrm{loc}}(\langle \beta-\varepsilon, \beta+\varepsilon\rangle)$ that locally solves

$$\begin{cases} \tilde u'(t) = h(t, \tilde u(t)) & \text{(a.e. } t) \\ \tilde u(\beta) = u_\beta \end{cases}.$$

Let $v : \langle \alpha, \beta+\varepsilon\rangle \longrightarrow \mathbf{R}$ be the function defined by

$$v(t) := \begin{cases} u(t)\,, & t \in \langle \alpha, \beta] \\ \tilde u(t)\,, & t \in [\beta, \beta+\varepsilon\rangle \end{cases}.$$

By the preceding lemma $u \in \mathrm{W}^{1,\infty}_{\mathrm{loc}}(\langle \alpha, \beta])$, which together with $\tilde u \in \mathrm{W}^{1,\infty}_{\mathrm{loc}}(\langle \beta-\varepsilon, \beta+\varepsilon\rangle)$ implies that $v \in \mathrm{W}^{1,\infty}_{\mathrm{loc}}(\langle \alpha, \beta+\varepsilon\rangle)$. As v is clearly a solution of (ODE-h) this contradicts the fact that u is the maximal solution of (ODE-h).

Q.E.D.

Remark. The above theorem states that *if the maximal solution stops before time T, it is due to the blowup.*

∎

Theorem 8. *Let $p \in \langle 1, \infty]$, and $\mathsf{a} \in \mathrm{L}^{\infty}(\Omega; \mathbf{R}^d)$ be a function that satisfies*

$$(\exists\, A > 0)(\forall\, \mathbf{x}, \mathbf{y} \in \mathbf{R}^d) \quad |\mathsf{a}(t,\mathbf{x}) - \mathsf{a}(t,\mathbf{y})| \le A|\mathbf{x}-\mathbf{y}| \quad \text{(a.e. } t \in \langle 0, T\rangle)\,.$$

Furthermore, let $v \in \mathrm{L}^p(\mathbf{R}^d)$ and $f \in \mathrm{L}^1([0,T]; \mathrm{L}^p(\mathbf{R}^d))$. Then the Cauchy problem

$$\begin{cases} \partial_t U(t,\mathbf{x}) + \mathsf{a}(t,\mathbf{x}) \cdot \nabla U(t,\mathbf{x}) = f(t,\mathbf{x}) & \text{in} \quad \Omega \\ U(0,\cdot) = v \end{cases}$$

has unique weak solution U in class $\mathrm{L}^{\infty}([0,T]; \mathrm{L}^p(\mathbf{R}^d))$, in the sense

$$(\forall\, \varphi \in \mathrm{C}^1_c([0,T\rangle \times \mathbf{R}^d)) \quad -\int_0^T\!\!\int_{\mathbf{R}^d} U[\partial_t\varphi + \mathsf{div}\,(\varphi\mathsf{a})]d\mathbf{x}dt = \int_0^T\!\!\int_{\mathbf{R}^d} f\varphi d\mathbf{x}dt + \int_{\mathbf{R}^d} v\varphi(0,\cdot)d\mathbf{x}\,.$$

Additionally, there is a constant C_p depending on p, T and a, such that the estimate

$$\|U(t,\cdot)\|_{\mathrm{L}^p(\mathbf{R}^d)} \le C_p\Big(\|v\|_{\mathrm{L}^p(\mathbf{R}^d)} + \int_0^t \|f(s,\cdot)\|_{\mathrm{L}^p(\mathbf{R}^d)}ds\Big)\,, \qquad \text{(a.e. } t \in [0,T])$$

holds. In particular $C_\infty = 1$.

∎

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