Odjel za matematiku
Sveučilište Josipa Jurja Strossmayera u Osijeku

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Serije: Rukopisi u pripremi

Series: Technical Reports

Dimension Reduction for Damping Optimization in Linear Vibrating Systems

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Abstract

We consider a mathematical model of a linear vibrational system described by the second-order differential equation $M\ddot{x} + D\dot{x} + Kx = 0$, where M and K are positive definite matrices, called mass and stiffness, respectively. We consider the case where the damping matrix D is positive semidefinite. The main problem considered in the paper is the construction of an efficient algorithm for calculating an optimal damping. As optimization criterion we use the minimization of the average total energy of the system which is equivalent to the minimization of the trace of the solution of the corresponding Lyapunov equation $AX + XA^T = -I$, where A is the matrix obtained from linearizing the second-order differential equation. Finding the optimal D such that the trace of X is minimal is a very demanding problem, caused by the large number of trace calculations, which are required for bigger matrix dimensions. We propose a dimension reduction to accelerate the optimization process. We will present an approximation of the solution of the structured Lyapunov equation and a corresponding error bound for the approximation. Our algorithm for efficient approximation of the optimal damping is based on this approximation. Numerical results illustrate the effectiveness of our approach.

AMS subject classifications (MSC 2010). 74P10, 70J10, 15A24, 37J15.

Keywords. Vibrating system, Lyapunov equation, energy minimization, dimension reduction, error bounds.

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1 Introduction

We consider a mathematical model of a linear vibrational system:

$$M\ddot{x} + D\dot{x} + Kx = 0, \tag{1}$$

where $D = C_u + C_{ext}$ is the damping matrix, C_u represents the internal damping, and $C_{ext} = vC_0$ is a semidefinite matrix which describes external damping with viscosity v. The matrices M, D and K (called mass, damping and stiffness, respectively) are real, symmetric matrices of order n with M and K positive definite. The internal damping C_u is usually taken to be a small multiple of the critical damping that is,

$$C_u = \alpha C_{crit}, \qquad C_{crit} = M^{1/2}\sqrt{M^{-1/2}KM^{-1/2}}M^{1/2}, \quad \alpha = 2\text{–}10\%. \tag{2}$$

Equation (1) can be transformed to the phase space which yields a system of first order differential equations. For that purpose let Φ be a matrix that simultaneously diagonalizes M and K, i.e.,

$$\Phi^T K\Phi = \Omega^2 = \mathrm{diag}(\omega_1^2, \ldots, \omega_n^2) \quad \text{and} \quad \Phi^T M\Phi = I. \tag{3}$$

We can write the differential equation (1) in phase space as

$$\frac{d}{dt}\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} 0 & \Omega \\ -\Omega & -\Phi^T D\Phi \end{bmatrix}\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} \tag{4}$$

$$\text{or} \qquad \dot{y} = Ay,$$

where

$$A = \begin{bmatrix} 0 & \Omega \\ -\Omega & -\Phi^T D\Phi \end{bmatrix}, \quad y = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}, \tag{5}$$

for more details see [8, 16, 20, 21].

Now, we have the first order differential equation

$$\dot{y} = Ay,$$

with the solution

$$y = e^{At}y_0, \quad \text{where } y_0 \text{ containes the initial data.}$$

For (M, D, K) from our problem it can be shown that the eigenvalues of A are in the left half of the complex plane, see, e.g., [18, 22], that is, A is (asymptotically) stable or *Hurwitz*.

For given masses and stiffnesses we want to determine the "best" damping matrix D which insures optimal evanescence of each component of y. Such problems

require a certain optimization criterion. One criterion is the so-called spectral abscissa criterion, which requires that the maximal real part of the eigenvalues of the corresponding quadratic eigenvalue problem is minimized. The spectral abscissa is defined by $\mu(A) := \max_k \operatorname{Re} \lambda_k$, where λ_k are the complex eigenvalues of the corresponding quadratic matrix polynomial, i.e., they satisfy

$$(\lambda^2 M + \lambda D + K)x = 0. \tag{6}$$

Then the spectral abscissa criterion is equivalent to minimization of $\mu(A)$.

Here, we will use another criterion which is based on the minimization of the total energy of the system:

$$\int_0^\infty E(t)dt \quad \rightarrow \quad \min. \tag{7}$$

Criterion (7) depends on the initial data. In order to overcome this problem, we take the average over all initial states of the unit total energy and a given frequency range. It can be shown [16] that with this averaging our criterion is equivalent to

$$\operatorname{trace} X \quad \rightarrow \quad \min, \tag{8}$$

where X is the solution of the Lyapunov equation

$$AX + XA^T = -Z \tag{9}$$

and A is as in (5). In this paper we will consider the case when $Z = I$ which corresponds to the case when all eigenfrequencies of the undamped system are damped. The structure of Z has been studied in [16]. The problem of partial damping will be considered in a forthcoming paper.

Damping optimization using criterion (8) requires solving the Lyapunov equation (9) numerous times. For example, in Section 6 we will show that the determination of the optimal damping for an oscillator with 1500 masses and two dampers of the same viscosity requires solving more than 16 million Lyapunov equations with coefficient matrices of dimension 3000×3000. Solving such a large number of Lyapunov equations is very expensive. Thus our main problem will be the construction of an efficient solver for the Lyapunov equation (9). Our method is based on a dimension reduction.

There are several methods for dimension reduction of second-order systems. This problem is attacked using balanced truncation in [7, 10, 15]. Dimension reduction using Krylov methods is proposed in [3, 14]. Several methods for dimension reduction are also given in [1, 2, 6]. Unfortunately, for efficiency, these methods all require the right-hand side Z of (8) to be of low rank and are thus not applicable in the case considered here.

In our approach (using the structure of the system (1)), we construct the truncation matrix Q_r, with the column rank equal to r, such that system (1) can be approximated by two systems. One system will have a dense damping matrix of dimension r, this system is

$$M_r\ddot{y} + D_r\dot{y} + K_r y = 0, \tag{10}$$

where $M_r = Q_r^T M Q_r$, $D_r = Q_r^T D Q_r$, $K_r = Q_r^T K Q_r$; Q_r is an $n \times r$ matrix. The second one is obtained with the complimentary[1] truncation matrix Q_{n-r}, and the resulting system in the linearized form will have a block diagonal matrix of order $2(n-r)$. For this case we do not need to form the solution explicitly since in this case the trace of the solution of the corresponding Lyapunov equation can be expressed by an analytic formula.

The paper is organized as follows. The basic idea of dimension reduction is explained in Section 2. Generally, our approximation strategy for the solution of the Lyapunov equation is presented in Section 3. The corresponding error bound is given in Section 4. In Section 5 we present a method for the calculation of the optimal viscosity, this method uses results obtained in Sections 3 and 4. A numerical example is presented in Section 6.

Throughout the paper we will use the following notation. The symbol $\|\cdot\|$ stands for the standard 2-norm (Euclidian/spectral norm), $\|\cdot\|_F$ denotes the Frobenius norm, and $\mathrm{sep}(A, B)$ represents the separation of the matrices A and B. If p and q are vectors, then the notation $A(p, q)$, as used in MATLAB®, denotes the submatrix of A obtained by intersection of rows determined by the elements of p and columns determined by the elements of the vector q. Another MATLAB notation is also used: $i : j$ denotes the vector of integers from i to j.

2 Solving Lyapunov equations using dimension reduction: basic idea

For this approach we need the assumption that the system has an internal damping C_u, such that M, C_u, K can be simultaneously diagonalized. It is easy to show that C_u from (2) satisfies $\Phi^T C_u \Phi = \alpha\Omega$.

Furthermore, for $C_{ext} = 0$, the quadratic eigenvalue problem

$$(\lambda^2 M + \lambda C_u + K)x = 0 \tag{11}$$

corresponds to equation (1). Let Φ and Ω be matrices such that

$$\Phi^T K \Phi = \Omega^2 = \mathrm{diag}(\omega_1^2, \ldots, \omega_n^2), \quad \Phi^T M \Phi = I \quad \text{and} \quad \Phi^T C_u \Phi = \alpha\Omega, \tag{12}$$

[1] Complementary is meant in the sense that the columns of Q_{n-r} are orthogonal to those of Q_r with respect to the inner product induced by M.

where $\omega_1 \geq \omega_2 \geq \ldots \geq \omega_n$ are the eigenvalues of the undamped system $(\lambda^2 M + K)x = 0$ and the columns of Φ are eigenvectors of (11).

Before we show how we will construct the truncation matrix Q_r we will discuss the structure of the considered Lyapunov equation (9). After the perfect shuffle permutation P we obtain the following Lyapunov equation

$$A_P X_P + X_P A_P^T = -I\,, \tag{13}$$

where

$$A_P = A_0 + P^T \begin{bmatrix} 0 & 0 \\ 0 & C \end{bmatrix} P, \quad C = \Phi^T C_{ext} \Phi, \tag{14}$$

$$A_0 = \widehat{A}_1 \oplus \widehat{A}_2 \oplus \cdots \oplus \widehat{A}_n, \quad \widehat{A}_i = \begin{bmatrix} 0 & \omega_i \\ -\omega_i & -\alpha\omega_i \end{bmatrix}. \tag{15}$$

Our approach in based on the simple fact that if $C_{ext} = 0$, then the solution of the Lyapunov equation is simply given by

$$X_P = \widehat{X}_1 \oplus \widehat{X}_2 \oplus \cdots \oplus \widehat{X}_n,$$

where

$$\widehat{X}_i = \frac{1}{\omega_i} \begin{bmatrix} \frac{2+\alpha^2}{2\,\alpha} & -\frac{1}{2} \\ -\frac{1}{2} & \frac{1}{\alpha} \end{bmatrix}, \tag{16}$$

and the trace of the solution X_P is equal to the trace of X since P is a permutation matrix. This yields

$$\operatorname{trace} X = \operatorname{trace} X_P = (\frac{2}{\alpha} + \frac{\alpha}{2}) \sum_{i=1}^{n} \frac{1}{\omega_i}. \tag{17}$$

Furthermore, if $C_{ext} \neq 0$, then the matrix A_P in the Lyapunov equation (13) has the following form:

$$A_P = \begin{bmatrix} 0 & \omega_1 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 \\ -\omega_1 & -\alpha\omega_1 - c_{11} & 0 & -c_{12} & 0 & -c_{13} & \cdots & 0 & -c_{1n} \\ 0 & 0 & 0 & \omega_2 & 0 & 0 & \cdots & 0 & 0 \\ 0 & -c_{12} & -\omega_2 & -\alpha\omega_2 - c_{22} & 0 & -c_{23} & \cdots & 0 & -c_{2n} \\ 0 & 0 & 0 & 0 & 0 & \omega_3 & \cdots & 0 & 0 \\ 0 & -c_{13} & 0 & -c_{23} & -\omega_3 & -\alpha\omega_3 - c_{33} & \cdots & 0 & -c_{3n} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & \omega_n \\ 0 & -c_{1n} & 0 & -c_{2n} & 0 & -c_{3n} & \cdots & -\omega_n & -\alpha\omega_n - c_{nn} \end{bmatrix},$$

where $c_{ij} = (C)_{ij}$ (note that C is a symmetric matrix).

We will illustrate our main idea using the following example. If some part of the matrix C has small norm, for example if $\|C(:, r+1:n)\|$ (the last $n-r$ columns) is small, then we approximate A_P by

$$\widetilde{A}_P = \begin{bmatrix} \widetilde{A}_{11} & 0 \\ 0 & \widetilde{A}_{22} \end{bmatrix}, \tag{18}$$

where

$$\widetilde{A}_{11} = A_P(1:2r, 1:2r) \quad \text{and} \quad \widetilde{A}_{22} = \widehat{A}_{r+1} \oplus \cdots \oplus \widehat{A}_n, \tag{19}$$

and $\widehat{A}_i$ is defined in (15).

Now, the solution of the Lyapunov equation (13) can be approximated by the solution of the Lyapunov equation

$$\widetilde{A}_P \widetilde{X}_P + \widetilde{X}_P \widetilde{A}_P^T = -I\,, \tag{20}$$

where $\widetilde{A}_P$ is given by (18), (19) and $\widetilde{X}_P = X_r \oplus \widetilde{X}_{22}$. The block-diagonal part X_r is the solution of the Lyapunov equation

$$\widetilde{A}_{11} X_r + X_r \widetilde{A}_{11}^T = -I, \tag{21}$$

which corresponds to the Lyapunov equation obtained from the system (10) using $Q_r = \Phi(:, 1:r)$. Since, $\widetilde{X}_{22} = \widehat{X}_{r+1} \oplus \cdots \oplus \widehat{X}_n$ where $\widehat{X}_i$ is from (16) it follows that the solution of (9) has the form

$$\widetilde{X}_P = \begin{bmatrix} X_r & 0 \\ 0 & \widetilde{X}_{22} \end{bmatrix}.$$

Note that $\widetilde{X}_{22}$ is the solution which corresponds to the Lyapunov equation obtained from the system (10) using $Q_{n-r} = \Phi(:, r+1:n)$ and setting $C_{ext} = 0$.

Now, using formula (17), the trace of the solution (9) can be approximated by

$$\operatorname{trace} X \approx \operatorname{trace} X_r + (\frac{2}{\alpha} + \frac{\alpha}{2}) \sum_{i=r+1}^{n} \frac{1}{\omega_i}.$$

The question is: *when can we expect that the norm of some part of the matrix $C = \Phi^T C_{ext} \Phi$ will be small?*

Note that the elements of the matrix Φ are closely related to displacements of the corresponding modes, particularly the ith mode of the system is determined by the ith column of Φ. Mass displacements are sinusoidal with the same frequency and the extreme value of displacement for the ith mode at the kth mass is equal to Φ_{ki} (for more details see [12]).

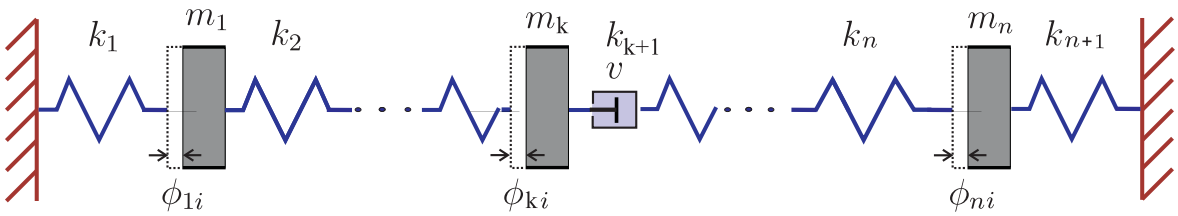

Figure 1: displacements for the ith mode are shown at their extreme values

As illustrative example consider the mechanical system shown in Figure 1, with one damper at position k with viscosity v. This means that $C = v\Phi^T e_k e_k^T \Phi$ and $c_{ij} = v\phi_{ki}\phi_{kj}$ where $\phi_{ij} = (\Phi)_{ij}$. From Figure 1 it can be seen that if some masses are larger then the others, the corresponding displacements will be smaller. That justifies our assumption that some entries of the matrix C will be small, indeed since $c_{ij} = v\phi_{ki}\phi_{kj}$ for small ϕ_{ki} and ϕ_{kj}, c_{ij} will be small, too.

Based on the presented ideas, we will show in the next section how one can find an approximate solution of the Lyapunov equation (9) in general.

3 An algorithm for the approximate solution of the Lyapunov equation (9)

We will present how we to calculate the approximation of the solution in the general case. Let the viscosity and damper positions be given, and set $C = \Phi^T C_{ext} \Phi$. Furthermore, let $\overline{p} \in \mathbb{R}^{n-r}$ be a vector of indices for which $n - r$ is the maximal (r is minimal) dimension such that $\|C(\overline{p}, :)\|_F$ is less or equal than some tolerance tol.

The vector $p \in \mathbb{R}^r$ is chosen such that $p \cup \overline{p} = \{1, 2, \ldots, n\}$. Then we define the vector $w \in \mathbb{R}^n$ with $w(i) = p(i)$ for $i = 1, \ldots, r$ and $w(i) = \overline{p}(i-r)$ for $i = r+1, \ldots, n$.

Now, we solve the Lyapunov equation

$$\widehat{P}^T A_P \widehat{P} \widehat{X}_P + \widehat{X}_P \widehat{P}^T A_P^T \widehat{P} = -I\,, \tag{22}$$

instead of the Lyapunov equation (13). In (22), the matrix $\widehat{P}$ is an additional permutation matrix such that the matrix $\widehat{P}^T A_P \widehat{P}$ has the following structure:

$$\widehat{A}_P = \widehat{P}^T A_P \widehat{P} = \begin{bmatrix} 0 & \omega_{w(1)} & \cdots & 0 & 0 \\ -\omega_{w(1)} & -\alpha\omega_{w(1)} - c_{w(1)w(1)} & \cdots & 0 & -c_{w(1)w(n)} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & \omega_{w(n)} \\ 0 & -c_{w(1)w(n)} & \cdots & -\omega_{w(n)} & -\alpha\omega_{w(n)} - c_{w(n)w(n)} \end{bmatrix}. \tag{23}$$

Since $\widehat{P}$ is a permutation matrix, it holds that $\operatorname{trace} \widehat{X}_P = \operatorname{trace} X$, where X is the solution of (9).

Now, the solution of the corresponding Lyapunov equation is approximated in the same way as in the previous section. That is, an approximation of the solution of the Lyapunov equation (9) can be obtained from the following Lyapunov equation

$$\widetilde{A}_P \widehat{X}_P + \widehat{X}_P \widetilde{A}_P^T = -I\,, \tag{24}$$

where

$$\widetilde{A}_P = \begin{bmatrix} \widetilde{A}_{11} & 0 \\ 0 & \widetilde{A}_{22} \end{bmatrix} \quad \text{with} \quad \begin{array}{l} \widetilde{A}_{11} = \widehat{A}_P(1:2r, 1:2r), \\ \widetilde{A}_{22} = \widehat{A}_{\overline{p}(1)} \oplus \cdots \oplus \widehat{A}_{\overline{p}(n-r)}, \end{array} \tag{25}$$

and $\widehat{A}_i$ is defined in (15) while $\widehat{A}_P$ is given in (23).

Let $\widetilde{X}_r$ be the solution of the Lyapunov equation

$$\widetilde{A}_{11} \widetilde{X}_r + \widetilde{X}_r \widetilde{A}_{11}^T = -I, \tag{26}$$

then the approximation of the solution of the Lyapunov equation is given by

$$\widetilde{X}_P = \begin{bmatrix} X_r & 0 \\ 0 & \widetilde{X}_{22} \end{bmatrix},$$

with $\widetilde{X}_{22} = \widehat{X}_{\overline{p}(r+1)} \oplus \cdots \oplus \widehat{X}_{\overline{p}(n)}$ and the matrices $\widehat{X}_i$ from (16).

As we are interested in the trace, now the trace of the solution (24) can be calculated using

$$\operatorname{trace} X \approx \operatorname{trace} \widetilde{X}_r + \big(\frac{2}{\alpha} + \frac{\alpha}{2}\big) \sum_{i=1}^{n-r} \frac{1}{\omega_{\overline{p}(i)}}, \tag{27}$$

where $\widetilde{X}_r$ is solution of the Lyapunov equation (26).

Equation (26) corresponds to the Lyapunov equation which is obtained from the reduced system (10) using the truncation matrix $Q_r = \Phi(:, p)$. Now $\widetilde{X}_{22}$ is the solution which corresponds to the Lyapunov equation obtained from the system (10) using $Q_{n-r} = \Phi(:, \overline{p})$ and setting $C_{ext} = 0$.

As it can be seen from (27), the approximation of trace X is obtained by summing the trace of the solution of the Lyapunov equation (26) obtained from the reduced system and the trace of the solution of the Lyapunov equation $\widetilde{A}_{22}\widetilde{X}_{22} + \widetilde{X}_{22}\widetilde{A}_{22}^T = -I$. ¿From the computational point of view, the most expensive part is solving the Lyapunov equation (26). Summarizing, we have reduced the problem of solving a $2n \times 2n$ Lyapunov equation to solving a Lyapunov equation of dimension $2r \times 2r$. In this sense we will call the parameter r the reduced dimension. The resulting method is summarized in Algorithm 1.

Note that in the optimization process we only need trace X instead of the solution X. Thus for optimization, instead of Step 5. of Algorithm 1 we will use the formula $\operatorname{trace} X = \operatorname{trace} X_r + (\frac{2}{\alpha} + \frac{\alpha}{2}) \sum_{i=1}^{n-r} \frac{1}{\omega_{\overline{p}(i)}}$.

In the following section we will derive an error bound for the approximation of the solution of the Lyapunov equation (9) obtained by the previous algorithm.

Algorithm 1 (Approximation solution of the Lyapunov equation (9))

Input: α, tol, Φ such that $\Phi^T K \Phi = \Omega^2 = \mathrm{diag}(\omega_1^2, \ldots, \omega_n^2)$ and $\Phi^T M \Phi = I$,
C (C contains information about the dampers' positions and viscosities).

Output: $X(C)$

1: Determine minimal r and vector $\overline{p} \in \mathbb{R}^{n-r}$ such that $\|C(\overline{p}, :)\|_F < tol$, $i = 1, \ldots, n-r$.

2: Determine vector $p \in \mathbb{R}^r$ such that $p \cup \overline{p} = \{1, 2, \ldots, n\}$.

3: $\Omega_r = \mathrm{diag}(\omega_{p(1)}, \omega_{p(2)}, \ldots, \omega_{p(r)})$, $\quad C_r = C(p, p)$.

4: Calculate X_r, where

$$A_r X_r + X_r A_r^T = -I, \quad A_r = \begin{bmatrix} 0 & \Omega_r \\ -\Omega_r & -\alpha\Omega_r - C_r \end{bmatrix}.$$

5: $\widetilde{X}_P = X_r \oplus \widehat{X}_{\overline{p}(r+1)} \oplus \cdots \oplus \widehat{X}_{\overline{p}(n)}$, where $\widehat{X}_i$ is given in (16).

6: $X(C) = P\widehat{P}\widetilde{X}_P\widehat{P}^T P^T$, where the matrix P is the perfect shuffle permutation and the permutation matrix $\widehat{P}$ is introduced in (22).

4 Error bound

Consider the Lyapunov equation (22) in the partitioned form

$$\begin{bmatrix} A_{11} & \varepsilon E \\ \varepsilon E^T & A_{22} \end{bmatrix} \begin{bmatrix} X_{11} & X_{12} \\ X_{12}^T & X_{22} \end{bmatrix} + \begin{bmatrix} X_{11} & X_{12} \\ X_{12}^T & X_{22} \end{bmatrix} \begin{bmatrix} A_{11}^T & \varepsilon E \\ \varepsilon E^T & A_{22}^T \end{bmatrix} = -I,$$

where $A = \begin{bmatrix} A_{11} & \varepsilon E \\ \varepsilon E^T & A_{22} \end{bmatrix}$, $X = \begin{bmatrix} X_{11} & X_{12} \\ X_{12}^T & X_{22} \end{bmatrix}$, $\varepsilon = \|A_{12}\|_F$ and $E = \frac{1}{\|A_{12}\|_F} \cdot A_{12}$.

By equalizing the corresponding blocks we obtain

$$A_{11}X_{11} + X_{11}A_{11}^T = -I - \varepsilon(EX_{12}^T + X_{12}E), \tag{28}$$
$$A_{22}X_{22} + X_{22}A_{22}^T = -I - \varepsilon(E^T X_{12} + X_{12}^T E^T), \tag{29}$$
$$A_{11}X_{12} + X_{12}A_{22}^T = -\varepsilon(EX_{22} + X_{11}E^T). \tag{30}$$

Since Equations (28) – (30) can be considered as perturbed Sylvester equations, we will need a perturbation bound for Sylvester equations of the form $AX - XB = C$. For our purpose we will use one from [13]: let

$$(A + \Delta A)(X + \Delta X) - (X + \Delta X)(B + \Delta B) = C + \Delta C$$

be the perturbed equation, then the following bound holds

$$\|\Delta X\|_F \leq \sqrt{3}\|P^{-1}\|\big((\alpha + \beta)\|X\|_F + \gamma\big)\eta, \tag{31}$$

where $P = I \otimes A - B^T \otimes I$ and $\eta = \max\left\{\frac{\|\Delta A\|_F}{\alpha}, \frac{\|\Delta B\|_F}{\beta}, \frac{\|\Delta C\|_F}{\gamma}\right\}$, while α, β and γ are scaling factors as in [13]. Usually one uses $\alpha = \|A\|_F, \beta = \|B\|_F$, and $\gamma = \|C\|_F$, but some other constants for α, β and γ can be used as well.

Note that

$$\|P^{-1}\| = \mathrm{sep}(A, B)^{-1}, \tag{32}$$

where $\mathrm{sep}(A, B)$ is the separation of the matrices A and B defined as

$$\mathrm{sep}(A, B) = \min_{X \neq 0} \frac{\|AX - XB\|_F}{\|X\|_F}.$$

First we will derive the bound for the solution X_{12} of the Sylvester equation (30). In our case we have $A_{22} = \widetilde{A}_{22} + \Delta A_{22}$ with $\widetilde{A}_{22}$ and $\widetilde{A}_{11}$ given in (25) and

$$\Delta A_{22} = \widehat{A}_P(2r+1:n, 2r+1:n) - \widetilde{A}_{22},$$

where $\widehat{A}_P$ is defined as in (23). Note that there is no perturbation in the matrix A_{11}, thus $\Delta A_{11} = 0$. Since ε is small, Equation (30) is a perturbation of the equation

$$A_{11}\widetilde{X}_{12} + \widetilde{X}_{12}\widetilde{A}_{22}^T = 0,$$

with solution $\widetilde{X}_{12} = 0$. Using the bound (31) one can obtain a bound for $X_{12} = \widetilde{X}_{12} + \Delta X_{12}$ with $\eta_1 = \max\left\{\frac{\|\Delta A_{22}\|_F}{\|\widetilde{A}_{22}\|_F}, \frac{\|\varepsilon(EX_{22}+X_{11}E^T)\|_F}{\gamma}\right\}$, where γ is an arbitrary small constant (since the right hand side of Equation (30) is zero): it holds

$$\|X_{12}\|_F = \|\Delta X_{12}\|_F \leq \sqrt{3}\|P_1^{-1}\|\eta_1\gamma,$$

where $P_1 = I \otimes A_{11} + \widetilde{A}_{22} \otimes I$. Since $\|E\|_F = 1$, using the triangle inequality we have

$$\eta_1 \leq \max\left\{\frac{\|\Delta A_{22}\|_F}{\|\widetilde{A}_{22}\|_F}, \varepsilon\frac{\|X_{22}\|_F + \|X_{11}\|_F}{\gamma}\right\}.$$

Setting γ small enough yields

$$\eta_1 \leq \varepsilon\frac{\|X_{22}\|_F + \|X_{11}\|_F}{\gamma},$$

thus we obtain

$$\|X_{12}\|_F = \|\widetilde{X}_{12} + \Delta X_{12}\|_F = \|\Delta X_{12}\|_F \leq \sqrt{3}\|P_1^{-1}\|\|(\|X_{22}\|_F + \|X_{11}\|_F)\varepsilon. \tag{33}$$

Furthermore, we will derive the relative perturbation bound for the Lyapunov equation (28). Note that (28) can be considered as perturbation of the Lyapunov

equation $A_{11}\widetilde{X}_{11} + \widetilde{X}_{11}A_{11}^T = -I$ with $X_{11} = \widetilde{X}_{11} + \Delta X_{11}$. Since there is no perturbation in A_{11}, in (31) we set $\alpha = \beta = 0$ and $\gamma = \|I\|_F = \sqrt{n}$. Then (31) yields

$$\begin{aligned} \|\Delta X_{11}\|_F &\leq \sqrt{3}\|P_2^{-1}\|\|\varepsilon(EX_{12}^T + X_{12}E)\|_F \\ &\leq 2\sqrt{3}\|P_2^{-1}\|\|X_{12}\|_F\varepsilon, \end{aligned} \tag{34}$$

where $P_2 = I \otimes A_{11} + A_{11} \otimes I$.

Now, from (32), (33) and (34) it follows

$$\frac{\|\Delta X_{11}\|_F}{\|X_{11}\|_F + \|X_{22}\|_F} \leq 6\frac{1}{\operatorname{sep}(A_{11}, -\widetilde{A}_{22}^T)}\frac{1}{\operatorname{sep}(A_{11}, -A_{11}^T)}\varepsilon^2. \tag{35}$$

Similarly, one can obtain the relative perturbation bound for the Lyapunov equation (29). Let (29) be the perturbed equation related to $\widetilde{A}_{22}\widetilde{X}_{22} + \widetilde{X}_{22}\widetilde{A}_{22}^T = -I$, where $X_{22} = \widetilde{X}_{22} + \Delta X_{22}$. Here $\alpha = \beta = \|\widetilde{A}_{22}\|_F$ and $\gamma = \sqrt{n}$, thus from (31) it follows

$$\|\Delta X_{22}\|_F \leq \sqrt{3}\|P_3^{-1}\|\big(2\|\widetilde{A}_{22}\|_F\|\widetilde{X}_{22}\|_F + \sqrt{n}\big)\eta_2, \tag{36}$$

where $P_3 = I \otimes \widetilde{A}_{22} + \widetilde{A}_{22} \otimes I$. Since $\|E\|_F = 1$, using the triangle inequality we have

$$\eta_2 \leq \max\left\{\frac{\|\Delta A_{22}\|_F}{\|\widetilde{A}_{22}\|_F}, \varepsilon\frac{\|2X_{12}\|_F}{\sqrt{n}}\right\}. \tag{37}$$

¿From (33) it follows that $\|X_{12}\|_F$ is bounded by an $\mathcal{O}(\varepsilon)$ term, thus for small ε

$$\max\left\{\frac{\|\Delta A_{22}\|_F}{\|\widetilde{A}_{22}\|_F}, \varepsilon\frac{\|2X_{12}\|_F}{\sqrt{n}}\right\} = \frac{\|\Delta A_{22}\|_F}{\|\widetilde{A}_{22}\|_F}.$$

Now, (36) can be written in the form

$$\|\Delta X_{22}\|_F \leq \sqrt{3}\|P_3^{-1}\|\big(2\|\widetilde{X}_{22}\|_F + \tfrac{\sqrt{n}}{\|\widetilde{A}_{22}\|_F}\big)\|\Delta A_{22}\|_F.$$

In order to express the previous bound in relative form, divide the above equation by $\|\widetilde{X}_{22}\|_F$ and express $\|P_3^{-1}\|$ in terms of sep, then we have

$$\frac{\|\Delta X_{22}\|_F}{\|\widetilde{X}_{22}\|_F} \leq \frac{\sqrt{3}}{\operatorname{sep}(\widetilde{A}_{22}, -\widetilde{A}_{22}^T)}\left(2 + \frac{\sqrt{n}}{\|\widetilde{A}_{22}\|_F\|\widetilde{X}_{22}\|_F}\right)\|\Delta A_{22}\|_F. \tag{38}$$

Note that $\|\widetilde{X}_{22}\|_F$ and $\|\widetilde{A}_{22}\|_F$ are easy to calculate since $\widetilde{X}_{22}$ and $\widetilde{A}_{22}$ are block diagonal matrices with 2×2 blocks on their diagonals which can be expressed by analytic formulas.

In summary, we obtain an error bound for the solution of the Lyapunov equation (22) computed by Algorithm 1. As we are interested in the trace, we only provide the bounds for the diagonal blocks of the solution.

Theorem 4.1. *With $A_{11} = A_r$, where A_r is given in Algorithm 1, $\widetilde{A}_{22} = \widehat{A}_{\overline{p}(1)} \oplus \cdots \oplus \widehat{A}_{\overline{p}(n-r)}$, $\Delta A_{22} = C(\overline{p}, \overline{p})$, and X_{ii} $(i = 1, 2)$ being the exact solutions of (28), (29), $\widetilde{X}_{ii}$ their approximations computed by Algorithm 1, the error matrices $\Delta X_{ii} = X_{ii} - \widetilde{X}_{ii}$ satisfy*

$$\frac{\|\Delta X_{11}\|_F}{\|X_{11}\|_F + \|X_{22}\|_F} \leq 6 \frac{1}{\operatorname{sep}(A_{11}, -\widetilde{A}_{22}^T)} \frac{1}{\operatorname{sep}(A_{11}, -A_{11}^T)} \varepsilon^2, \tag{39}$$

$$\frac{\|\Delta X_{22}\|_F}{\|\widetilde{X}_{22}\|_F} \leq \frac{\sqrt{3}}{\operatorname{sep}(\widetilde{A}_{22}, -\widetilde{A}_{22}^T)} \big(2 + \frac{\sqrt{n}}{\|\widetilde{A}_{22}\|_F \|\widetilde{X}_{22}\|_F}\big) \|\Delta A_{22}\|_F, \tag{40}$$

where $\varepsilon = \|C(p, \overline{p})\|_F$ and the vectors $\overline{p}$ and p are calculated in Steps 1. and 2. of Algorithm 1, respectively.

The right-hand sides in (39) and (40) thus also provide the desired, though conservative, bounds for the traces computed using the approximate Lyapunov solution rather than the exact ones. These bounds do not account for numerical errors made due to roundoff during the actual computations in finite-precision arithmetic, but if the analytical formula (16) is used for the $\widehat{X}_i$, X_r is computed by a numerically stable algorithm like the Bartels-Stewart method [4], and ε is significantly larger than machine precision (which will usually be the case in applications), than the bounds (39) and (40) will dominate the numerical errors by far.

We also note that $\|\widetilde{X}_{22}\|_F = \frac{1}{2}\sqrt{(6 + \frac{8}{\alpha^2} + \alpha^2) \sum_{k=1}^{n-r} \omega_{\overline{p}(k)}^{-2}}$.

To make the bounds (35) and (38) operable we must estimate $\operatorname{sep}(\cdot, \cdot)$. This can be done for example using algorithms from [9, 11]. In [9], for the estimation of $\operatorname{sep}(A, B)$, one must solve a Sylvester equation with coefficient matrices A and B. For solving this Sylvester equation with standard solvers (for example, the Bartels-Stewart algorithm [4]), the main costs are the calculation of the Schur decompositions. In our case, we need Schur decompositions of the matrices A_{11} and $\widetilde{A}_{22}$, and these have already been obtained since in the calculation of the approximation of the solution we have already solved Lyapunov equations with the matrices A_{11} and $\widetilde{A}_{22}$. The availability of Schur decompositions cannot be utilized using the Lyapunov solvers in the MATLAB Control System Toolbox™, but this is possible, e.g. in SLICOT (Subroutine Library In COntrol Theory[2]) [17] and the SLICOT Basic Systems and Control Toolbox for MATLAB [5]. Employing the Schur decomposition usually accelerates the Lyapunov solve by a factor of 5 or more. Furthermore, the block diagonal structure of $\widetilde{A}_{22}$ can be used, too, in the estimation of the separation. Thus, separation estimation can be done efficiently.

In the following section we will propose an algorithm which uses these error bounds in the determination of the optimal viscosity for given dampers' positions.

[2]See `http://www.slicot.org`.

5 Calculating the optimal viscosity

The algorithm for viscosity optimization calculates the trace approximation using Algorithm 1. The main idea is to determine the reduced dimension efficiently for a given accuracy. For that purpose we will use the new error bounds (39) and (40).

During the optimization process we do not check the error bounds, except when we determine a suboptimal viscosity. Then we calculate the error bounds (39) and (40), and if the errors are small enough, we stop the optimization process (then our suboptimal viscosity is close to optimal) or if errors are too large we repeat the optimization process with a smaller tolerance, that is, with a larger reduced dimension r. All this is included in Algorithm 2.

Algorithm 2 (Computation of the optimal viscosity at given damper positions)

Input: α, $\kappa \geq 1$, machine precision $\mathbf{u}$, positions $d_1, d_2, \ldots, d_k$ of the dampers;
Φ such that $\Phi^T K \Phi = \Omega^2 = \mathrm{diag}(\omega_1^2, \ldots, \omega_n^2)$ and $\Phi^T M \Phi = I$;
the tolerances ε_1, ε_2 for the relative error bounds (39) and (40), resp.;
the starting tolerance tol_{start} for Algorithm 1;
a constant c_1 for scaling a tolerance ($c_1 < 1$).

Output: Optimal viscosities $\widehat{v}_1, \ldots, \widehat{v}_k$.

1: $tol = tol_{start}$
2: **while** $tol > \kappa \cdot u$ **do**
3: Find suboptimal viscosities with an optimization algorithm (e.g., Nelder-Mead as in the MATLAB function `fminsearch`), in the optimization process calculate the trace $X(v_1, \ldots, v_k)$ using Algorithm 1 with tolerance tol, and denote them with $\widehat{v}_1, \ldots, \widehat{v}_k$.
4: Calculate the right-hand sides of the bounds (39) and (40) for the suboptimal viscosities $\widehat{v}_1, \ldots, \widehat{v}_k$, and denote them by b_1 and b_2, resp.
5: **if** $b_1 < \varepsilon_1$ **and** $b_2 < \varepsilon_2$ **then**
6: **return** optimal viscosities $\widehat{v}_1, \ldots, \widehat{v}_k$
7: **break**
8: **else**
9: $tol = c_1 \cdot tol$
10: **end if**
11: **end while**

Note that in Step 3., each time we start the optimization process for determining optimal viscosities, and in order to accelerate the optimization process our algorithm should use the information of suboptimal viscosities obtained in the previous step. For example, if we optimize with the Nelder-Mead method in Step 3., then in the optimization process the starting values are the suboptimal viscosities obtained in the previous step, except for the first time when we take some fixed starting viscosities.

6 Numerical experiments

In this section we will compare the application of the new algorithm for viscosity optimization using dimension reduction with the same optimization method without dimension reduction, based on the Bartels-Stewart Lyapunov solver. For that purpose we will consider the mechanical system shown in Figure 2 with two dampers of the same viscosity and $3d+1$ masses, consisting of three rows of masses with $d+1$ springs. Each row has springs of the same stiffness equal to k_1, k_2, k_3, respectively. On the left-hand side, rows of the springs are connected to the fixed base, and on the right-hand side they are connected to the last mass (m_{3d+1}), which is connected to the fixed base with the spring with stiffness k_4.

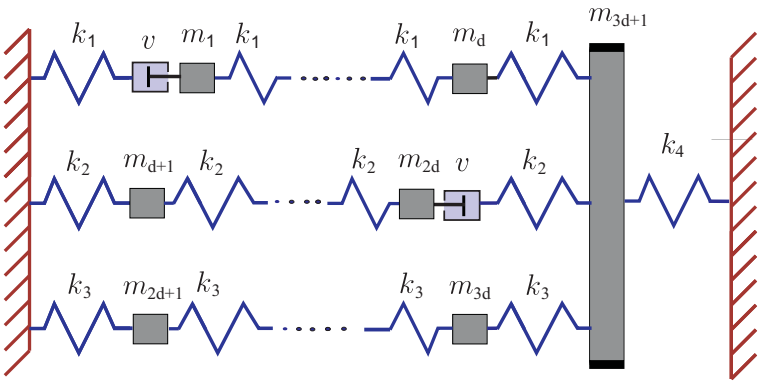

Figure 2: $3d+1$ mass oscillator

The mathematical model for the considered vibrational system is given by Equation (1) where the mass matrix is

$$M = \operatorname{diag}(m_1, m_2, \ldots, m_n).$$

The stiffness matrix is defined as

$$K = \begin{bmatrix} K_{11} & & & -\kappa_1 \\ & K_{22} & & -\kappa_2 \\ & & K_{33} & -\kappa_3 \\ -\kappa_1^T & -\kappa_2^T & -\kappa_3^T & k_1+k_2+k_3+k_4 \end{bmatrix},$$

where

$$K_{ii} = k_i \begin{bmatrix} 2 & -1 & & & \\ -1 & 2 & -1 & & \\ & \ddots & \ddots & \ddots & \\ & & -1 & 2 & -1 \\ & & & -1 & 2 \end{bmatrix}, \quad \kappa_i = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ k_i \end{bmatrix}, \quad i = 1, 2, 3.$$

In our example we will consider the following configuration

$$\begin{aligned} d &= 500, \quad n = 3d + 1 = 1501, \\ m_k &= k, \quad k = 1, \ldots, n, \\ k_1 &= 1, \quad k_2 = 50, \quad k_3 = 100, \quad k_4 = 200. \end{aligned}$$

Our example has two dampers with the same viscosity v. Thus the damping matrix is equal to

$$D = C_u + C_{ext}, \quad \text{where } C_u = 0.01 \cdot M^{1/2} \sqrt{M^{-1/2} K M^{-1/2}} M^{1/2}, \;\; C_{ext} = v e_i e_i^T + v e_j e_j^T.$$

Note that damping optimization for the considered mechanical system with 2 dampers of the same viscosity leads to the problem of discrete optimization of $n(n-1)/2$ different positions (here we use symmetry). For each position one must calculate the optimal viscosity and for this we usually need to solve Lyapunov equations of dimension $2n \times 2n$ more than 15 times. This means that for our configuration for damping optimization we need to solve more than 16 million Lyapunov equations with matrices of dimension $3,002$. On the other hand, the new algorithm for calculation of the optimal viscosity at given dampers' positions reduces the dimension of the Lyapunov equation at each dampers' position and accelerates the optimization process.

Since the considered optimization process is extremely demanding, instead of performing complete optimization (consisting of all dampers' positions) we will relax the problem. We will compare our algorithm with the standard algorithm without dimension reduction on the equidistant mesh of dampers' positions:

$$i = 1 : 70 : n, \quad j = i + 5 : 70 : n,$$

where i is the position of the first damper and j is the position of the second damper, which results in 253 different positions.

In this example, we have two dampers of the same viscosity. Thus in Step 3. of Algorithm 2, for the optimization process, we will use the MATLAB function `fminbnd` with termination tolerances for the viscosity and for the function values equal to 10^{-4}.

Furthermore, in our application we have noticed that in Algorithm 2 if we set $\varepsilon_1 = \varepsilon_2 = 0.2$, then the relative errors for the approximation is good enough (relative errors are presented in Figure 3). The starting tolerance in Algorithm 2 is $tol_{start} = 3 \cdot 10^{-5}$, the constant for scaling the tolerances is $c_1 = 0.2$, and we use $\kappa = 10^4$.

The optimization process with `fminbnd` begins in the interval $[10^{-4}, 10^3]$. Once we have obtained the suboptimal viscosity $\widehat{v}_{opt}$ (in Step 3. of Algorithm 2), we check the error bounds in Step 5. of Algorithm 2. If the error bounds are small enough, the optimization process will be terminated (the suboptimal viscosity is then close

enough to the optimal one). On the other hand, if the errors are not small enough we continue the process with the interval $[\widehat{v}_{opt} - \widehat{v}_{opt} \cdot p,\ \widehat{v}_{opt} + \widehat{v}_{opt} \cdot p]$, where we set $p = 0.01$.

Note that if `fminbnd` finds a suboptimal viscosity $\widehat{v}_{opt}$ at one of the interval boundaries, we will expand the interval around $\widehat{v}_{opt}$, that is we will restart the optimization process in the interval $[\widehat{v}_{opt} - \widehat{v}_{opt} \cdot p,\ \widehat{v}_{opt} + \widehat{v}_{opt} \cdot p]$.

In the following figures we have plotted data which corresponds to the 100 smallest traces sorted by magnitude. The first result corresponds to the optimal position (the position which corresponds to the smallest trace).

Figure 3 shows relative errors for the optimal viscosity and the optimal trace. Relative errors for trace are calculated using $|\operatorname{trace} X - \operatorname{trace} \widehat{X}|/\operatorname{trace} X$, where $\operatorname{trace} X$ is the optimal trace for the given position obtained with the algorithm without dimension reduction, and $\operatorname{trace} \widehat{X}$ is the approximation of the optimal trace calculated with Algorithm 1. Similarly, relative errors for the optimal viscosity are calculated by $|\widehat{v}_{opt} - v_{opt}|/v_{opt}$, where $\widehat{v}_{opt}$ is the optimal viscosity obtained by Algorithm 2 and v_{opt} is the exact optimal viscosity obtained by optimization without dimension reduction.

Both algorithms (with and without dimension reduction) obtain the same optimal damper position $(i, j) = (211, 426)$. The optimal viscosity at this position according to algorithm without dimension reduction is 32.75013 with corresponding trace is $299,0313.07995$, while with Algorithm 2, we obtain 32.74978 for the viscosity and $299,0322.73886$ for the trace.

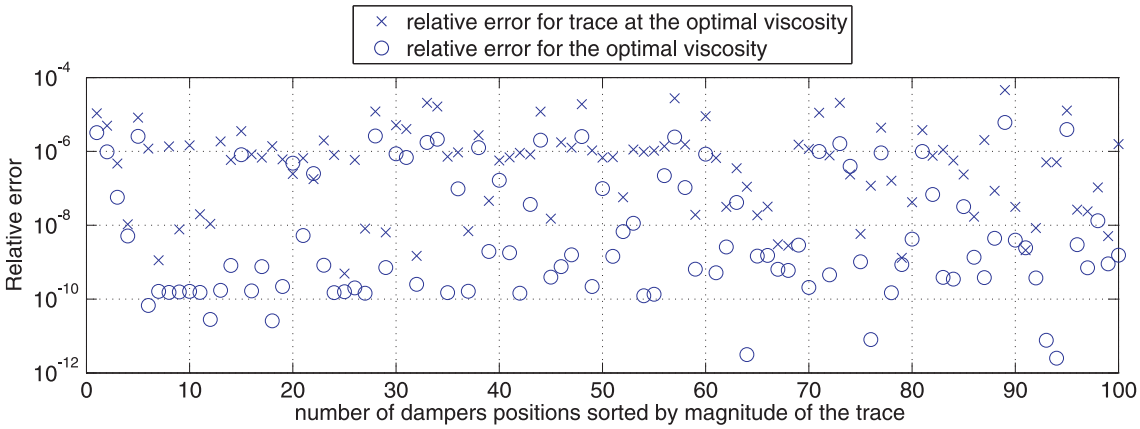

Figure 3: Relative errors

In the optimization process with the function `fminbnd`, tolerances were equal to 10^{-4}, thus relative errors in Figure 3 are in terms of tolerances for the optimization process. For all these results the while loop in Algorithm 2 just needed two iterations in order to satisfy inequalities in Step 5.

Figure 4 shows the percentage of dimension reduction obtained when computing the 100 smallest traces. Each point in Figure 4 corresponds to such a tolerance *tol* in Algorithm 2 that the inequalities in Step 5. of Algorithm 2 are satisfied.

We can see that the reduced dimension varies from 27 % to 86 % of the starting dimension. It is important to note that at the positions which have the smallest traces, the reduced dimension is best. This is good because in calculating the optimal dampers position with some heuristic, such as the one introduced in [19], we must calculate the optimal viscosity in a large number of dampers' positions which are close to the optimal damper position.

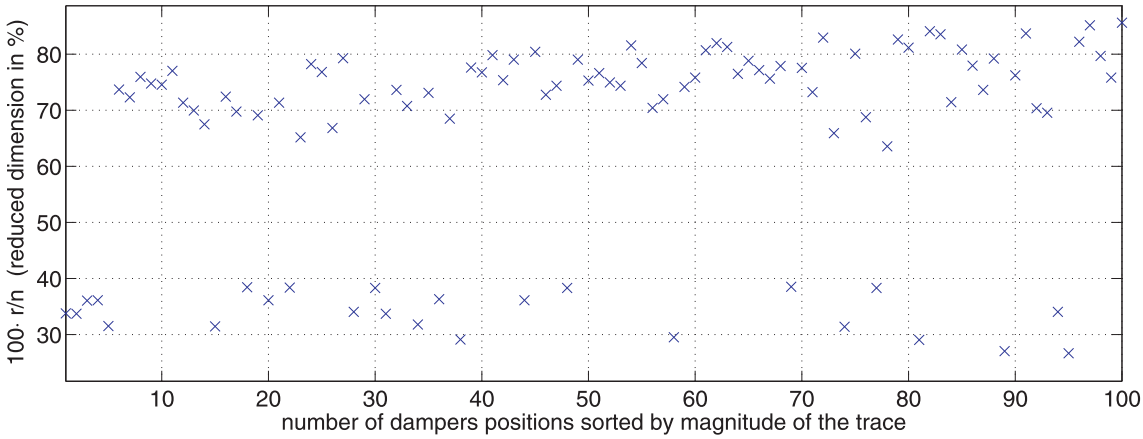

Figure 4: Reduced dimension

In order to demonstrate how much the reduced dimension accelerates the optimization process, in Figure 5 we show the time ratio for calculating the optimal viscosity at given dampers' positions with Algorithm 2 and by the algorithm without dimension reduction. These times were calculated using an Intel(R) Core(TM) i7 CPU 920 with 12GB of RAM and 8 MB cache. That is, near the optimal trace, the new algorithm is about 15 times faster than the one without dimension reduction. Though for many positions, the times are comparable or only a slight acceleration is observed, there are numerous instances where a significant time savings is achieved, so that altogether, a considerably shorter computation time for the whole optimization process is achieved.

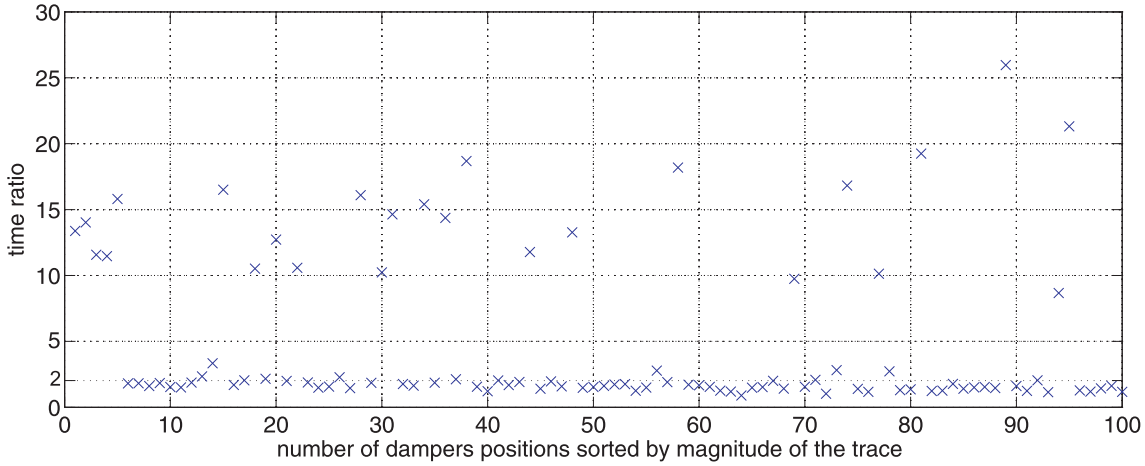

Figure 5: Time ratio

7 Conclusions

When finding the optimal positions of dampers in a mechanical vibrating system based on energy minimization, numerous Lyapunov equations have to be solved in order to find the traces of their solutions as these represent the energy contained in the system. In this paper, we have suggested a dimension reduction technique that can significantly accelerate the computation of these traces. The idea is based on exploiting the structure of the coefficient matrices of the resulting Lyapunov equations and considering the terms introduced by the damping as perturbations. Error bounds obtained from perturbation theory then guide us to those parts of the matrix that can be neglected so that parts of the trace calculation can then be obtained from analytical formulas and only a Lyapunov equation of much smaller size needs to be solved. Numerical experiments confirm the ability of this approximation technique to accelerate the optimization process significantly while ensuring that we still find the optima. Further results on partial damping will be reported elsewhere.

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