Problem 1 (Snježana Majstorović, University of J. J. Strossmayer in Osijek, Croatia). For $n \in \mathbb{N}$ let A_n be a sequence of matrices defined with the following reccurence relation

$$\begin{cases} A_1 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \\ \\ A_n = \begin{bmatrix} A_{n-1} & \mathrm{I} \\ \mathrm{I} & A_{n-1} \end{bmatrix}, \ n \geq 2, \end{cases}$$

with I being identity matrix. Prove that A_n has $n+1$ distinct integer eigenvalues $\lambda_0 < \lambda_1 < \cdots < \lambda_n$ and for $k = 0, 1, \ldots, n$ the multiplicity of λ_k is $\binom{n}{k}$.

Solution. For each $n \in \mathbb{N}$, matrix A_n is symmetric $2^n \times 2^n$ matrix with elements from the set $\{0, 1\}$, so that all elements on the main diagonal are equal to zero. We can write

$$A_n = I_{2^{n-1}} \otimes A_1 + A_{n-1} \otimes I_2, \tag{1}$$

where $\otimes$ is binary operation over the space of matrices, defined for arbitrary $B \in \mathbb{R}^{n \times p}$ and $C \in \mathbb{R}^{m \times s}$ as

$$B \otimes C := \begin{bmatrix} b_{11}C & b_{12}C & \ldots & b_{1p}C \\ b_{21}C & b_{22}C & \ldots & b_{2p}C \\ \vdots & & & \\ b_{n1}C & b_{n2}C & \ldots & b_{np}C \end{bmatrix}_{nm \times ps}.$$

Lemma 1. If $B \in \mathbb{R}^{n \times n}$ has eigenvalues λ_i, $i = 1, \ldots, n$ and $C \in \mathbb{R}^{m \times m}$ has eigenvalues μ_j, $j = 1, \ldots, m$, then $B \otimes C$ has eigenvalues $\lambda_i \mu_j$, $i = 1, \ldots, n$, $j = 1, \ldots, m$. If B and C are diagonalizable, then $B \otimes C$ has eigenvectors $y_i \otimes z_j$, with (λ_i, y_i) and (μ_j, z_j) being eigenpairs of B and C, respectively.

Proof 1. Let (λ, y) and (μ, z) be eigenpairs of B and C, respectively, where $\lambda, \mu \in \mathbb{R}$ and $y, z \in \mathbb{R}^n$. Then

$$(B \otimes C)(y \otimes z) = By \otimes Cz = \lambda y \otimes \mu z = \lambda\mu(y \otimes z).$$

□

If we take (λ, y) to be an eigenpair of A_1 and (μ, z) to be an eigenpair of A_{n-1}, then from (1) and Lemma 1 we get

$$\begin{aligned} A_n(z \otimes y) &= (I_{2^{n-1}} \otimes A_1 + A_{n-1} \otimes I_2)(z \otimes y) \\ &= (I_{2^{n-1}} \otimes A_1)(z \otimes y) + (A_{n-1} \otimes I_2)(z \otimes y) \\ &= (\lambda + \mu)(z \otimes y). \end{aligned}$$

So the entire spectrum of A_n can be obtained from eigenvalues of A_{n-1} and A_1: we calculate the sum of each eigenvalue of A_{n-1} with each eigenvalue A_1. Since the spectrum of A_1 is $\sigma(A_1) = \{-1, 1\}$, we get

$$\sigma(A_2) = \{-1 + (-1), -1 + 1, 1 + (-1), 1 + 1\} = \{-2, 0^{(2)}, 2\}$$

Let us assume that A_{n-1} has eigenvalues

$$-(n-1), -(n-1) + 2, -(n-1) + 4, \ldots, (n-1) - 4, (n-1) - 2, n - 1$$

with multiplicities $\binom{n-1}{0}, \binom{n-1}{1}, \binom{n-1}{2}, \ldots, \binom{n-1}{n-1}$, respectively. Then, eigenvalues of A_n are

$$-(n-1) + 1, -(n-1) + 2 + 1, -(n-1) + 4 + 1, \ldots, (n-1) - 4 + 1, (n-1) - 2 + 1, n - 1 + 1$$

with multiplicities $\binom{n-1}{0}, \binom{n-1}{1}, \binom{n-1}{2}, \ldots, \binom{n-1}{n-1}$, respectively, and

$$-(n-1)-1, -(n-1)+2-1, -(n-1)+4-1, \ldots, (n-1)-4-1, (n-1)-2-1, n-1-1$$

with multiplicities $\binom{n-1}{0}, \binom{n-1}{1}, \binom{n-1}{2}, \ldots, \binom{n-1}{n-1}$, respectively. By a simple calculation we conclude that A_n has $n+1$ distinct integer eigenvalues $-n, -n+2, -n+4, \ldots, n-4, n-2, n$ with multiplicities $\binom{n}{0}, \binom{n}{1}, \binom{n}{2}, \ldots, \binom{n}{n}$, respectively.

Remark. Matrix A_n is adjacency matrix of hypercube graph Q_n. In graph theory, the hypercube graph Q_n is the graph formed from the vertices and edges of an n-dimensional hypercube. It has 2^n vertices, $2^{n-1}n$ edges, and is a n-regular graph. The hypercube graph Q_n can be constructed by creating a vertex for each n-digit binary number, with two vertices adjacent if and only if their binary representations differ in a single digit.