UVIS - ISPITNI ROK 18. 2. 2021.

1.

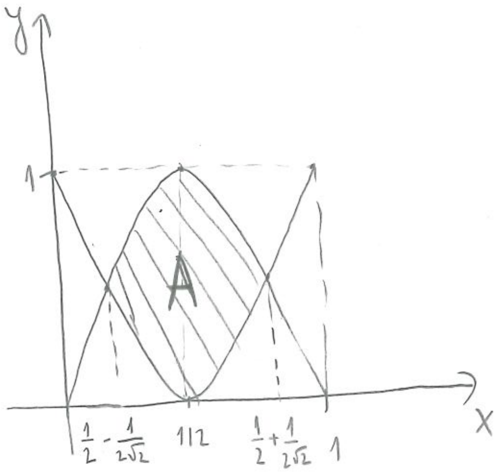
- $A_3 \supseteq A_1 \cap A_2 \Rightarrow P(A_3) \geq P(A_1 \cap A_2)$ (1)

- Pokažimo da vrijedi $P(A_1 \cap A_2) \geq P(A_1) + P(A_2) - 1$ (2)

Kako je $P(A_1 \cap A_2) = P(A_1) + P(A_2) - \underbrace{P(A_1 \cup A_2)}_{\leq 1} \geq P(A_1) + P(A_2) - 1$ //

Iz (1) i (2) slijedi tvrdnja zadatka!

2. $A = \{(x,y) \in [0,1] \times [0,1] : y \geq 4(x-1/2)^2, \ y \leq -4(x-1/2)^2 + 1\}$, $\Omega = [0,1] \times [0,1]$



- $\lambda(\Omega) = (1-0)(1-0) = 1$

- $$\lambda(A) = \int_{\frac{1}{2} - \frac{1}{2\sqrt{2}}}^{\frac{1}{2} + \frac{1}{2\sqrt{2}}} \left(-4(x-1/2)^2 + 1 - 4(x-1/2)^2\right) dx$$

$$= \int_{\frac{1}{2} - \frac{1}{2\sqrt{2}}}^{\frac{1}{2} + \frac{1}{2\sqrt{2}}} \left(1 - 8(x-1/2)^2\right) dx$$

$$= \left(x - \frac{8}{3}(x-1/2)^3\right) \Bigg|_{\frac{1}{2} - \frac{1}{2\sqrt{2}}}^{\frac{1}{2} + \frac{1}{2\sqrt{2}}} = \ldots = \frac{\sqrt{2}}{3} \ //$$

$$\Rightarrow P(A) = \frac{\lambda(A)}{\lambda(\Omega)} = \frac{\sqrt{2}/3}{1} = \frac{\sqrt{2}}{3} \ //$$

3. $X \sim \mathcal{N}(5, 4)$, $Y = |X-5| \Rightarrow f_X(x) = \frac{1}{2\sqrt{2\pi}} e^{-\frac{(x-5)^2}{8}}$

- $f_Y(y) = ?$ (Kako $g(x) = |x-5|$ nije bijekcija ne možemo koristiti teorem o bijektivnoj transformaciji!)

Krenimo od funkcije distribucije:

$$F_Y(y) = P(Y \leq y) = P(|X-5| \leq y) = P(5-y \leq X \leq 5+y) = F_X(5+y) - F_X(5-y), \ y \geq 0$$

$$\Rightarrow f_Y(y) = \frac{d}{dy} F_Y(y) = \frac{d}{dy}\left(F_X(5+y) - F_X(5-y)\right) = f_X(5+y) - f_X(5-y) \cdot (-1)$$

$$= \frac{1}{2\sqrt{2\pi}}\left(e^{-\frac{y^2}{8}} + e^{-\frac{y^2}{8}}\right) = \frac{1}{\sqrt{2\pi}} e^{-\frac{y^2}{8}}, \ y \geq 0$$

$$\Rightarrow \quad f_Y(y) = \begin{cases} \frac{1}{\sqrt{2\pi}} e^{-\frac{y^2}{8}}, & y \geq 0 \\ 0, & y < 0 \end{cases}$$

- $$E[Y] = \int_{\mathbb{R}} y f_Y(y)\,dy = \int_0^{+\infty} \frac{1}{\sqrt{2\pi}} y e^{-\frac{y^2}{8}}\,dy = \left| \begin{array}{l} y^2/8 = t \\ \frac{y\,dy}{4} = dt \end{array} \right| = \int_0^{+\infty} \frac{1}{\sqrt{2\pi}} 4 e^{-t}\,dt$$

$$= \frac{4}{\sqrt{2\pi}} (-e^{-t}) \Big|_0^{+\infty} = 2\sqrt{\frac{2}{\pi}} \quad //$$

(4.) $\mathcal{R}(X) = \{(k-1)^2,\ k \in \mathbb{N}\}$, $P(X=(k-1)^2) = \frac{2^{-k}}{k \ln 2}$, $k \in \mathbb{N}$

- $$E[X] = \sum_{k\in\mathbb{N}} (k-1)^2 \cdot \frac{2^{-k}}{k \ln 2} = \sum_{k\in\mathbb{N}} (k^2 - 2k + 1) \cdot \frac{2^{-k}}{k \ln 2} = \sum_{k\in\mathbb{N}} \left(k - 2 + \frac{1}{k}\right) \frac{2^{-k}}{\ln 2}$$

$$= \sum_{k\in\mathbb{N}} k \frac{2^{-k}}{\ln 2} - 2 \sum_{k\in\mathbb{N}} \frac{2^{-k}}{\ln 2} + \underbrace{\sum_{k\in\mathbb{N}} \frac{2^{-k}}{\ln 2 \cdot k}}_{=1} \quad \left(\text{jer je } \sum_{k\in\mathbb{N}} p_k = 1\right)$$

$$= \frac{2}{\ln 2} - \frac{2}{\ln 2} + 1 = 1 \quad //$$

(5.)
- X – broj realiziranih prostih brojeva, $\mathcal{R}(X) = \{0,1,2\}$
- Y – broj realiziranih parnih brojeva, $\mathcal{R}(Y) = \{0,1,2\}$

(· Prosti: 2, 3, 5
· Parni: 2, 4, 6)

distribucija sl. vek. (X,Y):

$X \backslash Y$	0	1	2	
0	$\frac{1}{36}$	$\frac{4}{36}$	$\frac{4}{36}$	$\frac{1}{4}$
1	$\frac{4}{36}$	$\frac{10}{36}$	$\frac{4}{36}$	$\frac{1}{2}$
2	$\frac{4}{36}$	$\frac{4}{36}$	$\frac{1}{36}$	$\frac{1}{4}$
	$\frac{1}{4}$	$\frac{1}{2}$	$\frac{1}{4}$	1

- marginalne distribucije:

$$X \overset{d}{=} Y = \begin{pmatrix} 0 & 1 & 2 \\ \frac{1}{4} & \frac{1}{2} & \frac{1}{4} \end{pmatrix} \Rightarrow X, Y \sim \mathcal{B}\left(2, \tfrac{1}{2}\right)$$

- koeficijent korelacije:

$E[X] = E[Y] = 2 \cdot \frac{1}{2} = 1$; $\operatorname{Var} X = \operatorname{Var} Y = \frac{1}{2}$;

$E[X \cdot Y] = \ldots = \frac{30}{36} = \frac{5}{6}$

$$\Rightarrow \rho_{X,Y} = \frac{\operatorname{Cov}(X,Y)}{\sqrt{\operatorname{Var}X \cdot \operatorname{Var}Y}} = \frac{E[XY] - E[X] \cdot E[Y]}{\sqrt{\operatorname{Var}X \cdot \operatorname{Var}Y}} = -\frac{1/6}{1/2} = \boxed{-\frac{1}{3}}$$

- $X_{|\{Y=1\}} = \,?$

$$\left.\begin{array}{l} P(X=0 \mid Y=1) = \dfrac{P(X=0, Y=1)}{P(Y=1)} = \dfrac{4/36}{1/2} = \dfrac{2}{9} \\[2ex] P(X=1 \mid Y=1) = \dfrac{P(X=1, Y=1)}{P(Y=1)} = \dfrac{10/36}{1/2} = \dfrac{5}{9} \\[2ex] P(X=2 \mid Y=1) = \dfrac{P(X=2, Y=1)}{P(Y=1)} = \dfrac{4/36}{1/2} = \dfrac{2}{9} \end{array}\right\}$$

$$X_{|\{Y=1\}} \overset{d}{=} \begin{pmatrix} 0 & 1 & 2 \\ \frac{2}{9} & \frac{5}{9} & \frac{2}{9} \end{pmatrix}$$

4.[(*)] (alternativno rješenje uz pomoć funkcija izvodnica)

- Neka je Y slučajna varijabla sa slikom $\mathcal{R}(Y) = \mathbb{N}$ & $P(Y=k) = \frac{2^{-k}}{k \ln 2}$.

  Nas zanima očekivanje s.v. $X = (Y-1)^2$

  $$\Rightarrow E[X] = E[(Y-1)^2] = EY^2 - 2E[Y] + 1 = \mathrm{Var}\,Y + (E[Y])^2 - 2E[Y] + 1 \quad (*)$$

- Sada je dovoljno odrediti prva dva momenta s.v. Y i uvrstiti ih u $(*)$

  Koristimo f.i. vjerojatnosti:

  $$g_Y(z) = \sum_{k=0}^{\infty} p_k z^k = \sum_{k=1}^{\infty} \frac{2^{-k}}{k \ln 2} z^k = \sum_{k=1}^{\infty} \frac{(2^{-1}z)^k}{k \ln 2} = \frac{-\ln(1-2^{-1}z)}{\ln 2}$$

  $$\left(\ln(1-x) = -\sum_{k=1}^{\infty} \frac{x^k}{k} \right) \qquad = -\frac{\ln\left(\frac{2-z}{2}\right)}{\ln 2}$$

  $$\Rightarrow g_Y'(z) = -\frac{1}{\ln 2} \cdot \frac{2}{2-z} \cdot \left(-\frac{1}{2}\right) = \frac{1}{\ln 2 \cdot (2-z)} = \frac{(2-z)^{-1}}{\ln 2}$$

  $$\Rightarrow g_Y''(z) = (2-z)^{-2} (\ln 2)^{-1}$$

  $$\Rightarrow E[Y] = g_Y'(1) = \frac{1}{\ln 2}; \quad \mathrm{Var}\,Y = g_Y''(1) + g_Y'(1) - g_Y'(1)^2 = (\ln 2)^{-1} + (\ln 2)^{-1} - (\ln 2)^{-2}$$

  $$\Rightarrow E[X] = \cancel{(\ln 2)^{-1}} + \cancel{(\ln 2)^{-1}} - \cancel{(\ln 2)^{-2}} + \cancel{(\ln 2)^{-2}} - 2\cancel{(\ln 2)^{-1}} + 1$$

  $$= 1$$