UVIS - ISPITNI ROK 4.2.2021.

1.

- A i B nezavisni $\Rightarrow P(A|B^c)=P(A)$; $P(B^c|A)=P(B^c)$; $P(B|A^c)=P(B)$; $P(A^c|B)=P(A^c)$ (*)
- $P(A^c\setminus B^c)=P(A^c)-P(A^c\cap B^c)$; $P(B^c\setminus A^c)=P(B^c)-P(B^c\cap A^c)$ (**)
- A i B jednako vjerojatni $\Rightarrow P(A)=P(B)$ (***)

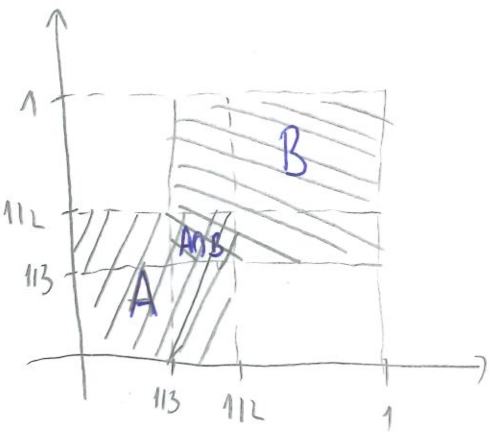
$$\Rightarrow \bullet P(A|B^c)+P(B^c|A)+P(A^c\setminus B^c)\overset{(*)}{=}P(A)+P(B^c)+P(A^c)-P(A^c\cap B^c)$$
$$=1+P(B^c)-P(A^c\cap B^c) \quad (1)$$

$$\bullet P(B|A^c)+P(A^c|B)+P(B^c\setminus A^c)\overset{(**)}{=}P(B)+P(A^c)+P(B^c)-P(A^c\cap B^c)$$
$$=1+P(A^c)-P(A^c\cap B^c)\overset{(***)}{=}1+P(B^c)-P(A^c\cap B^c) \quad (2)$$

(1) i (2) $\Rightarrow$ tvrdnja zadatka //

2.

$x,y\in[0,1]\Rightarrow \Omega=[0,1]\times[0,1]$



$$P(A|B)=\frac{P(A\cap B)}{P(B)}=\frac{\frac{\lambda(A\cap B)}{\cancel{\lambda(\Omega)}}}{\frac{\lambda(B)}{\cancel{\lambda(\Omega)}}}=\frac{\lambda(A\cap B)}{\lambda(B)}$$

- $\lambda(A\cap B)=$ (površina kvadrata) $=(1/2-1/3)^2=\frac{1}{36}$
- $\lambda(B)=$ (površina kvadrata) $=(1-1/3)^2=\frac{4}{9}$

$$\Rightarrow P(A|B)=\frac{1/36}{4/9}=\boxed{\frac{1}{16}}$$ //

3.

$$X\sim\mathcal{E}(2)\Rightarrow F_X(x)=\begin{cases}0, & x<0\\ 1-e^{-2x}, & x\geq 0\end{cases}$$

$Y=\sqrt{X}$

$$\Rightarrow \bullet F_Y(y)=P(Y\leq y)=P(\sqrt{X}\leq y)=\begin{cases}0, & y<0\\ P(\sqrt{X}\leq y), & y\geq 0\end{cases}=\begin{cases}0, & y<0\\ P(|X|\leq y^2), & y\geq 0\end{cases}=\begin{cases}0, & y<0\\ P(-y^2\leq X\leq y^2), & y\geq 0\end{cases}$$

$$=\begin{cases}0, & y<0\\ F_X(y^2)-F_X(-y^2), & y\geq 0\end{cases}=\begin{cases}0, & y<0\\ 1-e^{-2y^2}, & y\geq 0\end{cases}$$

• $f_Y(y) = \frac{d}{dy} F_Y(y) = \begin{cases} 0, & y<0 \\ -e^{-2y^2}(-4y), & y \geq 0 \end{cases} = \begin{cases} 0, & y<0 \\ 4ye^{-2y^2}, & y \geq 0 \end{cases}$

• $E[Y] = \int_{\mathbb{R}} y f_Y(y)\,dy = \int_0^{+\infty} 4y^2 e^{-2y^2}\,dy = \left| \begin{matrix} 2y^2 = t \\ 4y\,dy = dt \end{matrix} \right| = 2 \cdot \int_0^{+\infty} t e^{-t} \frac{1}{4 \cdot \sqrt{\frac{t}{2}}}\,dt = \frac{\sqrt{2}}{2} \int_0^{+\infty} t^{1/2} e^{-t}\,dt$

$= \frac{\sqrt{2}}{2} \Gamma\left(\frac{3}{2}\right) = \frac{\sqrt{2}}{2} \cdot \frac{\sqrt{\pi}}{2} = \frac{1}{2}\sqrt{\frac{\pi}{2}}$ //

4.

• 30 vrijednih kupona, 70 bezvrijednih kupona

• X - sl. varijabla kojom modeliramo broj uspjeha u n izvlačenja kupona

$\Rightarrow X \sim \mathcal{H}(n, 30, 100) \Rightarrow P(X=k) = \frac{\binom{30}{k}\binom{70}{n-k}}{\binom{100}{n}}$

• Y - sl. varijabla kojom modeliramo dobitak / gubitak igrača

$\Rightarrow Y = 2X - n$, za $\boxed{n=10}$ // $\quad P(X=k) = \frac{\binom{30}{k}\binom{70}{10-k}}{\binom{100}{10}}$

$\Rightarrow Y = 2X - 10 \Rightarrow \mathcal{R}(Y) = \{-10, -8, -6, -4, -2, 0, 2, 4, 6, 8, 10\}$

$\Rightarrow P(Y=-10) = P(X=0);\ P(Y=-8) = P(X=1);\ P(Y=-6) = P(X=2);\ \ldots\ P(Y=10) = P(X=10)$

$\Rightarrow$ distribucija s.v. Y:

$$Y = \begin{pmatrix} -10 & -8 & -6 & -4 & -2 & 0 & 2 & 4 & 6 & 8 & 10 \\ \frac{\binom{30}{0}\binom{70}{10}}{\binom{100}{10}} & \frac{\binom{30}{1}\binom{70}{9}}{\binom{100}{10}} & \frac{\binom{30}{2}\binom{70}{8}}{\binom{100}{10}} & \frac{\binom{30}{3}\binom{70}{7}}{\binom{100}{10}} & \frac{\binom{30}{4}\binom{70}{6}}{\binom{100}{10}} & \frac{\binom{30}{5}\binom{70}{5}}{\binom{100}{10}} & \frac{\binom{30}{6}\binom{70}{4}}{\binom{100}{10}} & \frac{\binom{30}{7}\binom{70}{3}}{\binom{100}{10}} & \frac{\binom{30}{8}\binom{70}{2}}{\binom{100}{10}} & \frac{\binom{30}{9}\binom{70}{1}}{\binom{100}{10}} & \frac{\binom{30}{10}\binom{70}{0}}{\binom{100}{10}} \end{pmatrix}$$

5.

• bacamo sim. kockicu dva puta za redom

• X - s.v. kojom modeliramo broj realiziranih trojki

• Y - s.v. kojom modeliramo broj realiziranih dvojki

$\Rightarrow \mathcal{R}(X) = \mathcal{R}(Y) = \{0, 1, 2\}$

$\Rightarrow X \overset{d}{=} Y = \begin{pmatrix} 0 & 1 & 2 \\ \frac{25}{36} & \frac{10}{36} & \frac{1}{36} \end{pmatrix} \Rightarrow X, Y \sim \mathcal{B}(2, 1/6)$

$P(X=0, Y=0) = P(\{$nije pala niti trojka niti četvorka$\})$

X\Y	0	1	2	M_X
0	$\frac{16}{36}$	$\frac{8}{36}$	$\frac{1}{36}$	$\frac{25}{36}$
1	$\frac{8}{36}$	$\frac{2}{36}$	0	$\frac{10}{36}$
2	$\frac{1}{36}$	0	0	$\frac{1}{36}$
M_Y	$\frac{25}{36}$	$\frac{10}{36}$	$\frac{1}{36}$	1

uvjetna dist. $X|\{Y=1\}$

• $P(X=0|Y=1) = P(X=0, Y=1) / P(X=1) = (8/36)/(10/36) = 4/5$
• $P(X=1|Y=1) = P(X=1, Y=1) / P(X=1) = (2/36)/(10/36) = 1/5$
• $P(X=2|Y=1) = P(X=2, Y=1) / P(X=1) = 0 / (10/36) = 0$

$X|\{Y=1\} \overset{d}{=} \begin{pmatrix} 0 & 1 \\ \frac{4}{5} & \frac{1}{5} \end{pmatrix}$

- $\rho_{X,Y} = \dfrac{Cov(X,Y)}{\sqrt{Var X \cdot Var Y}} = \dfrac{E[XY] - E[X] \cdot E[Y]}{\sqrt{Var X \cdot Var Y}}$ (*)

- $E[X] = E[Y] = 2 \cdot 1/6 = 1/3$, $Var X = Var Y = 2 \cdot 1/6 (1 - 1/6) = \dfrac{5}{18}$

- $E[XY] = \dfrac{2}{36}$ //

$\Rightarrow$ u (*): $\rho_{X,Y} = \dfrac{2/36 - (1/3)^2}{\sqrt{(5/18)^2}} = \dfrac{-2/36}{5/18} = -\dfrac{2}{10} = \boxed{-\dfrac{1}{5}}$ //