UVIS - ISPITNI ROK 14.6.2021.

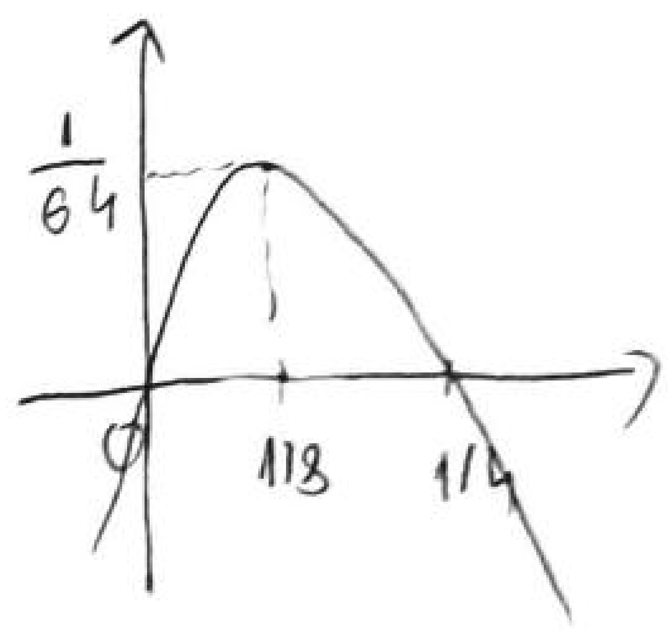
1. · $(\Omega, \mathcal{F}, P)$ vjer. prostor, $A, B \in \mathcal{F}$, $P(A\cap B) = \frac{1}{2}$, $P(A \cup B) = \frac{3}{4}$ (*)

Kako je $P(A\cup B) = P(A\setminus B) + P(A\cap B) + P(B\setminus A) \overset{(*)}{\Rightarrow} P(A\setminus B) + P(B\setminus A) = \frac{3}{4} - \frac{1}{2} = \frac{1}{4}$ (**)

Stoga je $P(A\setminus B)\cdot P(B\setminus A) \overset{(**)}{=} P(A\setminus B)\cdot\left(\frac{1}{4} - P(A\setminus B)\right)$

Obzirom da je $P(A\setminus B) \in [0,1]$, a graf funkcije $f(x) = x\left(\frac{1}{4}-x\right)$ parabola s maksimumom u tjemenu $\left(\frac{1}{8}, \frac{1}{64}\right)$

$\Rightarrow f(x) = x\left(\frac{1}{4}-x\right) \le \frac{1}{8}\left(\frac{1}{4}-\frac{1}{8}\right) = \frac{1}{64}$, $\forall x \in \mathbb{R}$



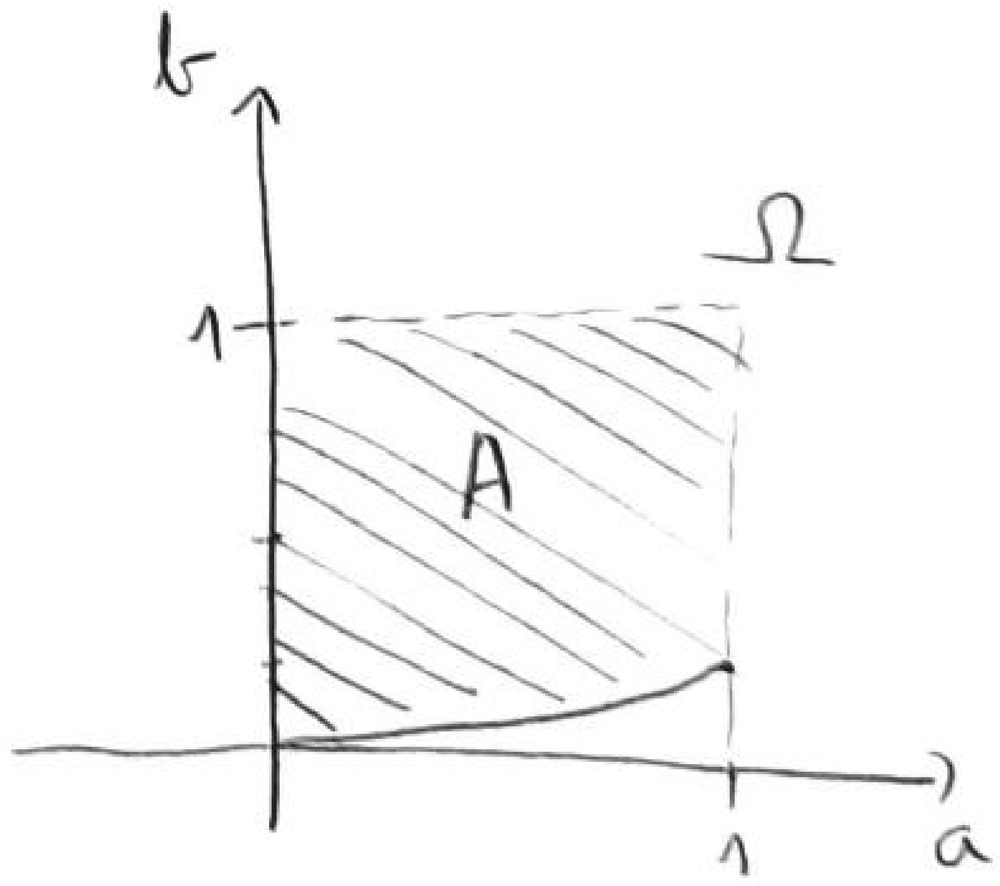
$\Rightarrow P(A\setminus B)\cdot P(B\setminus A) \le \frac{1}{64}$ //

2. $a, b \in [0,1]$ slučajno odabrani brojevi

· Bikvadratna jednadžba $x^4 + ax^2 + b = 0$ ima četiri rješenja, a jednadžbu rješavamo supstitucijom $t = x^2$ čime početnu jed. transformiramo u $t^2 + at + b = 0$, a ova kvadratna jednadžba ima par kompleksno konjugiranih rješenja ako je pripadna diskriminanta $D = a^2 - 4b < 0$. Tada su rješenja početne jednadžbe čisto kompleksni brojevi ($\mathrm{Im}(x) \neq 0$).

$\Rightarrow b > \frac{a^2}{4}$, $A = \left\{(a,b) \in [0,1]^2 : b > \frac{a^2}{4}\right\}$

$\Rightarrow P(A) = \frac{\lambda(A)}{\lambda(\Omega)} = \frac{1 - \int_0^1 \frac{a^2}{4}\,da}{1\cdot 1} = 1 - \frac{a^3}{12}\Big|_0^1 = 1 - \frac{1}{12} = \frac{11}{12}$ //



3. · $f_X(x) = \begin{cases} \frac{a}{x^{a+1}}, & x \ge 1 \\ 0, & x < 1 \end{cases}$, $a > 0$

$Y = \ln X$, $f_Y(y) = ?$ $\Rightarrow f_Y(y) = \begin{cases} f_X(g^{-1}(y)) \cdot \left|[g^{-1}(y)]'\right|, & y \in \mathcal{R}(g) \\ 0, & y \notin \mathcal{R}(g) \end{cases}$

$E[Y^5] = ?$

$\left(g(x) = \ln x \Rightarrow g^{-1}(x) = e^x\right)$

$$\Rightarrow f_Y(y) = \begin{cases} f_X(e^y)\cdot\left|[e^y]'\right|, & y\in\mathcal{R}(y) \\ 0, & y\notin\mathcal{R}(y)\end{cases} = \begin{cases} \dfrac{a}{(e^y)^{a+1}}\cdot e^y, & e^y\geq 1 \\ 0, & e^y<1\end{cases}$$

$$= \begin{cases} a\, e^{-ay}, & y\geq 0 \\ 0, & y<0 \end{cases} \Rightarrow Y\sim \mathcal{E}xp(a) \ //$$

peti moment: $E[Y^5] = \int_0^\infty y^5 a e^{-ay}\,dy = \left| \begin{matrix} ay=t \\ a\,dy = dt \end{matrix}\right| = \frac{1}{a^5}\int_0^\infty t^{\overset{6-1}{5}} e^{-t}\,dt = \frac{\Gamma(6)}{a^5}$

$$= \frac{5!}{a^5} = \frac{120}{a^5} \ //$$

(4.) X - broj posjetitelja muzeja u jednom danu, $X\sim\mathcal{P}(1) \Rightarrow P(X=k) = \frac{e^{-1}}{k!}$, $k\in\mathbb{N}_0$

Y - dnevna zarada muzeja

$$\Rightarrow Y = \begin{cases} 20(X-2), & X>2 \\ 0, & X\leq 2\end{cases} = 20(X-2)\,\mathbb{1}_{\{X>2\}}$$

(Ako je bilo X posjetitelja, prva dva imaju besplatne karte i ne doprinose zaradi, a svaki posjetitelj plaća 20 kn kartu. Svaki od $(X-2)$ osobe plaća 20 kn. Ako bi bilo samo 0, 1 ili 2 posjetitelja, nema zarade, odnosno iznosi 0)

$\Rightarrow \mathcal{R}(Y) = \{0, 20, 40, \dots\}$

$P(Y=0) = P(X\in\{0,1,2\}) = e^{-1}\left(\frac{1}{0!}+\frac{1}{1!}+\frac{1}{2!}\right) = \frac{5}{2}e^{-1}$

$P(Y=20) = P(X=3) = \frac{e^{-1}}{3!} = \frac{e^{-1}}{6}$

$\vdots$

$P(Y=k) = P\left(X=\frac{k}{20}+2\right) = \frac{e^{-1}}{\left(\frac{k}{20}+2\right)!}$, $k=20l$, $l\in\mathbb{N}$

$= e^{-1}/(l+2)!$

$\vdots$

distribucija sl. var. Y

$$\Rightarrow \begin{pmatrix} 0 & 20 & \dots & 20l & \dots \\ \frac{5}{2}e^{-1} & \frac{e^{-1}}{6} & \dots & \frac{e^{-1}}{(l+2)!} & \dots \end{pmatrix} (*)$$

$$E[Y] = E\left[20(X-2)\mathbb{1}_{\{X>2\}}\right] = \sum_{k=0}^{\infty} 20(k-2)\mathbb{1}_{\{k>2\}}\frac{e^{-1}}{k!} = \sum_{k=3}^{\infty} 20(k-2)\frac{e^{-1}}{k!}$$

$$= \sum_{k=0}^{\infty} 20(k-2)\frac{e^{-1}}{k!} - \sum_{k=0}^{2} 20(k-2)\frac{e^{-1}}{k!} = E\left[20(X-2)\right] - 20e^{-1}\left(\frac{-2}{0!} - \frac{1}{1!} - \frac{0}{2!}\right)$$

$$= 20(\underbrace{E[X]}_{1} - 2) + 20e^{-1}\cdot 3 = 20(1-2) + 60e^{-1} = 20(3e^{-1}-1) \;//$$

$$\approx 2.07 \text{ kn} \;//$$

(Alternativno, očekivanje se može izračunati na temelju ranije dobivene tablice distribucije (*):

$$E[Y] = \sum_{k\in\mathcal{R}(Y)} k\cdot P(Y=k) = 0\cdot P(Y=0) + \sum_{l=1}^{\infty} 20l\cdot\frac{e^{-1}}{(l+2)!} = 20e^{-1}\sum_{l=1}^{\infty}\frac{l}{(l+2)!} =$$

$$= 20e^{-1}\sum_{t=3}^{\infty}\frac{t-2}{t!} = 20\left[\sum_{t=3}^{\infty} t\cdot\frac{e^{-1}}{t!} - \sum_{t=3}^{\infty}\frac{2e^{-1}}{t!}\right] =$$

$$= 20\left[\underbrace{\sum_{t=0}^{\infty} t\frac{e^{-1}}{t!}}_{EX} - \sum_{t=0}^{2} t\frac{e^{-1}}{t!} - \underbrace{\sum_{t=0}^{\infty}\frac{2e^{-1}}{t!}}_{2\cdot 1} + \sum_{t=0}^{2}\frac{2e^{-1}}{t!}\right]$$

$$= 20\left[EX - \frac{e^{-1}}{1!} - \frac{2e^{-1}}{2!} - 2\cdot 1 + 2e^{-1} + 2e^{-1} + e^{-1}\right]$$

$$= 20\cdot\left[1 - \cancel{e^{-1}} - e^{-1} - 2 + 2e^{-1} + 2e^{-1} + \cancel{e^{-1}}\right] = 20(3e^{-1}-1)$$)

(5.) X, Y nezavisne diskretne sl. var.

X\Y	-1	0	1	
1	0.05	0.1	p_{13}	p_1
2	p_{22}	0.2	p_{23}	p_2
3	p_{31}	0.1	0.1	p_3
	q_1	q_2	q_3	

$\Rightarrow q_2 = 0.1 + 0.2 + 0.1 = 0.4$

$p_2\cdot q_2 = 0.2$ (zbog nezavisnosti je $p_i q_j = p_{ij}$)

$\Rightarrow p_2 = 1/2$, $p_3 = 0.1/q_2 = 1/4$, $p_1 = 0.1/q_2 = 1/4$

$\Rightarrow p_{31} = p_3 - 0.1 - 0.1 = 0.25 - 0.2 = 0.05$

$\Rightarrow p_{13} = p_1 - 0.05 - 0.1 = 0.25 - 0.05 - 0.1 = 0.1$

$\Rightarrow p_{22} + 0.2 + p_{23} = p_2 \Rightarrow p_{22} + p_{23} = 0.3$

(ili $\sum p_{ij} = 1 \Rightarrow p_{22} + p_{23} = 0.3$)

Ponovo zbog nezavisnosti je $p_3 \cdot q_3 = 0.1 \Rightarrow q_3 = 0.1/0.25 = 2/5 \Rightarrow q_1 = 1/5$
& $p_2 \cdot q_1 = p_{22} \Rightarrow p_{22} = \frac{1}{2} \cdot \frac{1}{5} = \frac{1}{10} \Rightarrow p_{23} = 0.2$

Dakle, tablica distribucije je

X\Y	-1	0	1	
1	0.05	0.1	0.1	0.25
2	0.1	0.2	0.2	0.5
3	0.05	0.1	0.1	0.25
	0.2	0.4	0.4	1

koeficijent korelacije:

$\rho_{X,Y} = 0$ jer je

$$\rho_{X,Y} = \frac{\overbrace{Cov(X,Y)}^{=0}}{\sqrt{VarX \cdot VarY}} = 0$$

($Cov(X,Y) = 0$ zbog nezavisnosti sl. var. X i Y)

(Koristili smo činjenice:

- $\sum_i p_{ij} = q_j$ (suma elemenata po stupcu daje marginalnu razd. sl. var. Y
- $\sum_j p_{ij} = p_i$ (suma elemenata po retku daje marginalnu razd. sl. var. X
- $\sum_{i,j} p_{ij} = 1$
- Nezavisnost s.v. X i Y $\Leftrightarrow p_i \cdot q_j = p_{ij}, \forall i,j$)