UVIS - ISPITNI ROK 16.9.2021.

1. E - skup studenata koji znaju engleski
F - skup studenata koji znaju francuski
N - skup studenata koji znaju njemački

$k(E)=28$, $k(F)=42$, $k(N)=30$ | $k(E\cap F)=10$, $k(F\cap N)=5$, $k(E\cap N)=8$ | $k(E\cap N\cap F)=3$

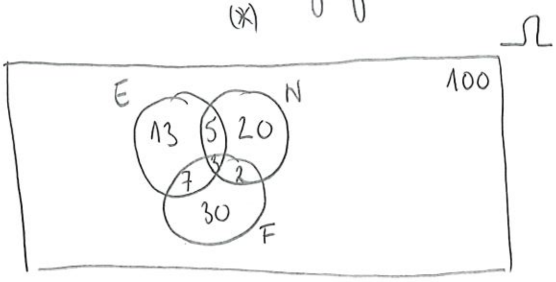
Ω - skup svih studenata iz kojeg na sl način biramo jednog

$P((E\cup F\cup N)^c)=?$

$k(\Omega)=100$

Vennov dijagram:

(*)



$$k(E\cup F\cup N)=k(E)+k(F)+k(N)-k(E\cap F)-k(F\cap N)-k(E\cap N)+k(E\cap N\cap F)$$

$$=28+42+30-10-5-8+3=80$$ (alternativno, može se izračunati i uz pomoć (*))

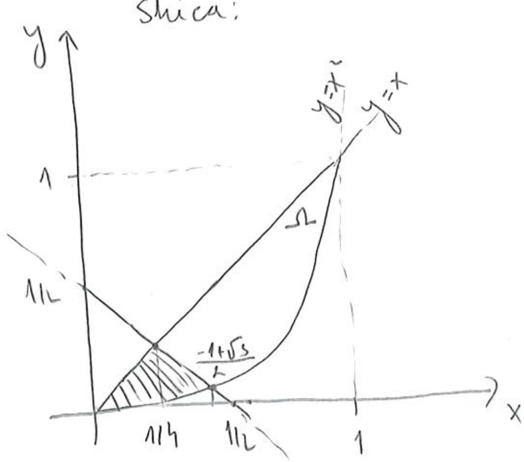
$$\Rightarrow k((E\cup F\cup N)^c)=k(\Omega)-k(E\cup N\cup F)=100-80=20$$

$$\Rightarrow P((E\cup N\cup F)^c)=\frac{k((E\cup N\cup F)^c)}{k(\Omega)}=\frac{20}{100}=0.2 \;//$$

(ili $P((E\cup N\cup F)^c)=1-P(E\cup N\cup F)=1-\frac{80}{100}=\frac{20}{100}$)

2. $\Omega=\{(x,y)\in\mathbb{R}^2 : x^2\le y\le x\}$, $B=\{(x,y)\in\Omega : y\le -x+1/2\}$

skica:



$$\cdot\ \lambda(\Omega)=\int_0^1 (x-x^2)\,dx=\left(\frac{x^2}{2}-\frac{x^3}{3}\right)\Big|_0^1=\frac{1}{6}$$

$$\cdot\ \lambda(A)=\int_0^{1/4}(x-x^2)\,dx+\int_{1/4}^{\frac{-1+\sqrt{3}}{2}}(-x+1/2-x^2)\,dx$$

$$=\left(\frac{x^2}{2}-\frac{x^3}{3}\right)\Big|_0^{1/4}+\left(-\frac{x^2}{2}+\frac{x}{2}-\frac{x^3}{3}\right)\Big|_{1/4}^{\frac{-1+\sqrt{3}}{2}}$$

$$=-\frac{1}{16}-\frac{(\sqrt{3}-1)^2}{8}+\frac{\sqrt{3}-1}{2}-\frac{(\sqrt{3}-1)^3}{24}\approx 0.037182$$

$$\Rightarrow P(A)=\frac{\lambda(A)}{\lambda(\Omega)}\approx\frac{0.037182}{1/6}=0.223092$$

③ $f_X(x) = \frac{1}{96} x^{b-1} e^{-\frac{x}{2}} \mathbb{1}_{[0,+\infty)}(x)$, $b \in \mathbb{N}$

a) $b = ?$

Normiranost funkcije f: $\int_{\mathbb{R}} f_X(x)dx = 1$

$$\Rightarrow \int_0^{+\infty} \frac{1}{96} x^{b-1} e^{-\frac{x}{2}} dx = \begin{vmatrix} \frac{x}{2} = t \\ dx = 2dt \end{vmatrix} = \int_0^{+\infty} \frac{1}{96} 2^{b-1} t^{b-1} e^{-t} 2\,dt = \frac{2^{b-1}}{48} \int_0^{+\infty} t^{b-1} e^{-t} dt$$

$$= \frac{2^{b-1}}{48} \Gamma(b) = 1 \Rightarrow \Gamma(b) = \frac{48}{2^{b-1}} = 3 \cdot 2^{5-b} \ (*)$$

Kako je $b \in \mathbb{N} \Rightarrow b = 4$ jer je $\Gamma(4) = 3! = 6$ & $3 \cdot 2^{5-4} = 6$ //

(Lako se vidi da za $b \geq 6$ izraz s desne strane (*) nije prirodan broj,

a $\Gamma(b) = (b-1)!$ za $b \in \mathbb{N}$) $\Rightarrow f_X(x) = \frac{1}{96} x^3 e^{-\frac{x}{2}} \mathbb{1}_{[0,+\infty)}(x)$

b) $Y = 1/X$, $f_Y(y) = ?$, $E[Y] = ?$

$g(x) = 1/x \Rightarrow g^{-1}(x) = 1/x$; $[g^{-1}(x)]' = -\frac{1}{x^2}$

$$\Rightarrow \cdot f_Y(y) = \begin{cases} f_X(g^{-1}(y)) \cdot |[g^{-1}(y)]'| , & y \in \mathcal{R}(g) \\ 0 , & y \notin \mathcal{R}(g) \end{cases} = \begin{cases} f_X(1/y) \cdot \frac{1}{y^2} , & y \in \mathcal{R}(g) \\ 0 , & y \notin \mathcal{R}(g) \end{cases}$$

$$= \begin{cases} \frac{1}{96} y^{-3} e^{-\frac{1}{2y}} \cdot \frac{1}{y^2} , & y > 0 \\ 0 , & y \leq 0 \end{cases} = \begin{cases} \frac{1}{96} y^{-5} e^{-\frac{1}{2y}} , & y > 0 \\ 0 , & y \leq 0 \end{cases}$$

$$\cdot\ E[Y] = \int_{\mathbb{R}} y f_Y(y) dy = \int_0^{+\infty} \frac{1}{96} y^{-4} e^{-\frac{1}{2y}} dy = \begin{vmatrix} \frac{1}{2y} = t \\ -\frac{1}{2y^2} dy = dt \end{vmatrix} = -\int_{\infty}^{0} \frac{1}{12} t^2 e^{-t} dt$$

$$= \int_0^{+\infty} \frac{1}{12} t^{3-1} e^{-t} dt = \frac{\Gamma(3)}{12} = \frac{2}{12} = \frac{1}{6} \ //$$

(4.) X - broj karata s vrijednošću 8, Y - broj karata s vrijednošću manjom od 10

$R(X) = R(Y) = \{0,1,2\}$

· tablica distribucije

$X \backslash Y$	0	1	2	
0	$\left(\frac{20}{32}\right)^2$	$\frac{320}{32^2}$	$\frac{64}{32^2}$	$784/1024$
1	0	$\frac{160}{32^2}$	$\frac{64}{32^2}$	$224/1024$
2	0	0	$\left(\frac{4}{32}\right)^2$	$16/1024$
	$\frac{400}{1024}$	$\frac{480}{1024}$	$\frac{144}{1024}$	

$\Rightarrow$

$X \backslash Y$	0	1	2	
0	$\frac{25}{64}$	$\frac{5}{16}$	$\frac{1}{16}$	$\frac{49}{64}$
1	0	$\frac{5}{32}$	$\frac{1}{16}$	$\frac{14}{64}$
2	0	0	$\frac{1}{64}$	$\frac{1}{64}$
	$\frac{25}{64}$	$\frac{15}{32}$	$\frac{9}{64}$	

· marginalne distribucije:

$$X \overset{d}{=} \begin{pmatrix} 0 & 1 & 2 \\ \frac{49}{64} & \frac{14}{64} & \frac{1}{64} \end{pmatrix}, \quad Y \overset{d}{=} \begin{pmatrix} 0 & 1 & 2 \\ \frac{25}{64} & \frac{15}{32} & \frac{9}{64} \end{pmatrix}$$

· X i Y nisu nezavisni jer npr. $P(X=2, Y=0) = 0$ & $P(X=2) = 16/1024$, $P(Y=0) = 400/1024$

$$\Rightarrow P(X=2, Y=0) \neq P(X=2) \cdot P(Y=0) \; //$$

· $P(XY=2) = P(X=1, Y=2) + P(X=2, Y=1) = \frac{64}{1024} + 0 = \frac{64}{1024} = \frac{1}{16} \; //$

(5) · početni ulog: 30 kn, zarada 5 kn za svaku plavu izvučenu kuglicu

· U kutiji 15 bijelih i 30 plavih kuglica $\Rightarrow$ izvlačimo s vraćanjem dok ne izvučemo bijelu kuglicu

X - s.v. kojom modeliramo broj izvučenih kuglica

$$\Rightarrow X \sim G\left(\frac{1}{3}\right) \Rightarrow P(X=k) = \left(\frac{2}{3}\right)^{k-1} \cdot \frac{1}{3}, \; k \in \mathbb{N}$$

Y - s.v. kojom modeliramo zaradu

$$\boxed{Y} = \underbrace{-30}_{\text{poč. ulog}} + \underbrace{5}_{\text{za svaku plavu dobijemo 5 kn.}} \cdot \underbrace{(X-1)}_{\text{broj izvučenih plavih}} = 5X - 35 \Rightarrow P(Y = 5k-35) = P(X=k) = \left(\frac{2}{3}\right)^{k-1} \cdot \left(\frac{1}{3}\right)$$

$$\Rightarrow R(Y) = \{-30, -25, \ldots, -5, 0, 5, 10, \ldots\}$$

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$$\Rightarrow Y = \begin{pmatrix} -30 & -25 & -20 & \cdots & 0 & 5 & \cdots \\ \frac{1}{3} & \frac{2}{9} & \frac{4}{27} & \cdots & \left(\frac{2}{3}\right)^6 \cdot \frac{1}{3} & \left(\frac{2}{3}\right)^7 \cdot \frac{1}{3} & \cdots \end{pmatrix}$$