UVIS - drugi kolokvij 2016./2017.
GRUPA (A)

(2.) X - sl. varijabla kojom modeliramo broj izvučenih crnih kuglica

a) $X \sim \mathcal{H}(3,4,10) \Rightarrow P(X=2) = \dfrac{\binom{4}{2}\binom{6}{1}}{\binom{10}{3}} = \dfrac{3}{10}$, $E[X] = 3 \cdot \dfrac{4}{10} = \dfrac{6}{5}$

b) $X \sim \mathcal{B}(3, \frac{2}{5}) \Rightarrow P(X=2) = \binom{3}{2}\left(\frac{2}{5}\right)^2\left(\frac{3}{5}\right)^1 = \dfrac{36}{125}$, $E[X] = 3 \cdot \dfrac{2}{5} = \dfrac{6}{5}$ //

(3.) X - sl. var. kojom modeliramo prirodan broj realiziran na automatu

• $\mathcal{R}(X) = \mathbb{N}$, $P(X=n) = 3/4^n$

• računamo funkciju izvodnicu s.v. X:

$$g_X(z) = \sum_{k=1}^{\infty} z^k \cdot P(X=k) = \sum_{k=1}^{\infty} z^k \cdot \frac{3}{4^k} = 3\sum_{k=1}^{\infty}\left(\frac{z}{4}\right)^k = 3 \cdot \frac{\frac{z}{4}}{1-\frac{z}{4}} = \frac{3z}{4-z}$$

znamo: $E[X] = g_X'(1)$ & $\operatorname{Var} X = g_X''(1) + g_X'(1) - (g_X'(1))^2$ (*)

kako je $g_X'(z) = \dfrac{12}{(4-z)^2}$ & $g_X''(z) = \dfrac{24}{(4-z)^3} \Rightarrow g_X'(1) = \dfrac{4}{3}$, $g_X''(1) = \dfrac{8}{9}$

$$\overset{(*)}{\Rightarrow} E[X] = \frac{4}{3}, \quad \operatorname{Var} X = \frac{8}{9} + \frac{4}{3} - \left(\frac{4}{3}\right)^2 = \frac{4}{9}$$

(4.) X s.v. s funkcijom gustoće $f(x;\alpha) = \begin{cases} \frac{1}{\Gamma(\alpha)} 2^\alpha x^{\alpha-1} e^{-2x}, & x>0 \\ 0, & x \le 0 \end{cases}$, $\alpha > 0$

a) • $E[X] = \int_{\mathbb{R}} x f(x;\alpha)\,dx = \dfrac{2^\alpha}{\Gamma(\alpha)} \int_0^{+\infty} x^\alpha e^{-2x}\,dx = \begin{vmatrix} 2x=t \\ 2dx=dt \end{vmatrix} = \dfrac{2^\alpha}{\Gamma(\alpha)} \int_0^{+\infty} \dfrac{t^\alpha}{2^\alpha} e^{-t} \dfrac{dt}{2}$

$$= \frac{1}{\Gamma(\alpha)} \cdot \frac{1}{2} \int_0^{+\infty} t^\alpha e^{-t}\,dt = \frac{1}{\Gamma(\alpha)} \cdot \frac{1}{2} \cdot \Gamma(\alpha+1) = \frac{\alpha}{2}$$

• $\operatorname{Var} X = EX^2 - (E(X))^2 = EX^2 - \dfrac{\alpha^2}{4}$ (*)

• $EX^2 = \int_{\mathbb{R}} x^2 f(x;\alpha)\,dx = \dfrac{2^\alpha}{\Gamma(\alpha)} \int_0^{+\infty} x^{\alpha+1} e^{-2x}\,dx = \ldots = \dfrac{1}{\Gamma(\alpha)} \cdot \dfrac{1}{4} \int_0^{+\infty} t^{\alpha+1} e^{-t}\,dt = \dfrac{\Gamma(\alpha+2)}{\Gamma(\alpha)} \cdot \dfrac{1}{4}$

$$= \frac{(\alpha+1)\Gamma(\alpha+1)}{\Gamma(\alpha)} \cdot \frac{1}{4} = (\alpha+1) \cdot \alpha \cdot \frac{1}{4} //$$

$\overset{(*)}{\Rightarrow} \quad \mathrm{Var}\,X = \frac{(\alpha+1)\alpha}{4} - \frac{\alpha^2}{4} = \frac{\alpha^2+\alpha-\alpha^2}{4} = \frac{\alpha}{4} \,//$

(5.) $X \sim U(2,4)$, $Y = e^X$, $f_Y(y) = ?$, $F_Y(y) = ?$ $\qquad f_X(x) = \begin{cases} \frac{1}{2}, & x \in \langle 2,4 \rangle \\ 0, & \text{inače} \end{cases}$

$g(x) = e^x$, $Y = g(X)$, $g^{-1}(x) = \ln x$

- $f_Y(y) = \begin{cases} f_X(g^{-1}(y)) \left| [g^{-1}(y)]' \right|, & y \in \mathcal{R}(y) \\ 0, & \text{inače} \end{cases} = \begin{cases} f_X(\ln y) \cdot \left| \frac{1}{y} \right|, & y \in \mathcal{R}(y) \\ 0, & \text{inače} \end{cases}$

$= \begin{cases} \frac{1}{2} \cdot \left| \frac{1}{y} \right|, & \ln y \in \langle 2,4 \rangle \\ 0, & \text{inače} \end{cases} = \begin{cases} \frac{1}{2y}, & y \in \langle e^2, e^4 \rangle \\ 0, & \text{inače} \end{cases}$

- $F_Y(y) = \int_{-\infty}^{y} f_Y(t)\,dt = \begin{cases} 0, & y \le e^2 \\ \frac{\ln y - 2}{2}, & y \in \langle e^2, 4 \rangle \\ 1, & y \ge e^2 \end{cases}$ jer je za

- $y \le e^2$: $\int_{-\infty}^{y} f_Y(t)\,dt = 0$
- $y \in \langle e^2, e^4 \rangle$: $\int_{-\infty}^{y} f_Y(t)\,dt = \frac{\ln y - 2}{2}$
- $y \ge e^4$: $\int_{-\infty}^{y} f_Y(t)\,dt = 1$

(6.) X - s.v. kojom modeliramo ukupnu težinu bombona, $\mathcal{R}(X) = \{8, 9, 10\}$

Y - s.v. kojom modeliramo broj izvučenih žutih bombona, $\mathcal{R}(Y) = \{0, 1, 2\}$

- razdioba:

X\Y	0	1	2	
8	0	0	1/6	1/6
9	0	5/9	0	5/9
10	5/18	0	0	5/18
	5/18	5/9	1/6	1

- marginalne distribucije:

$X \overset{d}{=} \begin{pmatrix} 8 & 9 & 10 \\ \frac{1}{6} & \frac{5}{9} & \frac{5}{18} \end{pmatrix}$, $Y \overset{d}{=} \begin{pmatrix} 0 & 1 & 2 \\ \frac{5}{18} & \frac{5}{9} & \frac{1}{6} \end{pmatrix}$

- $X_{|\{Y=2\}} \overset{d}{=} \begin{pmatrix} 8 & 9 & 10 \\ 1 & 0 & 0 \end{pmatrix}$

- $\mathrm{Cov}(X,Y) = \frac{276}{36} - \frac{328 \cdot 32}{(36)^2} = -\frac{35}{81}$

$\Rightarrow \mathrm{Cov}(X,Y) \neq 0 \Rightarrow X$ i Y su korelirane!