LU općeg oblika

Rješava se sustav $Ax=b$ gdje je $A \in \mathbb{C}^{n\times n}$ i $b \in \mathbb{C}^n$ tj.

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & & & \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix}, \quad b = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix}$$

<u>1. korak</u>

$$A_1 = L_1 \cdot A \quad \text{gdje je} \quad L_1 = \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \\ -\frac{a_{21}}{a_{11}} & 1 & 0 & \cdots & 0 \\ -\frac{a_{31}}{a_{11}} & 0 & 1 & 0 \cdots & 0 \\ \vdots & \vdots & & & \\ -\frac{a_{n1}}{a_{nn}} & 0 & \cdots & 0 & 1 \end{bmatrix}$$

* Želimo $a_{21} = a_{31} = \cdots = a_{n1} = 0$!

$$A_1 = \begin{bmatrix} 1 & 0 & \cdots & 0 \\ -\frac{a_{21}}{a_{11}} & 1 & & 0 \\ -\frac{a_{31}}{a_{11}} & 0 & 1 \; 0 \cdots & 0 \\ \vdots & & & \\ -\frac{a_{n1}}{a_{11}} & 0 & \cdots & 1 \end{bmatrix} \begin{bmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{nn} \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2n} \\ a_{31} & a_{32} & a_{33} & \cdots & a_{3n} \\ \vdots & & & & \\ a_{n1} & a_{n2} & a_{n3} & \cdots & a_{nn} \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ 0 & -\frac{a_{21}a_{12}}{a_{11}} + a_{22} & \cdots & & -\frac{a_{21}a_{1n}}{a_{11}} + a_{2n} \\ 0 & -\frac{a_{31}a_{12}}{a_{11}} + a_{32} & \cdots & & -\frac{a_{31}a_{1n}}{a_{11}} + a_{3n} \\ \vdots & & & & \\ 0 & -\frac{a_{n1}a_{12}}{a_{11}} + a_{n2} & \cdots & & -\frac{a_{n1}a_{1n}}{a_{11}} + a_{nn} \end{bmatrix}$$

$$= \begin{bmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ 0 & a_{22}^{(1)} & a_{23}^{(1)} & \cdots & a_{2n}^{(1)} \\ 0 & a_{32}^{(1)} & a_{33}^{(1)} & \cdots & a_{3n}^{(1)} \\ \vdots & & & & \\ 0 & a_{n2}^{(1)} & a_{n3}^{(1)} & \cdots & a_{nn}^{(1)} \end{bmatrix}$$

tj. $a_{22}^{(1)} = -\frac{a_{21}a_{12}}{a_{11}} + a_{22}$

itd....

* Matrica L_1 se može zapisati ovako:

$$L_1 = I - u_1 e_1^T, \quad \text{gdje je} \quad u_1 = \left[0 \;\; \frac{a_{21}}{a_{11}} \;\; \frac{a_{31}}{a_{11}} \;\cdots\; \frac{a_{n1}}{a_{11}}\right]^T$$

$$= \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & 0 \cdots & 0 \\ \vdots & & & \\ 0 & \cdots & 0 & 1 \end{bmatrix} - \begin{bmatrix} 0 \\ \frac{a_{21}}{a_{11}} \\ \vdots \\ \frac{a_{n1}}{a_{11}} \end{bmatrix} \begin{bmatrix} 1 & 0 & \cdots & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & 0 \cdots & 0 \\ \vdots & & & \\ 0 & 0 & \cdots & 1 \end{bmatrix} - \begin{bmatrix} 0 & 0 & \cdots & 0 \\ \frac{a_{21}}{a_{11}} & 0 & \cdots & 0 \\ \vdots & & & \\ \frac{a_{n1}}{a_{11}} & 0 & \cdots & 0 \end{bmatrix}$$

✓

Dalje, želimo $a_{32}^{(2)} = a_{42}^{(2)} \cdots = a_{n2}^{(2)} = 0$!

$$A_2 = L_2 A_1 = L_2 \cdot L_1 A_1$$

$$L_2 = \begin{bmatrix} 1 & 0 & \cdots & & 0 \\ 0 & 1 & 0 & \cdots & 0 \\ 0 & -\frac{a_{32}^{(2)}}{a_{22}^{(2)}} & 1 & \cdots & 0 \\ \vdots & & & & \\ 0 & -\frac{a_{n2}^{(2)}}{a_{22}^{(2)}} & 0 & \cdots & 1 \end{bmatrix} \quad \text{tj. } L_2 = I - u_2 e_2^T \text{ gdje je}$$

$$u_2 = \begin{bmatrix} 0 & 0 & \frac{a_{32}^{(2)}}{a_{22}^{(2)}} & \cdots & \frac{a_{n2}^{(2)}}{a_{22}^{(2)}} \end{bmatrix}$$

POMNOŽIMO

$$A_2 = \begin{bmatrix} 1 & 0 & \cdots & & 0 \\ 0 & 1 & 0 & \cdots & 0 \\ 0 & -\frac{a_{32}^{(2)}}{a_{22}^{(2)}} & 1 & \cdots & 0 \\ \vdots & & & & \\ 0 & -\frac{a_{n2}^{(2)}}{a_{22}^{(2)}} & 0 & \cdots & 1 \end{bmatrix} \begin{bmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ 0 & a_{22}^{(2)} & a_{23}^{(2)} & \cdots & a_{2n}^{(2)} \\ 0 & a_{32}^{(2)} & a_{33}^{(2)} & \cdots & a_{3n}^{(2)} \\ \vdots & & & & \\ 0 & a_{n2}^{(2)} & a_{n3}^{(2)} & \cdots & a_{nn}^{(2)} \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} & \cdots & & a_{1n} \\ 0 & a_{22}^{(2)} & a_{23}^{(2)} & \cdots & a_{2n}^{(2)} \\ 0 & 0 & a_{33}^{(3)} & \cdots & a_{3n}^{(3)} \\ 0 & 0 & a_{43}^{(3)} & \cdots & a_{4n}^{(3)} \\ \vdots & & & & \\ 0 & 0 & a_{n3}^{(3)} & \cdots & a_{nn}^{(3)} \end{bmatrix}$$

i tako nastavljamo postupak! Dalje treba $a_{43}^{(3)} = \cdots = a_{n3}^{(3)} = 0$!

...

Što se dešava s vektorom b?

Primjetimo! Radimo ovakav postupak.

$$L_1 \mid Ax = b \Rightarrow \underbrace{L_1 A}_{A_1} x = \underbrace{L_1 b}_{b^{(2)}}$$

$$b^{(2)} = \begin{bmatrix} 1 & 0 & \cdots & & 0 \\ -\frac{a_{21}}{a_{11}} & 1 & 0 & \cdots & 0 \\ -\frac{a_{31}}{a_{11}} & 0 & 1 & \cdots & 0 \\ \vdots & & & & \\ -\frac{a_{n1}}{a_{11}} & 0 & \cdots & & 1 \end{bmatrix} \begin{bmatrix} b_1 \\ b_2 \\ b_3 \\ \vdots \\ b_n \end{bmatrix} = \begin{bmatrix} b_1 \\ -\frac{a_{21} b_1}{a_{11}} + b_2 \\ \vdots \\ -\frac{a_{n1} b_1}{a_{11}} + b_n \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2^{(2)} \\ \vdots \\ b_n^{(2)} \end{bmatrix}$$

pa dalje

$$L_2 \mid A_1 x = b^{(2)} \Rightarrow \underbrace{L_2 A_1}_{A_2} x = \underbrace{L_2 b^{(2)}}_{b^{(3)}}$$

slično

$$b^{(3)} = \begin{bmatrix} b_1 \\ b_2^{(2)} \\ b_3^{(3)} \\ \vdots \end{bmatrix}$$