Max Planck Institute for Dynamics of Complex Technical Systems
Computational Methods in Systems and Control Theory

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# Model Order Reduction via System Balancing

**Exercise 1** (Controllability of dynamical systems)
Let $A \in \mathbb{R}^{n\times n}$ and $B \in \mathbb{R}^{n\times m}$. Show that the following statements are equivalent:

a) the pair $(A, B)$ is controllable
(i.e. for all times $t_0, t_1 \in \mathbb{R}$, $t_0 < t_1$, and states $x_0, x_1 \in \mathbb{R}^n$, there exists $u(t)$ such that the solution of the initial value problem $\dot{x}(t) = Ax(t) + Bu(t)$, $x(t_0) = x_0$, satisfies $x(t_1) = x_1$),

b) the controllability matrix $\mathcal{C} = \begin{bmatrix} B & AB & \cdots & A^{n-1}B \end{bmatrix} \in \mathbb{R}^{n\times nm}$ has full rank $n$,

c) the controllability Gramian

$$P(t) = \int_0^t e^{A\tau} B B^T e^{A^T \tau} \,\mathrm{d}\tau$$

is positive definite for all $t > 0$.

**Exercise 2** (The (infinite) controllability Gramian and a Lyapunov equation)
Let $A \in \mathbb{R}^{n\times n}$ be stable and $Q \in \mathbb{R}^{n\times n}$. Prove that

$$X = \int_0^\infty e^{At} Q e^{A^T t} \,\mathrm{d}t$$

is the unique solution of the Lyapunov equation

$$AX + XA^T + Q = 0.$$

**Exercise 3** (Stability, controllability, and the Lyapunov equation)
Let $A \in \mathbb{R}^{n\times n}$ and $B \in \mathbb{R}^{n\times m}$. Prove that, of the following three statements, any two imply the third:

a) $A$ is a stable matrix,

b) the pair $(A, B)$ is controllable,

c) the Lyapunov equation $AP + PA^T + BB^T = 0$ has a positive definite solution $P$.

**Exercise 4** (Properties of the matrix sign function)
For $Z \in \mathbb{C}^{n\times n}$ with no eigenvalues on the imaginary axis and a Jordan canonical form

$$Z = S \begin{bmatrix} J^+ & \\ & J^- \end{bmatrix} S^{-1},$$

where $J^+ \in \mathbb{C}^{k\times k}$ and $J^- \in \mathbb{C}^{(n-k)\times(n-k)}$ respectively have eigenvalues in $\mathbb{C}_+$ and $\mathbb{C}_-$, we define the matrix sign function as

$$Z = S \begin{bmatrix} I_k & \\ & -I_{n-k} \end{bmatrix} S^{-1}.$$

Show that:

a) the matrix sign function is well-defined,

b) $\operatorname{sign}(T^{-1}ZT) = T^{-1}\operatorname{sign}(Z)T$ for all nonsingular $T \in \mathbb{C}^{n\times n}$,

c) if $Z$ is stable, then $\operatorname{sign}(Z) = -I_n$ and $\operatorname{sign}(-Z) = I_n$,

d) $\operatorname{sign}(Z)^2 = I_n$, i.e. $\operatorname{sign}(Z)$ is a square root of the identity matrix,

e) the Newton iteration $Z_0 = Z$, $Z_{i+1} = \frac{1}{2}(Z_i + Z_i^{-1})$, $i = 0, 1, 2, \ldots$, is a Newton iteration applied to the function $F(X) = X^2 - I$.

**Exercise 5** (Solving Sylvester equations via the matrix sign function)
Consider the Sylvester equation

$$AX + XB + C = 0, \tag{1}$$

with $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{m\times m}$, and $C \in \mathbb{R}^{n\times m}$. Assume that $A$ and $B$ are stable matrices and that $X$ is the solution of the equation (1).

a) Show that

$$\operatorname{sign}\left(\begin{bmatrix} A & C \\ 0 & -B \end{bmatrix}\right) = \begin{bmatrix} -I_n & 2X \\ 0 & I_m \end{bmatrix}.$$

**Hint:** Compute $T^{-1}\begin{bmatrix} A & C \\ 0 & -B \end{bmatrix}T$, for $T = \begin{bmatrix} I_n & X \\ 0 & I_m \end{bmatrix}$.

b) Show that instead of iterating on $\begin{bmatrix} A & C \\ 0 & -B \end{bmatrix}$, one can compute $X$ via an iteration on $A, B, C$.

**Exercise 6** (Implementing a Lyapunov equation solver)
Our goal here is to implement a solver, using matrix sign function Newton iteration, for the Lyapunov equation

$$AX + XA^T + W = 0, \tag{2}$$

where $A \in \mathbb{R}^{n\times n}$ is a stable matrix.

a) Derive the iteration method

$$\begin{aligned} A_0 = A, \quad A_{i+1} &= \frac{1}{2}\left(A_i + A_i^{-1}\right), \\ W_0 = W, \quad W_{i+1} &= \frac{1}{2}\left(W_i + A_i^{-1}W_iA_i^{-T}\right), \end{aligned}$$

for the equation (2) using the solution of the Exercise 5 b). Show that $A_i \to -I_n$ and $W_i \to 2X$.

b) Implement a function `lyap_sgn`, applying the above iteration, with the matrices $A$ and $W$, the maximum number of iterations `maxit`, and the tolerance `tol` for the stopping criterion $\|A_i + I_n\|_F <$ `tol` as inputs.

c) Test your implementation on random examples by computing the relative error

$$\frac{\|AX + XA^T + W\|_F}{\|W\|_F}$$

and plotting how $\|A_i + I_n\|_F$ varies across iterations. Check if the approximate solution you find is symmetric (e.g. by computing $\|X - X^T\|_F$) for symmetric $W$.

**Exercise 7** (Model reduction by balanced truncation)
Here we apply the balanced truncation method to the Clamped Beam model from the NICONET benchmark collection (you need to download `beam.mat` from [1]).

a) Compute the controllability and observability Gramians by solving the Lyapunov equations

$$\begin{aligned} AP + PA^T + BB^T &= 0, \\ A^TQ + QA + C^TC &= 0, \end{aligned}$$

using the function `lyap_sgn` you implemented in the previous Exercise.

b) Compute factorizations $P = S^T S$ and $Q = R^T R$.

c) Compute the singular value decomposition $SR^T = U\Sigma V^T$.

d) Plot the Hankel singular values.

e) Find the reduced order model $(A_r, B_r, C_r) = (W_r^T A V_r, W_r^T B, C V_r)$, where

$$V_r = S^T U(:, 1:r)\Sigma(1:r, 1:r)^{-\frac{1}{2}},$$
$$W_r = R^T V(:, 1:r)\Sigma(1:r, 1:r)^{-\frac{1}{2}},$$

for some $r$.

f) Draw the log-log plots of $\omega \mapsto |H(i\omega)|$ and $\omega \mapsto |H_r(i\omega)|$, where

$$H(s) = C(sI_n - A)^{-1}B,$$
$$H_r(s) = C_r(sI_r - A_r)^{-1}B_r,$$

are the transfer functions of the original and reduced model. Use 1000 logarithmically distributed sample points over the frequency interval $\omega \in [10^{-2}, 10^4]$.

g) Draw the log-log plot of $\omega \mapsto |H(i\omega) - H_r(i\omega)|$, same as in f), with a horizontal line for the upper bound of the $\mathcal{H}_\infty$-error using Hankel singular values.

**Exercise 8** (Balancing-free square root (BFSR) method)
For numerical reasons, the balancing-free square root (BFSR) algorithm is preferred to the method used in the previous Exercise. The difference is in the part e).

a) Compute the projection matrices

$$V_r = P_1 \text{ and } W_r = Q_1 \left(P_1^T Q_1\right)^{-1},$$

where

$$S^T U_1 = P_1 \widehat{R} \text{ and } R^T V_1 = Q_1 \widetilde{R},$$

with $P_1, Q_1 \in \mathbb{R}^{n\times r}$ orthogonal and $\widehat{R}, \widetilde{R} \in \mathbb{R}^{r\times r}$ upper-triangular.

b) Show that the reduced order system is equivalent to a balanced system and that it satisfies the same error bound as the one obtained by the standard square root balanced truncation method.

**Exercise 9** (Low-rank Lyapunov equation solver)
It is possible to combine parts a) and b) in Exercise 7.

a) For the Lyapunov equation

$$AX + XA^T + BB^T = 0,$$

derive the iteration method

$$A_0 = A, \quad A_{i+1} = \frac{1}{2}\left(A_i + A_i^{-1}\right),$$
$$B_0 = B, \quad B_{i+1} = \frac{1}{\sqrt{2}}\begin{bmatrix} B_i & A_i^{-1}B_i \end{bmatrix},$$

by setting $W_i = B_i B_i^T$ in Exercise 6 a).

b) Since $B_{i+1}$ has the twice the number of column as $B_i$, it is necessary to include column compression in the iterations. Implement a function `col_comp` that will perform this for an arbitrary matrix, using rank-revealing LQ decomposition or SVD, with specified error tolerance.

c) Implement a Lyapunov equation solver `lyap_sgn_fac`, using the above iterations with column compression.

**Exercise 10** (Solving algebraic Riccati equations via the matrix sign function)
Motivated by balancing-related methods such as LQG balanced truncation, let us consider the algebraic Riccati equation

$$AX + XA^T - XFX + G = 0,$$

where $A \in \mathbb{R}^{n\times n}$ and $F = F^T,\ G = G^T \in \mathbb{R}^{n\times n}$ are symmetric positive semi-definite matrices and $(A, F)$ is stabilizable. Let

$$M = \begin{bmatrix} A & G \\ F & -A^T \end{bmatrix}$$

and assume that the matrix sign function of $M$ is partitioned as

$$\operatorname{sign}(M) = \begin{bmatrix} Z_{11} & Z_{12} \\ Z_{21} & Z_{22} \end{bmatrix}.$$

Show that

$$\begin{bmatrix} I_n - Z_{11} \\ Z_{21} \end{bmatrix} X = \begin{bmatrix} Z_{12} \\ I_n - Z_{22} \end{bmatrix}.$$

**Hint:** First show that

$$M = \begin{bmatrix} I_n - XQ & X \\ -Q & I_n \end{bmatrix} \begin{bmatrix} A - XF & 0 \\ 0 & -(A - XF)^T \end{bmatrix} \begin{bmatrix} I_n - XQ & X \\ -Q & I_n \end{bmatrix}^{-1},$$

where $Q$ solves $(A - XF)^T Q + Q(A - XF) + F = 0$. Then make use of the properties of the matrix sign function.

# References

[1] http://slicot.org/20-site/126-benchmark-examples-for-model-reduction

## Solutions

**1** $\boxed{\text{a)} \Rightarrow \text{b)}}$ Proof by contraposition. Assume that the rank of $\mathcal{C}$ is less then $n$. We have (w.l.o.g. $t_0 = 0$)

$$x_1 = e^{At_1}x_0 + \int_0^{t_1} e^{A(t_1-t)}Bu(t)\,\mathrm{d}t.$$

Then (using Cayley-Hamilton theorem)

$$\begin{aligned}
x_1 - e^{At_1}x_0 &= \int_0^{t_1} e^{A(t_1-t)}Bu(t)\,\mathrm{d}t = \int_0^{t_1} \sum_{i=0}^{\infty} \frac{1}{i!}(t_1-t)^i A^i Bu(t)\,\mathrm{d}t = \sum_{i=0}^{\infty} A^i B \int_0^{t_1} \frac{1}{i!}(t_1-t)^i u(t)\,\mathrm{d}t \\
&= \sum_{i=0}^{n-1} A^i B\alpha_i \int_0^{t_1} (t_1-t)^i u(t)\,\mathrm{d}t.
\end{aligned}$$

Therefore, $x_1 - e^{At_1}x_0$ is a linear combination of the columns of $\mathcal{C}$, so there exists $x_1$ for which this cannot be satisfied.

$\boxed{\text{b)} \Rightarrow \text{c)}}$ Proof by contraposition. Assume that $P(t)$ is singular for some $t > 0$. Then there is a nonzero vector $v$ such that $P(t)v = 0$. Then also $v^*P(t)v = 0$, so

$$\int_0^t v^* e^{A\tau}BB^T e^{A^T\tau}v\,\mathrm{d}\tau = 0.$$

It follows that $v^*e^{A\tau}B = 0$ for all $\tau \in [0,t]$. By differentiation at $\tau = 0$, it follows $v^*A^iB$ for all $i = 0, 1, \ldots, n-1$. Therefore, $v^*\mathcal{C} = 0$, so $\mathcal{C}$ doesn't have full rank.

$\boxed{\text{c)} \Rightarrow \text{a)}}$ Choose the input

$$u(t) = B^T e^{A^T(t_1-t)}P(t_1)^{-1}\left(-e^{At_1}x_0 + x_1\right).$$

Then the final state is

$$\begin{aligned}
x(t_1) &= e^{At_1}x_0 + \int_0^{t_1} e^{A(t_1-t)}BB^T e^{A^T(t_1-t)}P(t_1)^{-1}\left(-e^{At_1}x_0 + x_1\right)\mathrm{d}t \\
&= e^{At_1}x_0 + \left(\int_0^{t_1} e^{A(t_1-t)}BB^T e^{A^T(t_1-t)}\,\mathrm{d}t\right)P(t_1)^{-1}\left(-e^{At_1}x_0 + x_1\right) \\
&= e^{At_1}x_0 - e^{At_1}x_0 + x_1 \\
&= x_1.
\end{aligned}$$

**2**

$$\begin{aligned}
AX + XA^T + Q &= A\int_0^{\infty} e^{At}Qe^{A^Tt}\,\mathrm{d}t + \left(\int_0^{\infty} e^{At}Qe^{A^Tt}\,\mathrm{d}t\right)A^T + Q \\
&= \int_0^{\infty}\left(Ae^{At}Qe^{A^Tt} + e^{At}Qe^{A^Tt}A^T\right)\mathrm{d}t + Q \\
&= \int_0^{\infty}\frac{\mathrm{d}}{\mathrm{d}t}\left(e^{At}Qe^{A^Tt}\right)\mathrm{d}t + Q \\
&= \left(e^{At}Qe^{A^Tt}\right)\Big|_0^{\infty} + Q \\
&= -Q + Q \\
&= 0.
\end{aligned}$$

**3** $\boxed{\text{a), b)} \Rightarrow \text{c)}}$ a) implies $P = \int_0^\infty e^{At}BB^Te^{A^Tt}\,\mathrm{d}t$, b) implies $P > 0$ (using the same argument as in the first Exercise).

$\boxed{\text{a), c)} \Rightarrow \text{b)}}$ a) and c) imply $P = \int_0^\infty e^{At}BB^Te^{A^Tt}\,\mathrm{d}t > 0$. Assuming that b) is false, from $v^*\mathcal{C} = 0$ for a nonzero $v$, it follows that $v^*e^{At}B = 0$ using Cayley-Hamilton, which then implies that $v^*Pv = 0$, a contradiction.

$\boxed{\text{b), c)} \Rightarrow \text{a)}}$ Let $x^*A = \lambda x^*$. Then $A^Tx = \overline{\lambda}x$. We have

$$\begin{aligned} x^*APx + x^*PA^Tx + x^*BB^Tx &= 0, \\ \lambda x^*Px + \overline{\lambda}x^*Px &= -x^*BB^Tx, \\ (\lambda + \overline{\lambda})x^*Px &= -\|x^*B\|, \\ 2\operatorname{Re}\lambda &= -\frac{\|x^*B\|}{x^*Px}. \end{aligned}$$

If $x^*B = 0$, then $x^*\mathcal{C} = 0$, so $(A, B)$ is uncontrollable, which is a contradiction. Therefore $\operatorname{Re}\lambda < 0$.

**5** a) We see that

$$\begin{bmatrix} I_n & -X \\ 0 & I_m \end{bmatrix}\begin{bmatrix} A & C \\ 0 & -B \end{bmatrix}\begin{bmatrix} I_n & X \\ 0 & I_m \end{bmatrix} = \begin{bmatrix} A & XB + C \\ 0 & -B \end{bmatrix}\begin{bmatrix} I_n & X \\ 0 & I_m \end{bmatrix} = \begin{bmatrix} A & AX + XB + C \\ 0 & -B \end{bmatrix} = \begin{bmatrix} A & 0 \\ 0 & -B \end{bmatrix}.$$

Therefore,

$$\begin{aligned} \operatorname{sign}\left(\begin{bmatrix} A & C \\ 0 & -B \end{bmatrix}\right) &= \begin{bmatrix} I_n & X \\ 0 & I_m \end{bmatrix}\operatorname{sign}\left(\begin{bmatrix} A & 0 \\ 0 & -B \end{bmatrix}\right)\begin{bmatrix} I_n & -X \\ 0 & I_m \end{bmatrix} = \begin{bmatrix} I_n & X \\ 0 & I_m \end{bmatrix}\begin{bmatrix} -I_n & 0 \\ 0 & I_m \end{bmatrix}\begin{bmatrix} I_n & -X \\ 0 & I_m \end{bmatrix} \\ &= \begin{bmatrix} -I_n & X \\ 0 & I_m \end{bmatrix}\begin{bmatrix} I_n & -X \\ 0 & I_m \end{bmatrix} = \begin{bmatrix} -I_n & 2X \\ 0 & I_m \end{bmatrix}. \end{aligned}$$

b) Iterating over $\begin{bmatrix} A & C \\ 0 & -B \end{bmatrix}$ looks like

$$\begin{aligned} \begin{bmatrix} A_{i+1} & C_{i+1} \\ 0 & -B_{i+1} \end{bmatrix} &= \frac{1}{2}\left(\begin{bmatrix} A_i & C_i \\ 0 & -B_i \end{bmatrix} + \begin{bmatrix} A_i & C_i \\ 0 & -B_i \end{bmatrix}^{-1}\right) = \frac{1}{2}\left(\begin{bmatrix} A_i & C_i \\ 0 & -B_i \end{bmatrix} + \begin{bmatrix} A_i^{-1} & A_i^{-1}C_iB_i^{-1} \\ 0 & -B_i^{-1} \end{bmatrix}\right) \\ &= \frac{1}{2}\begin{bmatrix} A_i + A_i^{-1} & C_i + A_i^{-1}C_iB_i^{-1} \\ 0 & -B_i - B_i^{-1} \end{bmatrix}. \end{aligned}$$

Thus we find

$$\begin{aligned} A_{i+1} &= \frac{1}{2}\left(A_i + A_i^{-1}\right), \\ B_{i+1} &= \frac{1}{2}\left(B_i + B_i^{-1}\right), \\ C_{i+1} &= \frac{1}{2}\left(C_i + A_i^{-1}C_iB_i^{-1}\right). \end{aligned}$$

**10**

$$\left[\begin{array}{cc|cc} I_n - XQ & X & I_n & 0 \\ -Q & I_n & 0 & I_n \end{array}\right] \sim \left[\begin{array}{cc|cc} I_n & 0 & I_n & -X \\ -Q & I_n & 0 & I_n \end{array}\right] \sim \left[\begin{array}{cc|cc} I_n & 0 & I_n & -X \\ 0 & I_n & Q & I_n - QX \end{array}\right]$$

$$\begin{aligned} &\begin{bmatrix} I_n & -X \\ Q & I_n - QX \end{bmatrix}\begin{bmatrix} A & G \\ F & -A^T \end{bmatrix}\begin{bmatrix} I_n - XQ & X \\ -Q & I_n \end{bmatrix} \\ &= \begin{bmatrix} A - XF & G + XA^T \\ QA + F - QXF & QG - A^T + QXA^T \end{bmatrix}\begin{bmatrix} I_n - XQ & X \\ -Q & I_n \end{bmatrix} \\ &= \begin{bmatrix} A - XF & 0 \\ QA + F - QXF - FXQ + A^TQ & FX - A^T \end{bmatrix} \\ &= \begin{bmatrix} A - XF & 0 \\ (A - XF)^TQ + Q(A - XF) + F & -(A - XF)^T \end{bmatrix} \\ &= \begin{bmatrix} A - XF & 0 \\ 0 & -(A - XF)^T \end{bmatrix} \end{aligned}$$

$$
\begin{aligned}
\operatorname{sign}\left(\begin{bmatrix} A & G \\ F & -A^T \end{bmatrix}\right) &= \begin{bmatrix} I_n - XQ & X \\ -Q & I_n \end{bmatrix} \begin{bmatrix} -I_n & 0 \\ 0 & I_n \end{bmatrix} \begin{bmatrix} I_n & -X \\ Q & I_n - QX \end{bmatrix} \\
&= \begin{bmatrix} -I_n + XQ & X \\ Q & I_n \end{bmatrix} \begin{bmatrix} I_n & -X \\ Q & I_n - QX \end{bmatrix} \\
&= \begin{bmatrix} -I_n + 2XQ & 2X - 2XQX \\ 2Q & I_n - 2QX \end{bmatrix}
\end{aligned}
$$

$$
\begin{gathered}
-I_n + 2XQ = Z_{11},\ 2X - 2XQX = Z_{12},\ 2Q = Z_{21},\ I_n - 2QX = Z_{22} \\
-I_n + XZ_{21} = Z_{11},\ 2X - XZ_{21}X = Z_{12},\ 2Q = Z_{21},\ I_n - Z_{21}X = Z_{22} \\
-I_n + XZ_{21} = Z_{11},\ 2X - (I_n + Z_{11})X = Z_{12},\ 2Q = Z_{21},\ I_n - Z_{21}X = Z_{22} \\
-I_n + XZ_{21} = Z_{11},\ (I_n - Z_{11})X = Z_{12},\ 2Q = Z_{21},\ Z_{21}X = I_n - Z_{22}
\end{gathered}
$$