On the Robbins problem for cyclic polygons

DRAGUTIN SVRTAN

University of Zagreb
Department of Mathematics
Bijenička cesta 30
10000 Zagreb, Croatia
dsvrtan@math.hr

In a masterfully written paper (published in 1828 in Crelle's Journal) A. F. Möbius studied some properties of the polynomial equations for the circumradius of arbitrary cyclic polygons (convex and nonconvex) and showed the existence of a polynomial of degree $\delta_n = \frac{n}{2}\binom{n-1}{\lfloor (n-1)/2 \rfloor} - 2^{n-2}$ that relates the square of a circumradius (r^2) of a cyclic polygon to the squared side lengths. His approach is based, by a clever use of trigonometry, on the rationalization (in terms of the squared sines) of the sine of a sum of n angles (peripheral angles of a cyclic polygon). In this way one obtains a polynomial relating the circumradius to the side lengths squared. These polynomials, known also as generalized Heron $r-$polynomials, are a kind of generalized (symmetric) multivariable Chebyshev polynomials and are quite difficult to be computed explicitly. By an argument involving series expansions (cf. [7]) he proved that the r^2-degree for cyclic $n-$polygon is equal to δ_n. Möbius also obtained for the squared area a rational function in $r^2, a_1, a_2, \ldots, a_n$ involving partial derivatives, with respect to side length variables, of all the coefficients of the Heron $r-$polynomial.

About ten years ago David Robbins ([2], [3]) obtained, for the first time, concise explicit formulas for the areas of cyclic pentagons and hexagons (he mentioned that he computed also the circumradius polynomials for cyclic pentagons and hexagons but was not able to put either formula into a sensible compact form). In [2] two general conjectures (Conjecture1 and Conjecture2), naturally extending nice Möbius product formulas for the leading and constant terms for pentagons and hexagons are given. We shall give a proof of these conjectures up to $n = 8$.

One of the Additional Conjectures of Robbins, stating that the degree of the minimal A-polynomial equation for cyclic $n-$polygons $\alpha_n(16S^2, a_1^2, \ldots, a_n^2) = 0$, (i.e. of the generalized Heron $A-$ polynomial), is equal to δ_n was established in [9] first (by relating it to the Sabitov theory of volume polynomials of polyhedra, see nice survey article by Pak) and later in [7] (obtained by reviving the argument of Möbius and reproving the Robbins lower bound on the degrees of minimal polynomials).

In Robbins work a method of undetermined coefficients is used for pentagons (70 unknowns) and hexagons (134 unknowns). This method seems to be inadequate for heptagons because one would need to handle a linear system with 143307 undetermined coefficients. By using a clever substitution (Robbins t_i's) he was able to write the pentagon and hexagon area equations in a compact form. He wrote his formulas also as a discriminant of some (still mysterious) qubic. In [7] a formula for the area polynomial for heptagons and octagons is found in the form of a quotient of two resultants, one of which could be expanded explicitly so far. This exiting result was finished by two of the Robbins collaborators just few months later after Robbins passed away.

Another approach, which uses elimination of diagonals in cyclic polygons, is treated at length in [10] where among numerous results one also finds an explicit derivation of the Robbins area polynomial for pentagons by using some general properties, developed in that paper, together with a little use of one undetermined coefficient. In [4], the Robbins pentagon area formula was obtained intrinsically with a simpler system of equations by a direct elimination (and MAPLE of course) with no assistance of undetermined coefficient method. Also for hexagon (a much harder case) an intrinsic proof was found in [6].

In this talk we illustrate yet another approach to the Robbins problem, especially well suited for obtaining Heron r–polynomials. We have discovered that Robbins problem is somehow related to a Wiener–Hopf factorization. We first associate a Laurent polynomial L_P to a cyclic polygon P,which is invariant under similarity of cyclic polygons (it is a kind of "conformal invariant"). Then there exists a (Wiener-Hopf) factorization of L_P into a product of two polynomials, $\gamma_+(1/z)$ and $\gamma_-(z)$, (in our case it will be $\gamma_- = \gamma_+ =: \gamma$) providing a complex realization of P is given. The factorization (i.e. $\gamma(z)$) is then given in terms of the elementary symmetric functions e_k of the vertex quotients, if we regard vertices of (a realization of) P as complex numbers of equal moduli $(= r)$. For $e_k's$, viewed as the unknowns, we then obtain a system of n quadratic equations, arising from our Wiener-Hopf factorization, with $n - 1$ unknowns (note that e_n is necessarily equal to 1 as a product of all the vertex quotients (we call this a "cocycle property" or simply "cocyclicity")). The consistency condition (obtained by eliminating all $e_k, k = 1..n - 1$) for our "overdetermined" system will then give a relation between the coefficients of our conformal invariant L_P, which in turn will be nothing but the equation relating the inverse square radius of P with the side lengths squared.

In the course of these investigations we found another type of substitutions by expressing the coefficients of L_P in terms of the inverse radius squared (ρ) and the elementary symmetric functions of side lengths squared. By using this substitutions, our Heron ρ-polynomials get remarkably small coefficients. Further simplifications we have obtained by doing computations in some quadratic algebraic extensions. In such quadratic extensions we can simplify our original system (having all but one equations quadratic) by replacing two quadratic equations by two linear ones). Also the final result can be written in a more compact form $\rho_n = A_n^2 - \Delta_n B_n^2$ (a Pell equation). Thus the number of terms is the final formula is roughly a square root of the number of terms in the fully expanded formula. With such tricks we have obtained so far , down to earth, explicit formulas for Heron $\rho-$polynomials, up to $n = 8$.

For tangential polygons the Heron r–polynomials can be handled more easily and more generally in terms of tangential segments of a polygon instead of lengths of its sides.

References and Literature for Further Reading

[1] A. F. Möbius, Ueber die Gleichungen, mittelst welcher aus der Seiten eines in einen Kreis zu beschriebenden Vielecks der Halbmesser des Kreises un die Flähe des Vielecks gefunden werden, *Crelle's Journal*, 3:5–34. 1828.

[2] D. P. Robbins, Areas of polygons inscribed in a circle, *Discrete & Computational Geometry*, 12:223–236, 1994.

[3] D. P. Robbins, Areas of polygons inscribed in a circle, *Amer. Math. Monthly* 102 (1995), 523–530

[4] D. Svrtan, D. Veljan and V. Volenec, Geometry of pentagons: from Gauss to Robbins, math.MG/0403503

[5] D. Svrtan, A new approach to rationalization of surds, submitted

[6] D. Svrtan, Intrinsic Proofs for Area and Circumradius of Cyclic Hexagons. Solving Equations for Arbitrary Cyclic Polygons, Abstract: MATH/CHEM/COMP 2005, Dubrovnik, June 20–25, 2005, p. 70

[7] F. Miller Maley, David P. Robbins, Julie Roskies, On the areas of cyclic and semicyclic polygons, math.MG/0407300v1

[8] I. Pak, The area of cyclic polygons: Recent progress on Robbins' conjectures, math.MG/0408104

[9] M. Fedorchuk and I. Pak, Rigidity and polynomial invariants of convex polytopes., *Duke Math. J.*, to appear

[10] V. V. Varfolomeev, Inscribed polygons and Heron Polynomials, *Sbornik: Mathematics*, 194 (3):311–331, 2003.

[11] V. V. Varfolomeev, Galois groups of Heron-Sabitov polynomials for pentagons inscribed in a circle, *Sbornik: Mathematics*, 195 (3):3–16, 2004.