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On non-coercive mixed problems for parameter-dependent elliptic operators

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Abstract. We consider a non-coercive mixed boundary value problem in a bounded domain D of $\mathbb{R}^n$ for a second order parameter-dependent elliptic differential operator $A(x,\partial,\lambda)$ with complex-valued essentially bounded measured coefficients and complex parameter λ. The differential operator is assumed to be of divergent form in D, the boundary operator $B(x,\partial)$ is of Robin type with possible pseudo-differential components on ∂D. The boundary of D is assumed to be a Lipschitz surface. Under these assumptions the pair $(A(x,\partial,\lambda),B)$ induces a holomorphic family of Fredholm operators $L(\lambda): H^+(D) \to H^-(D)$ in suitable Hilbert spaces $H^+(D)$, $H^-(D)$ of Sobolev type. If the argument of the complex-valued multiplier of the parameter in $A(x,\partial,\lambda)$ is continuous, then we prove that the operators $L(\lambda)$ are continuously invertible for all λ with sufficiently large modulus $|\lambda|$ on each ray on the complex plane $\mathbb{C}$ where the operator $A(x,\partial,\lambda)$ is parameter-dependent elliptic. We also describe reasonable conditions for the system of root functions related to the family $L(\lambda)$ to be (doubly) complete in the spaces $H^+(D)$, $H^-(D)$ and the Lebesgue space $L^2(D)$.
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1. Introduction

The notion of a parameter-dependent elliptic operator provides a useful link between the theories of boundary value problems for parabolic and elliptic operators (see, for instance, [3]). Investigating a boundary value problem for a parameter-dependent elliptic operator $A(x,\partial,\lambda)$ on a ray in the complex plane, first one aims to prove the continuous invertibility in proper functional spaces $H^+(D)$, $H^-(D)$ of the corresponding family $L(\lambda): H^+(D) \to H^-(D)$ of the operators for all λ with sufficiently large modulus $|\lambda|$ on the ray (see [3, 7, 8, 17]). The next step is to prove the (multiple) completeness of the corresponding root functions associated with the parameter-dependent family (see, for instance, [10, 12, 16, 22]). Actually, this provides a justification for the application of Galerkin type methods and a numerical solution of the problem. For elliptic (coercive) problems, the results of this type are well known. The investigation is usually based on the classical methods of functional analysis and the theory of partial differential equations (see [1, 5, 10, 12, 16], and many others).

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For domains with smooth boundaries, the standard Shapiro-Lopatinsky conditions with a parameter and their generalizations are usually imposed (see [3, 7, 8]). The spectral theory in non-smooth domains usually depends upon hard analysis near singularities on the boundary (see, for instance, [4, 20]).

Recently, the classical approach was adapted for investigation of spectral properties of non-coercive mixed problems for strongly elliptic operators in Lipschitz domains (see [18, 19]). An essential part of the approach is the analysis in spaces of negative smoothness. We use this method to prove that under reasonable assumptions the non-coercive operator pencil $L(\lambda) : H^+(D) \to H^-(D)$ has almost the same properties as a coercive one. Also, an example related to non-coercive mixed problems for a strongly elliptic two-dimensional Lamé system is considered.

2. A Fredholm holomorphic family of mixed problems

Let D be a bounded domain in Euclidean space $\mathbb{R}^n$ with Lipschitz boundary ∂D. We consider complex-valued functions defined in the domain D. We write $L^q(D)$ for the space of all (equivalence classes of) measurable functions u in D, such that the Lebesgue integral of $|u|^q$ over D is finite. As usual, this scale continues to include the case $q = \infty$, too. We denote by $H^1(D)$ the Sobolev space and by $H^s(D)$, $0 < s < 1$ the Sobolev-Slobodetskii spaces.

Consider a second order differential operator

$$A(x, \partial, \lambda)u = -\sum_{i,j=1}^{n} \partial_i(a_{i,j}(x)\partial_j u) + \sum_{j=1}^{n} a_j(x)\partial_j u + a_0(x)u + E(\lambda)u$$

in the domain D with a complex parameter λ; here $x = (x_1, \dots, x_n)$ are the coordinates in $\mathbb{R}^n$, $\partial_j = \frac{\partial}{\partial x_j}$ and

$$E(\lambda)u = \lambda \left(\sum_{j=1}^{n} a_j^{(1)}(x)\partial_j u + a_0^{(1)}(x)u \right) + \lambda^2 a_0^{(2)}(x)u.$$

The coefficients $a_{i,j}$, a_j, $a_j^{(1)}$, $a_0^{(1)}$, $a_0^{(2)}$ are assumed to be complex-valued functions of class $L^\infty(D)$. We suppose that the matrix $\mathfrak{A}(x) = (a_{i,j}(x))_{\substack{i=1,\dots,n \\ j=1,\dots,n}}$ is Hermitian and that it satisfies

$$\sum_{i,j=1}^{n} a_{i,j}(x)\overline{w}_i w_j \geq 0 \text{ for all } (x, w) \in \overline{D} \times \mathbb{C}^n, \tag{1}$$

$$\sum_{i,j=1}^{n} a_{i,j}(x)\xi_i \xi_j \geq m_0 |\xi|^2 \text{ for all } (x, \xi) \in \overline{D} \times (\mathbb{R}^n \setminus \{0\}), \tag{2}$$

where m_0 is a positive constant independent of x and ξ. Estimate (2) is nothing but the statement that the operator $A(x, \partial, 0)$ is strongly elliptic. Since the coefficients of the operator and the functions under consideration are complex-valued, the matrix $\mathfrak{A}(x)$ can be degenerate. In particular, inequalities (1) and (2) are weaker than the (strong) coerciveness of the Hermitian form, i.e., the existence of a constant m_0 such that

$$\sum_{i,j=1}^{n} a_{i,j}(x)\,\overline{w}_i w_j \geq m_0\,|w|^2 \text{ for all } (x,w) \in \overline{D} \times (\mathbb{C}^n \setminus \{0\}). \tag{3}$$

We consider the following Robin type boundary operator

$$B = b_1(x) \sum_{i,j=1}^{n} a_{i,j}(x)\,\nu_i \partial_j + \partial_\tau + B_0,$$

where b_1 is a bounded function on ∂D, $\nu(x) = (\nu_1(x), \dots, \nu_n(x))$ is the unit outward normal vector of ∂D at $x \in \partial D$, $\partial_\tau = \sum_{j=1}^{n} \tau_j(x)\partial_j$ is the tangential derivative with a tangential field $\tau = (\tau_1, \dots, \tau_n)$ on ∂D and B_0 is a densely defined linear operator in $L^2(\partial D)$ of "order" not exceeding 1. The function $b_1(x)$ is allowed to vanish on an open connected subset S of ∂D with piecewise smooth boundary ∂S and the vector τ vanishes identically on S.

To specify the operator B_0, fix a number $0 \leq \rho \leq 1/2$ and a bounded linear operator $\Psi : H^\rho(\partial D) \to L^2(\partial D)$. The range of ρ is motivated by trace and duality arguments. We will consider operator B_0 of the following form

$$B_0 = \chi_S u + b_1 \left(\Psi^* \Psi(u) + \delta B_0 \right),$$

where χ_S is the characteristic function of the set S on ∂D, $\Psi^* : L^2(\partial D) \to H^\rho(\partial D)$ is the adjoint operator for Ψ and δB_0 is a "low order" perturbation that will be described later. For $\rho = 0$, a typical operator Ψ is a zero order differential operator, i.e., it is given by $\Psi u = \psi u$, where ψ is a function on ∂D locally bounded away from ∂S. Then $(\Psi^* \Psi u)(x) = |\psi(x)|^2 u(x)$ is invertible provided that $|\psi(x)| \geq c > 0$. If ∂D is C^2-smooth, then a model operator Ψ is $\Psi = (1 + \Delta_{\partial D})^{\rho/2}$, where $\Delta_{\partial D}$ is the Laplace-Beltrami operator on the boundary.

Consider the following family of boundary value problems. Given data f in D and u_0 on ∂D, find a distribution u in D which satisfies

$$\begin{cases} A(x,\partial,\lambda)u = f & \text{in } D, \\ \quad B(x,\partial)u = u_0 & \text{at } \partial D. \end{cases} \tag{4}$$

If $\lambda = 0$ and Ψ is given by the multiplication on a function, this is a well known mixed problem of Zaremba type (see [23]). It can be handled in a standard way in Sobolev type spaces associated with Hermitian forms or in Hölder spaces and Sobolev spaces using the potential methods (for the coercive case see [13, 23, 15] and elsewhere). In the non-coercive case, the methods should be more subtle (see, for instance, [2, 19]) because of the lack of regularity of its solutions near the boundary of the domain. In [19], the method, involving non-negative Hermitian forms, was adopted to study problem (4) in non-coercive cases with a zero order differential operator Ψ. Namely, denote by $C^1(\overline{D}, S)$ the subspace of $C^1(\overline{D})$ consisting of those functions whose restriction to the boundary vanishes on $\overline{S}$. Let $H^1(D,S)$ be the closure of $C^1(\overline{D}, S)$ in $H^1(D)$. This space is Hilbert under the induced norm. Since on S the boundary operator reduces to $B = \chi_S$ and $\chi_S(x) \neq 0$ for $x \in S$, the functions of $H^1(D)$ satisfying $Bu = 0$ on ∂D belong to $H^1(D,S)$.

Split a_0 into two parts $a_0 = a_{0,0} + \delta a_0$, where $a_{0,0}$ is a non-negative bounded function in D. Then, under reasonable assumptions, the Hermitian form

$$(u, v)_+ = \int_D \sum_{i,j=1}^n a_{i,j} \partial_j u \overline{\partial_i v}\, dx + (a_{0,0} u, v)_{L^2(D)} + (\Psi(u), \Psi(v))_{L^2(\partial D)}$$

defines an inner product on $H^1(D, S)$. Denote by $H^+(D)$ the completion of the space $H^1(D, S)$ with respect to the corresponding norm $\| \cdot \|_+$.

To study problem (4) we need an embedding theorem for the space $H^+(D)$.

Theorem 1. *Let the coefficients $a_{i,j}$ be C^∞ in a neighbourhood X of the closure of D, inequalities (1), (2) hold and there is a constant $c_1 > 0$, such that*

$$\|\Psi u\|_{L^2(\partial D)} \geq c_1 \|u\|_{H^\rho(\partial D)} \text{ for all } u \in H^1(\partial D, S). \tag{5}$$

If there is a positive constant c_2, such that $a_{0,0} \geq c_2$ in D or the operator A is strongly elliptic in a neighbourhood X of $\overline{D}$ and

$$\int_X \sum_{i,j=1}^n a_{i,j} \partial_j u \overline{\partial_i u}\, dx \geq m_1 \|u\|^2_{L^2(X)}$$

for all $u \in C^\infty_{\mathrm{comp}}(X)$, with $m_1 > 0$ a constant independent of u, then the space $H^+(D)$ is continuously embedded into $H^s(D)$, where s is given by

$$s = \begin{cases} 1/2 - \epsilon \text{ with } \epsilon > 0, & \text{if } \rho = 0, \\ 1/2, & \text{if } \rho = 0 \text{ and } \partial D \in C^2, \\ 1/2 + \rho, & \text{if } 0 < \rho \leq 1/2. \end{cases}$$

Proof. It is similar to the proof of [19, Theorem 2.5] corresponding to the case where $\rho = 0$ and Ψ is given by the multiplication on a function. □

Of course, under the coercive estimate (3), the space $H^+(D)$ is continuously embedded into $H^1(D)$. However, in general, the embedding described in Theorem 1 is rather sharp (see [19, Remark 5.1], [18] and §5 below). In particular, it may happen that the space $H^+(D)$ can not be continuously embedded into $H^{\rho+\epsilon}(D)$ with any $\epsilon > 0$. Thus the operator Ψ is introduced here in order to improve, if necessary, the smoothness of elements of $H^+(D)$ in the non-coercive case.

In order to pass to the generalized setting of the mixed problem, we need that all the derivatives $\partial_j u$ belong to $L^2(D)$ for an element $u \in H^+(D)$, at least if $s \leq 1/2$ in Theorem 1. However, if $0 < s < 1$, then the absence of coerciveness does not allow this. To cope with this difficulty we note that the operator $\sum_{i,j=1}^n \partial_i (a_{i,j} \partial_j \cdot)$ admits a factorisation, i.e., there is an $(m \times n)$-matrix $\mathfrak{D}(x) = (\mathfrak{D}_{i,j}(x))_{\substack{i=1,\dots,m \\ j=1,\dots,n}}$ of bounded functions in D, such that $(\mathfrak{D}(x))^* \mathfrak{D}(x) = \mathfrak{A}(x)$ for almost all $x \in D$. For example, one could take the standard non-negative selfadjoint square root $\mathfrak{D}(x) = \sqrt{\mathfrak{A}(x)}$ of the matrix $\mathfrak{A}(x)$. Then

$$\sum_{i,j=1}^n a_{i,j} \partial_j u \overline{\partial_i v} = (\mathfrak{D} \nabla v)^* \mathfrak{D} \nabla u = \sum_{k=1}^m \overline{\mathfrak{D}_k v}\, \mathfrak{D}_k u,$$

for all smooth functions u and v in $\overline{D}$, where ∇u is thought of as an n-column with entries $\partial_1 u, \dots, \partial_n u$, and $\mathfrak{D}_k u := \sum_{l=1}^n \mathfrak{D}_{k,l}(x)\partial_l u$, $k = 1, \dots, m$. Then, by the definition of the space $H^+(D)$, any term $\tilde{a}_k(x)\mathfrak{D}_k u$, $k = 1, \dots, m$, belongs to $L^2(D)$ if $u \in H^+(D)$ and $\tilde{a}_k \in L^\infty(D)$. Thus, if $0 < s < 1$, then we may confine ourselves to first order summands of the form

$$\sum_{k=1}^m \tilde{a}_k(x)\mathfrak{D}_k \text{ and } \sum_{k=1}^m \tilde{a}_k^{(1)}(x)\mathfrak{D}_k$$

instead of $\sum_{j=1}^n a_j(x)\partial_j$ and $\sum_{k=1}^n a_j^{(1)}(x)\partial_j$. For this purpose, we fix a factorization $\mathfrak{D}(x)$ of the matrix $\mathfrak{A}(x)$ and functions $\tilde{a}_k \in L^\infty(D)$, $\tilde{a}_k^{(1)} \in L^\infty(D)$, $k = 1, \dots, m$.

These considerations allow us to handle problem (4) with the use of the standard tools of functional analysis. Indeed, let $H^-(D)$ stand for the completion of space $H^+(D)$ with respect to the norm

$$\|u\|_- = \sup_{\substack{v \in H^+(D) \\ v \neq 0}} \frac{|(v, u)_{L^2(D)}|}{\|v\|_+}.$$

It is the dual space for $H^+(D)$ with respect to the pairing $\langle \cdot, \cdot \rangle : H^-(D) \times H^+(D) \to \mathbb{C}$ induced by the scalar product $(\cdot, \cdot)_{L^2(D)}$

$$\langle u, v \rangle = \lim_{\nu \to +\infty} (u_\nu, v)_{L^2(D)}, \quad u \in H^-(D), v \in H^+(D),$$

where $\{u_\nu\} \subset H^+(D)$ converges to u in $H^-(D)$, see [15]. Note that under the hypothesis of Theorem 1, the natural embedding $\iota : H^+(D) \to L^2(D)$ is continuous; it is compact if (5) holds. Let $\iota' : L^2(D) \to H^+(D)$ stand for the adjoint map for ι with respect to the pairing $\langle \cdot, \cdot \rangle$, i.e.,

$$\langle \iota' u, v \rangle = (u, \iota v)_{L^2(D)} \text{ for all } u \in L^2(D), v \in H^+(D).$$

Now, integration by parts leads to a weak formulation of problem (4): given $f \in H^-(D)$, find $u \in H^+(D)$, such that for all $v \in C^1(\overline{D}, S)$ we have

$$\begin{aligned}(u, v)_+ + \Big(\big(\sum_{j=k}^m \tilde{a}_k \mathfrak{D}_k + \delta a_0 + E(\lambda)\big)u, v\Big)_{L^2(D)}& \\ + \big((b_1^{-1}\partial_\tau + \delta B_0)u, v\big)_{L^2(\partial D \setminus S)} &=< f, v >. \qquad (6)\end{aligned}$$

By the Cauchy inequality, if

$$\Big| \big((b_1^{-1}\partial_\tau + \delta B_0)u, v\big)_{L^2(\partial D \setminus S)} \Big| \le c\, \|u\|_+ \, \|u\|_+$$

with a constant $c > 0$ independent of $u, v \in H^+(D)$, then (6) induces a holomorphic (with respect to $\lambda \in \mathbb{C}$) family $L(\lambda) : H^+(D) \to H^-(D)$ of bounded linear operators.

Denote by L_0 the operator $L(0)$ in the case where $\tau \equiv 0$, $\delta B_0 \equiv 0$, $\delta a_0 \equiv 0$, $\tilde{a}_k \equiv 0$, $k = 1, \dots m$. According to [19, Lemma 2.6], the operator $L_0 : H^+(D) \to$

$H^-(D)$ is continuously invertible and $\|L_0\| = \|L_0^{-1}\| = 1$. Then we can consider each operator $L(\lambda)$, $\lambda \in \mathbb{C}$, as a perturbation of L_0.

Actually, it is convenient to endow the space $H^-(D)$ with the scalar product

$$(u, v)_- = (L_0^{-1} u, L_0^{-1} v)_+ =< L_0^{-1} u, v >,\ u, v \in H^-(D) \tag{7}$$

coherent with the norm $\|\cdot\|_-$ see, for instance [19, p. 3316 and formula (2.2)].

Lemma 1. *Under the hypothesis of Theorem 1, if δB_0 maps $H^\rho(\partial D, S)$ continuously into $H^{-\rho}(\partial D)$, then the term $(\delta B_0 u, v)_{L^2(\partial D)}$ induces a bounded operator $\delta L_B : H^+(D) \to H^-(D)$. If δB_0 maps $H^\rho(\partial D, S)$ compactly into $H^{-\rho}(\partial D)$ then the operator δL_B is compact. In particular, if δB_0 is given by the multiplication on a function $\delta b_0 \in L^\infty(\partial D \setminus S)$, then*

1) δB_0 maps $H^\rho(\partial D, S)$ compactly into $H^{-\rho}(\partial D)$ for $0 < \rho \le 1/2$,

2) δB_0 maps $L^2(\partial D, S)$ continuously into $L^2(\partial D)$ for $\rho = 0$.

Proof. The proof is standard, cf. [19, Lemma 4.6]. □

Clearly, the linear span of the vectors

$$\tau_{i,j} = \vec{e}_j \nu_i(x) - \vec{e}_i \nu_j(x), \quad i > j,$$

coincides with the tangential plan at each point $x \in \partial D$ where it exists. Thus we may consider tangential partial differential operators of the following form:

$$\partial_\tau = \sum_{i>j} k_{i,j}(x) \partial_{\tau_{i,j}}.$$

Lemma 2. *Let $H^+(D)$ be continuously embedded into $H^1(D, S)$. If $k_{i,j}/b_1$ is of Hölder class $C^{0,\lambda}$ in the closure of $\partial D \setminus S$ for all $i > j$, with $\lambda > 1/2$, then the term $(b_1^{-1} \partial_\tau u, v)_{L^2(\partial D \setminus S)}$ induces a bounded operator $\delta L_\tau : H^+(D) \to H^-(D)$.*

Proof. The statement was proved in [19, Lemma 6.6]. □

Theorem 2. *Under the hypothesis of Theorem 1, let $\tau = 0$, unless $s = 1$. If either the term $(\delta B_0 u, v)_{L^2(\partial D)}$ induces a bounded operator δL_B from $H^+(D)$ to $H^-(D)$ with $\|\delta L_B + \delta L_\tau\| < 1$ or $\|\delta L_\tau\| < 1$ and the term $(\delta B_0 u, v)_{L^2(\partial D)}$ induces a compact operator from $H^+(D)$ to $H^-(D)$, then $\{L(\lambda)\}_{\lambda \in \mathbb{C}}$ is a holomorphic family of Fredholm operators of zero index.*

Proof. Follows from Lemmas 1 and 2 because $H^+(D)$ is compactly embedded into $L^2(D)$ under the hypothesis of Theorem 1. □

3. Mixed problems for parameter-dependent elliptic operators

To obtain the main theorem of this paper we recall that the operator $A(x, \partial, \lambda)$ is parameter-dependent elliptic on a ray $\Gamma = \{\arg(\lambda) = \varphi_\Gamma\}$ on the complex plane $\mathbb{C}$ if

$$\sum_{i,j=1}^{n} a_{i,j}(x)\zeta_i\zeta_j + \lambda \sum_{j=1}^{n} a_j^{(1)}(x)\zeta_j + \lambda^2 a_0^{(2)}(x) \neq 0 \tag{8}$$

for all $x \in \overline{D}$ and all $(\lambda, \zeta) \in (\Gamma \times \mathbb{R}^n) \setminus \{0, 0\}$.

In particular, if the operator $A(x, \partial, \lambda)$ is parameter-dependent elliptic on the ray Γ, then by taking $\zeta = 0$ and $\lambda \neq 0$ in (8) we obtain $a_0^{(2)}(x) \neq 0$ for all $x \in D$.

In the sequel we consider the case where $E(\lambda) = \lambda^2 a_0^{(2)}(x)$, the most common in applications. Let $\varphi_0(x) = \arg\left(a_0^{(2)}(x)\right)$. Denote by $C : H^+(D) \to H^-(D)$ the operator that is induced by the term $(a_0^{(2)}(x)u, v)_{L^2(D)}$.

Lemma 3. *Let*

$$a_0^{(2)} \neq 0 \text{ almost everywhere in } D. \tag{9}$$

Then the operator $C : H^+(D) \to H^-(D)$ is injective.

Lemma 4. *Suppose that the matrix $\mathfrak{A}(x)$ is Hermitian non-negative and (2) is fulfilled. If $E(\lambda) = \lambda^2 a_0^{(2)}$, then the operator $A(x, \partial, \lambda)$ is parameter-dependent elliptic on the ray Γ if and only if*

$$|a_0^{(2)}(x)| > 0 \text{ for all } x \in \overline{D}; \tag{10}$$

$$\cos(\varphi_0(x) + 2\varphi_\Gamma) > -1 \text{ for all } x \in \overline{D}. \tag{11}$$

If $|a_0^{(2)}(x)| \in C(\overline{D})$, then (10) is equivalent to the following

$$|a_0^{(2)}(x)| \geq \theta_0 > 0 \text{ for all } x \in \overline{D}; \tag{12}$$

similarly, if $\varphi_0(x) \in C(\overline{D})$, then (11) is equivalent to the following

$$\cos(\varphi_0(x) + 2\varphi_\Gamma) \geq \theta_1(\Gamma) = \theta_1 > -1 \text{ for all } x \in \overline{D}, \tag{13}$$

where the constants θ_0, θ_1 do not depend on x.

Clearly, under the hypothesis of Theorem 2 we can decompose

$$L(\lambda) = L_0 + \delta_c L + \delta_s L + \lambda^2 C,$$

where $\delta_c L : H^+(D) \to H^-(D)$ is a compact operator and $\delta_s L : H^+(D) \to H^-(D)$ is a bounded one. Moreover, the family $L(\lambda)$ is Fredholm if $\|\delta_s L\| < 1$.

Theorem 3. *Let either Ψ is given by the multiplication on a function $\psi \in L^\infty(\partial D)$ or $\partial D \in C^\infty$ and Ψ is a pseudodifferential operator on ∂D. Let also $E(\lambda) = \lambda^2 a_0^{(2)}$, the hypothesis of Theorem 2 be fulfilled, and (9) and (13) hold true. If $\varphi_0 \in C(\overline{D})$ and $\|\delta_s L\|^2 + (\max(0, -\theta_1(\Gamma)))^2 < 1$, then*

1) there is $\gamma_0 \in \Gamma$ such that the operators $L(\lambda) : H^+(D) \to H^-(D)$ are continuously invertible for all $\lambda \in \Gamma$ with $|\lambda| \geq |\gamma_0|$;

2) the operators $L(\lambda)$ are continuously invertible for all $\lambda \in \mathbb{C}$ except a discrete countable set $\{\lambda_\nu\}$ without limit points in $\mathbb{C}$.

Proof. We begin with the following lemma. Set $\eta(\Gamma) = \max(0, -\theta_1)$.

Lemma 5. *Under the hypothesis of Theorem 3, there is $k_0 \in \mathbb{N}$ such that for all $\lambda \in \Gamma$ with $|\lambda| \geq k_0$ we have*

$$\|(L_0 + \delta_s L + \lambda^2 C)u\|_- \geq \Big(\sqrt{1 - \eta^2(\Gamma)} - \|\delta_s L\|\Big) \|u\|_+ \text{ for all } u \in H^+(D)$$

and there are positive constants $p_1 = p_1(\varphi_\Gamma), q_1 = q_1(\varphi_\Gamma)$ such that

$$\|(L_0 + \delta_s L + \lambda^2 C)u\|_- \geq p_1 \|u\|_+ + q_1 |\lambda|^2 \|Cu\|_- \tag{14}$$

for all $u \in H^+(D)$ and $\lambda \in \Gamma$ with $|\lambda| \geq k_0$.

Proof. Given any $u \in H^+(D)$ a computation with the use of formula (7) shows that

$$\lambda^2 \langle Cu, u\rangle = |\lambda|^2 \int_D |a_0^{(2)}(x)| |u(x)|^2 e^{\sqrt{-1}(\varphi_0(x) + 2\varphi_\Gamma)}\, dx,$$

$$\begin{aligned} \|(L_0 + \lambda^2 C)u\|_-^2 &= \langle u + \lambda^2 L_0^{-1} Cu, (L_0 + \lambda^2 C)u\rangle^2 \\ &= \langle u, L_0 u\rangle + \langle \lambda^2 L_0^{-1} Cu, \lambda^2 Cu\rangle + \overline{\lambda}^2 \langle u, Cu\rangle + \lambda^2 \langle L_0^{-1} Cu, L_0 u\rangle \\ &= \|u\|_+^2 + |\lambda|^4 \|Cu\|_-^2 + \overline{\lambda}^2 \langle u, Cu\rangle + \lambda^2 (L_0^{-1} Cu, u)_+ \\ &= \|u\|_+^2 + |\lambda|^4 \|Cu\|_-^2 + \overline{\lambda}^2 \langle u, Cu\rangle + \lambda^2 \langle Cu, u\rangle \\ &= \|u\|_+^2 + |\lambda|^4 \|Cu\|_-^2 + 2\Re\Big(\lambda^2 \langle Cu, u\rangle\Big). \end{aligned} \tag{15}$$

Clearly, for $\lambda \in \Gamma$,

$$\Re\Big(\lambda^2 \langle Cu, u\rangle\Big) = |\lambda|^2 \int_D |a_0^{(2)}(x)| |u(x)|^2 \cos(\varphi_0(x) + 2\varphi_\Gamma)\, dx. \tag{16}$$

If $\theta_1 \in [0, 1]$, then $\eta(\Gamma) = 0$ and for all $u \in H^+(D)$ we immediately have:

$$\begin{aligned} \|(L_0 + \lambda^2 C)u\|_-^2 &\geq \|u\|_+^2 + |\lambda|^4 \|Cu\|_-^2, \\ \|(L_0 + \delta_s L + \lambda^2 C)u\|_- &\geq \|(L_0 + \lambda^2 C)u\|_- - \|\delta_s L u\|_- \\ &\geq \sqrt{\|u\|_+^2 + |\lambda|^4 \|Cu\|_-^2} - \|\delta_s L u\|_-. \end{aligned}$$

Then, for $\alpha \in [0, \pi/2]$ and non-negative numbers a, b, we have

$$\sqrt{a + b} \geq \sqrt{a} \cos(\alpha) + \sqrt{b} \sin(\alpha). \tag{17}$$

As $\|\delta_s L\| < \sqrt{1 - \eta^2(\Gamma)} = 1$, there is $\alpha_0 \in (0, \pi/2)$ such that

$$\|\delta_s L\| < \cos(\alpha_0)$$

In particular, this means that for all $u \in H^+(D)$ and all $\lambda \in \Gamma$ we have:

$$\begin{aligned}
\|(L_0+\delta_s L+\lambda^2 C)u\|_- &\geq \|u\|_+ - \|\delta_s L u\|_- \geq (1-\|\delta_s L\|)\|u\|_+, \\
\|(L_0+\delta_s L+\lambda^2 C)u\|_- &\geq \cos(\alpha_0)\|u\|_+ + \sin(\alpha_0)|\lambda|^2\|Cu\|_- - \|\delta_s L u\|_- \\
&\geq (\cos(\alpha_0)-\|\delta_s L\|)\|u\|_+ + \sin(\alpha_0)|\lambda|^2\|Cu\|_-,
\end{aligned}$$

i.e., the desired inequalities are true if $\theta_1 \in [0,1]$.

If $\theta_1 \in (-1,0)$, then, by (16) and (13),

$$\Re\Big(\lambda^2\langle Cu,u\rangle\Big) \geq -|\theta_1||\lambda|^2 \int_D |a_0^{(2)}(x)||u(x)|^2\, dx. \tag{18}$$

Let us prove that for any $\theta \in (-\theta_1,1]$ and $\gamma \in [0,1)$ with $\theta\sqrt{1-\gamma} > -\theta_1$ there is $k_0 \in \mathbb{N}$ such that

$$\|(L_0+\lambda^2 C)u\|^2 \geq \left(1-\theta^2\right)\|u\|_+^2 + \gamma|\lambda|^4\|Cu\|_-^2 \tag{19}$$

for all $u \in H^+(D)$ and all $\lambda \in \Gamma$ with $|\lambda| \geq k_0$. Indeed, we argue by contradiction. Let there be $\theta \in (|\theta_1|,1]$ and $\gamma \in [0,1)$ with $\theta\sqrt{1-\gamma} > |\theta_1|$ such that for each $k \in \mathbb{N}$ there are $u_k \in H^+(D)$ with $\|u_k\|_+ = 1$, and a number $\lambda_k \in \Gamma$ with $|\lambda_k| \geq k$ such that

$$\|(L_0+\lambda_k^2 C)u_k\|^2 < 1-\theta^2+\gamma|\lambda_k|^4\|Cu_k\|_-^2.$$

It follows from (15) and (16) that

$$\theta^2 + |\lambda_k|^4\|Cu_k\|_-^2(1-\gamma) + 2|\lambda_k|^2\int_D \cos(\varphi_0+2\varphi_\Gamma)|a_0^{(2)}(x)||u_k(x)|^2\, dx < 0,$$

i.e.,

$$\begin{aligned}
&(\theta - \sqrt{(1-\gamma)}|\lambda_k|^2\|Cu_k\|_-\ )^2 + \\
&2\left(\theta\sqrt{(1-\gamma)} + \frac{\int_D \cos(\varphi_0+2\varphi_\Gamma)|a_0^{(2)}(x)||u_k(x)|^2\, dx}{\|Cu_k\|_-}\right)|\lambda_k|^2\|Cu_k\|_- < 0,
\end{aligned} \tag{20}$$

for all $k \in \mathbb{N}$.

On the other hand, for all $u \in H^+(D)$ with $\|u\|_+ = 1$ we have

$$\|Cu\|_- = \|e^{2\sqrt{-1}\varphi_\Gamma}Cu\|_- \geq \left|(e^{\sqrt{-1}(\varphi_0+2\varphi_\Gamma)}|a_0^{(2)}|u,u)_{L^2(D)}\right|.$$

In particular, we have

$$\left|\frac{\int_D \cos(\varphi_0+2\varphi_\Gamma)|a_0^{(2)}(x)||u_k(x)|^2\, dx}{\|Cu_k\|_-}\right| \leq 1, \text{ for all } k \in \mathbb{N}.$$

Now, if the sequence $\{|\lambda_k|^2\|Cu_k\|_-\}$ is unbounded, then by extracting a subsequence $\{|\lambda_{k_j}|^2\|Cu_{k_j}\|_-\}$ tending to $+\infty$, dividing (20) by $|\lambda_{k_j}|^4\|Cu_{k_j}\|_-^2$ and passing to the limit with respect to $k_j \to +\infty$ we obtain $1 \leq 0$, a contradiction.

Let the sequence $\{|\lambda_k|^2\|Cu_k\|_-\}$ be bounded. Now, the weak compactness principle for Hilbert spaces yields that there is a subsequence $\{u_{k_j}\}$ weakly convergent to an element u_0 in the space $H^+(D)$. Then $\{Cu_{k_j}\}$ converges to Cu_0 in $H^-(D)$ because $C : H^+(D) \to H^-(D)$ is compact and $\{u_{k_j}\}$ converges to u_0 in $L^2(D)$ because the embedding $\iota : H^+(D) \to L^2(D)$ is compact, too. Since the sequence $\{\lambda^2_{k_j} Cu_{k_j}\}$ is bounded in $H^-(D)$ and $|\lambda_k| \to +\infty$ we conclude that $\{Cu_{k_j}\}$ converges to zero in $H^-(D)$. This means that $Cu_0 = 0$ and then $u_0 = 0$ because $a^{(2)}_0(x) \neq 0$ if (9) is fulfilled on the ray Γ and then the operator C is injective (see Lemma 3).

According to the compactness principle, we may consider the subsequences

$$\big\{|\lambda_{k_j}|^2\|Cu_{k_j}\|_-\big\} \quad \text{and} \quad \left\{-\frac{\int_D \cos(\varphi_0 + 2\varphi_\Gamma)|a^{(2)}_0(x)||u_{k_j}(x)|^2\, dx}{\|Cu_{k_j}\|_-}\right\}$$

as convergent to the limits $\alpha \geq 0$ and $\beta \in [-1, 1]$, respectively. Now it follows from (20) that

$$(\theta - \alpha)^2 + 2\alpha(\theta - \beta) \leq 0. \tag{21}$$

If $\alpha = 0$, then we have a contradiction because $\theta > 0$. If $\alpha > 0$ and $\beta \leq 0$, then $\theta - \beta > 0$ and we again have a contradiction.

Let $\alpha > 0$ and $\beta > 0$. If $\varphi_0 \in C(\overline{D})$, then, according to the Weierstraß Theorem, there is a polynomial sequence $\{P_i(x)\}$ approximating $\varphi_0(x)$ in this space. In particular, for each $\varepsilon > 0$, there is $i_\varepsilon \in \mathbb{N}$ such that

$$\max_{x\in\overline{D}} |1 - \cos(\varphi_0(x) - P_i(x))| < \varepsilon \text{ for all } i \geq i_\varepsilon.$$

Since $ue^{\sqrt{-1}P_i(x)} \in H^+(D)$, we see that for all $i \geq i_\varepsilon$

$$\begin{aligned}
\|Cu\|_- &\geq \frac{\left|(e^{\sqrt{-1}(\varphi_0(x)-P_i(x))}|a^{(2)}_0|u, u)_{L^2(D)}\right|}{\|ue^{\sqrt{-1}P_i}\|_+} \\
&\geq \frac{\left|(\cos(\varphi_0(x) - P_i(x))|a^{(2)}_0|u, u)_{L^2(D)}\right|}{\|ue^{\sqrt{-1}P_i}\|_+} \\
&\geq \frac{(1-\varepsilon)(|a^{(2)}_0|u, u)_{L^2(D)}}{\|ue^{\sqrt{-1}P_i}\|_+}.
\end{aligned}$$

Hence if $\varepsilon \in (0, 1)$, then

$$\limsup_{k_j\to\infty} \frac{(|a^{(2)}_0|u_{k_j}, u_{k_j})_{L^2(D)}}{\|Cu_{k_j}\|_-} \leq \frac{\limsup_{k_j\to\infty} \|u_{k_j}e^{\sqrt{-1}P_i}\|_+}{1-\varepsilon}, \text{ for all } i \geq i_\varepsilon.$$

On the other hand, as $|e^{\sqrt{-1}P_i}| = 1$, we conclude that

$$\begin{aligned}
\|ue^{\sqrt{-1}P_i}\|^2_+ =& \|u\|^2_+ + \|(\mathfrak{D}e^{\sqrt{-1}P_i})u\|^2_{L^2(D)} \\
&+ 2\Re\Big(((\mathfrak{D}e^{\sqrt{-1}P_i})u, e^{\sqrt{-1}P_i}\mathfrak{D}u)_{L^2(D)}\Big) \\
&+ \|\Psi(e^{\sqrt{-1}P_i}u)\|^2_{L^2(\partial D)} - \|\Psi(u)\|^2_{L^2(\partial D)},
\end{aligned} \tag{22}$$

for all $i \in \mathbb{N}$ and $u \in H^+(D)$. If Ψ is given by the multiplication on a function $\psi \in L^\infty(\partial D)$ then $\|\Psi(e^{\sqrt{-1}P_i}u)\|_{L^2(\partial D)} = \|\Psi(u)\|_{L^2(\partial D)}$. If $\partial D \in C^\infty$ and Ψ is a pseudodifferential operator of order ρ on ∂D, then, as the multiplication on a smooth function is a pseudodifferential operator of order zero, we conclude that the commutator $[\Psi, e^{\sqrt{-1}P_i}] = (\Psi \circ e^{\sqrt{-1}P_i} - e^{\sqrt{-1}P_i} \circ \Psi)$ is a pseudodifferential operator of order $(\rho - 1)$ on ∂D (see, for instance, [11]). By the conctruction of $\|\cdot\|_+$ and Theorem 1, the sequence $\{u_k\}$ is bounded in $H^\rho(\partial D)$ and then we can consider that the subsequence $\{u_{k_j}\}$ converges weakly to zero in this space. Then

$$\begin{aligned}\Big| \|\Psi(e^{\sqrt{-1}P_i}u)\|_{L^2(\partial D)} & - \|\Psi(u)\|_{L^2(\partial D)}\Big| \\ &= \Big| \|\Psi(e^{\sqrt{-1}P_i}u)\|_{L^2(\partial D)} - \|e^{\sqrt{-1}P_i}\Psi(u)\|_{L^2(\partial D)}\Big| \\ &\leq \|[\Psi, e^{\sqrt{-1}P_i}](u)\|_{L^2(\partial D)},\end{aligned}$$

for all $u \in H^+(D)$ and hence

$$\lim_{k_j \to \infty} \Big(\|\Psi(e^{\sqrt{-1}P_i}u_{k_j})\|_{L^2(\partial D)} - \|\Psi(u_{k_j})\|_{L^2(\partial D)} \Big) = 0 \tag{23}$$

because the operator $[\Psi, e^{\sqrt{-1}P_i}] : H^\rho(\partial D) \to L^2(\partial D)$ is compact by the Rellich Theorem. Thus, as $u_{k_j} \to 0$ in $L^2(D)$ and $\|u_{k_j}\|_+ = 1$, it follows from (22) and (23) that

$$\limsup_{k_j \to \infty} \|u_{k_j} e^{\sqrt{-1}P_i}\|_+ = 1, \text{ for all } i \in \mathbb{N}.$$

Therefore, if $\beta > 0$, then by (18)

$$\begin{aligned}\beta &= \lim_{k_j \to \infty} \frac{-\int_D \cos(\varphi_0 + 2\varphi_\Gamma)|a_0^{(2)}(x)||u_{k_j}(x)|^2\, dx}{\|Cu_{k_j}\|_-} \\ &\leq \limsup_{k_j \to \infty} |\theta_1| \frac{\int_D |a_0^{(2)}(x)||u_{k_j}(x)|^2\, dx}{\|Cu_{k_j}\|_-} \\ &\leq \frac{|\theta_1|}{1-\varepsilon} \text{ for each } \varepsilon \in (0,1).\end{aligned}$$

This means that $\theta - \beta > 0$ if $\theta > |\theta_1|$ and we again have a contradiction with (21). Thus, (19) is fulfilled.

Finally, as $\|\delta_s L\|^2 < 1 - \eta^2(\Gamma) = 1 - |\theta_1|^2$, we see that there are $\theta_2 \in (|\theta_1|, 1]$, $\gamma_0 \in [0,1)$ with $\theta_2\sqrt{1-\gamma_0} > |\theta_1|$ and $\alpha_1 \in (0, \pi/2)$ such that

$$\|\delta_s L\| < \cos(\alpha_1)\Big(1 - \theta_2\Big)^{1/2}.$$

Therefore, using (17), (19) we see that

$$\begin{aligned}\|(L_0 + \delta_s L + \lambda^2 C)u\|_- &\geq \sqrt{\Big(1-\theta_2^2\Big)\|u\|_+^2 + \gamma_0|\lambda|^4\|Cu\|_-^2} - \|\delta_s L u\|_- \\ &\geq \cos(\alpha_1)\Big(1-\theta_2^2\Big)^{1/2}\|u\|_+ + \sin(\alpha_1)\sqrt{\gamma_0}|\lambda|^2\|Cu\|_- - \|\delta_s L u\|_- \\ &\geq \Big(\cos(\alpha_1)\Big(1-\theta_2^2\Big)^{1/2} - \|\delta_s L\|\Big)\|u\|_+ + \sin(\alpha_1)\sqrt{\gamma_0}|\lambda|^2\|Cu\|_-,\end{aligned}$$

for all $u \in H^+(D)$ and all $\lambda \in \Gamma$ with $|\lambda| \geq k_0$. $\square$

We continue with the proof of property 1). For this purpose, using Lemma 5, we conclude that the operator $(L_0 + \delta_s L + \lambda^2 C)$ is continuously invertible if $\|\delta_s L\|^2 < 1 - \eta^2(\Gamma)$ and $\lambda \in \Gamma$ with $|\lambda| \geq k_0$. Hence we obtain

$$L(\lambda) = (I + \delta_c L(L_0 + \delta_s L + \lambda^2 C)^{-1})(L_0 + \delta_s L + \lambda^2 C) \tag{24}$$

for all $\lambda \in \Gamma$ with $|\lambda| \geq k_0$.

We will show that the operator $I + \delta_c L(L_0 + \delta_s L + \lambda^2 C)^{-1}$ is injective for all $\lambda \in \Gamma$ such that $|\lambda| \geq k_1$ with some $k_1 \in \mathbb{N}$ with $k_1 \geq k_0$. Indeed, we argue by contradiction. Suppose that for any $k \in \mathbb{N}$ there are $\lambda_k \in \Gamma$ with $|\lambda_k| \geq k$ and $f_k \in H^-(D)$, such that $\|f_k\|_- = 1$ and

$$(I + \delta_c L(L_0 + \delta_s L + \lambda_k^2 C)^{-1}) f_k = 0. \tag{25}$$

It follows from Lemma 5 that the sequence $u_k := (L_0 + \delta_s L + \lambda_k^2 C)^{-1} f_k$ is bounded in $H^+(D)$ for all $\lambda_k \in \Gamma$ with $|\lambda_k| \geq k_0$. Now the weak compactness principle for Hilbert spaces yields that there is a subsequence $\{f_{k_j}\}$ with the property that both $\{f_{k_j}\}$ and $\{u_{k_j}\}$ converge weakly in the spaces $H^-(D)$ and $H^+(D)$ to limits f and u, respectively. Since $\delta_c L$ is compact, it follows that the sequence $\{\delta_c L u_{k_j}\}$ converges to $\delta_c L u$ in $H^-(D)$, and so $\{f_{k_j}\}$ converges to f because of (25). Obviously, $\|f\|_- = 1$. In particular, we conclude that the sequence $\{\delta_c L(L_0 + \delta_s L + \lambda_{k_j}^2 C)^{-1} f_{k_j}\}$ converges to $(-f)$ whence

$$f = -\delta_c L\, u. \tag{26}$$

Further, on passing to the weak limit in the equality $f_{k_j} = (L_0 + \delta_s L + \lambda_{k_j}^2 C) u_{k_j}$ we obtain

$$f = L_0 u + \delta_s L u + \lim_{k_j \to \infty} \lambda_{k_j}^2\, C\, u_{k_j},$$

for the continuous operator $L_0 + \delta_s L : H^+(D) \to H^-(D)$ maps weakly convergent sequences to weakly convergent sequences.

As the operator C is compact, the sequence $\{C\, u_{k_j}\}$ converges to $C\, u$ in the space $H^-(D)$ and $C\, u \neq 0$, which is a consequence of (26) and the injectivity of C (see Lemma 3). This shows readily that the weak limit

$$\lim_{k_j \to \infty} \lambda_{k_j}^2\, C\, u_{k_j} = f - L_0 u - \delta_s L u$$

does not exist, a contradiction.

We have proved that the operator $I + \delta_c L(L_0 + \delta_s L + \lambda^2\, C)^{-1}$ is injective for all $\lambda \in \Gamma$ with $|\lambda| \geq k_1$. Since this is a Fredholm operator of index zero, it is continuously invertible. Hence, the operators $L(\lambda)$ are continuously invertible for all $\lambda \in \Gamma$ with sufficiently large $|\lambda|$.

Thus, $\{L^{-1}(\lambda) = (L_0 + \delta_c L + \delta_s L + \lambda^2 C)^{-1}\}_{\lambda \in \mathbb{C}}$ is a meromorphic family of Fredholm operators. Since there is a point γ where $L(\gamma)$ is continuously invertible, the operators $L(\lambda)$ are continuously invertible for all $\lambda \in \mathbb{C}$, except a discrete countable set $\{\lambda_\nu\}$ without limit points in $\mathbb{C}$ (see, for instance, [12] or [10]). $\square$

Corollary 1. *Let either Ψ is given by the multiplication on a function $\psi \in L^\infty(\partial D)$ or $\partial D \in C^\infty$ and let Ψ is a pseudodifferential operator on ∂D. Let also (9) hold true, $\varphi_0 \in C(\overline{D})$ and*

$$\Phi = \sup_{x,y\in\overline{D}} (\varphi_0(x) - \varphi_0(y)) < 2\pi.$$

Under the hypothesis of Theorem 1, for each compact operator $\delta_c L : H^+(D) \to H^-(D)$ and each bounded operator $\delta_s L : H^+(D) \to H^-(D)$ with

$$\|\delta_s L\|^2 + (\max(0, -\cos(\Phi/2)))^2 < 1, \tag{27}$$

the operators $L(\lambda) = L_0 + \delta_s L + \delta_c L + \lambda^2 C$ are continuously invertible for all $\lambda \in \mathbb{C}$, except a countable number of the characteristic values $\{\lambda_\nu\}$.

Proof. As $\varphi_0 \in C(\overline{D})$, the function admits maximal and minimal values $\Phi_1 = \min_{x\in\overline{D}} \varphi_0(x)$, $\Phi_2 = \max_{x\in\overline{D}} \varphi_0(x)$ and $\Phi = \Phi_2 - \Phi_1$. Hence, the statement follows from Theorem 3 applied to the ray $\Gamma_0 = \{\arg(\lambda) = -(\Phi_2 + \Phi_1)/4\}$ with $\theta_1(\Gamma_0) = \min_{x\in\overline{D}} \cos(\varphi_0(x) + 2\varphi_{\Gamma_0}) \geq \cos(\Phi/2) > -1$. □

4. On the completeness of root functions

We are interested in studying the completeness of root functions related to the mixed problem in Sobolev type spaces $H^+(D)$, $H^-(D)$.

For this purpose we recall some basic definitions. Suppose $\lambda_0 \in \mathbb{C}$ and $F(\lambda)$ is a holomorphic function in a punctured neighborhood of λ_0 which takes on its values in the space $\mathcal{L}(H_1, H_2)$ of bounded linear operators acting from a Hilbert space H_1 to a Hilbert space H_2. The point λ_0 is called a characteristic point of $F(\lambda)$ if there exists a holomorphic function $u(\lambda)$ in a neighborhood of λ_0 with values in H_1, such that $u(\lambda_0) \neq 0$ but $F(\lambda)u(\lambda)$ extends to a holomorphic function (with values in H_2) near the point λ_0 and vanishes at this point. As usual, we call $u(\lambda)$ a root function of the family $F(\lambda)$ at λ_0. If N is the order of zero of the holomorphic function $F(\lambda)u(\lambda)$ at the point λ_0, then we have

$$\sum_{j=0}^{m} F_{m-j} u_j = 0 \text{ for all } m \in \mathbb{Z}_+ \text{ with } 0 \leq m \leq N-1, \tag{28}$$

where $u_j = \frac{1}{j!}\frac{d^j u}{dz^j}(\lambda_0) \in H_1$ and $F_j = \frac{1}{j!}\frac{d^j F}{dz^j}(\lambda_0) \in \mathcal{L}(H_1, H_2)$, $j \in \mathbb{N}$. The vector u_0 is called an eigenvector of the family $F(\lambda)$ at the point λ_0 and the vectors u_j, $1 \leq j \leq N-1$, are said to be associated vectors for the eigenvector u_0. If the linear span of the set of all eigen- and associated vectors in the family $F(\lambda)$ is dense in H_1, one says that the root functions of the family $F(\lambda)$ are complete in H_1. The notion of root function of a holomorphic family is a generalization of the notion of a root vector of a linear operator. Namely, a non-zero element $u \in H$ is called a root vector of T corresponding to an eigenvalue $\mu_0 \in \mathbb{C}$ if $u \in D_{(T-\mu_0 I)^k}$, for all $1 \leq k \leq m$ and $(T - \mu_0 I)^m u = 0$ for some natural number m.

Note that under (9) the multiplication on the function $a_0^{(2)} \in L^\infty(D)$ induces a bounded injective operator in the space $L^2(D)$; it is continuously invertible under (12). We will denote this operator by C_0. Then we can factorize $C = \iota' C_0 \iota$.

Lemma 6. *If (9) is fulfilled, then for the holomorhic Fredholm family $L(\lambda) = L(0)+\lambda^2 C : H^+(D) \to H^-(D)$ the set of all its root functions coincides with the set of all root vectors of one of the following closed densely defined linear operators:*

$$C^{-1}L(\gamma) : H^+(D) \to H^+(D) \text{ and } L(\gamma)C^{-1} : H^-(D) \to H^-(D),$$

where $\gamma \in \mathbb{C}$ is an arbitrary point. Besides, if there is a point $\gamma_0 \in \mathbb{C}$, where the operator $L(\gamma_0) = L(0) + \lambda_0^2 C$ is continuously invertible, it also coincides with the set of all the root vectors of one of the following bounded linear operators:

$$L^{-1}(\gamma_0)C : H^+(D) \to H^+(D),\ CL^{-1}(\gamma_0) : H^-(D) \to H^-(D),$$
$$\iota L^{-1}(\gamma_0)\iota' C_0 : L^2(D) \to L^2(D).$$

Proof. Follows immediately from (28). □

To formulate the completeness results regarding parameter-dependent elliptic operators we need the notion of a compact operator of finite order. If $T : H \to H$ is compact, then the operator T^*T is compact, selfadjoint and non-negative. Hence it follows that T^*T possesses a unique non-negative selfadjoint compact square root $(T^*T)^{1/2}$ often denoted by $|T|$. By the Hilbert-Schmidt Theorem the operator $|T|$ has a countable system of non-negative eigenvalues $s_\nu(T)$ which are called the s-numbers of T. It is clear that if T is selfadjoint, then $s_\nu = |\mu_\nu|$, where $\{\mu_\nu\}$ is the system of eigenvalues of T. The operator T is said to belong to the Schatten class $\mathfrak{S}_p$, with $0 < p < \infty$, if

$$\sum_\nu |s_\nu(T)|^p < \infty.$$

A compact operator T is said to be of finite order if it belongs to the Schatten class $\mathfrak{S}_p$. The infinum ord (T) of such numbers p is called the order of T.

Let us denote by $\mathfrak{C} : H^+(D) \to H^-(D)$ a linear bounded operator induced by the term $(|a_0^{(2)}|u, v)_{L^2(D)}$. Note that under (9) the multiplication on the function $|a_0^{(2)}| \in L^\infty(D)$ induces a bounded injective selfadjont operator $\mathfrak{C}_0 : L^2(D) \to L^2(D)$; it is continuously invertible under (12). In the following theorem $h(\cdot, \cdot)$ stands for the Hermitian form

$$h(u, v) = (|a_0^{(2)}|u, v)_{L^2(D)}.$$

We note that under (9) it defines a scalar product on $L^2(D)$; this Hilbert space is denoted by $L^2_h(D)$. The corresponding norm is not stronger than $\|\cdot\|_{L^2(D)}$, it is equivalent to the original norm of this space if (12) is fulfilled.

Theorem 4. *Let (9) hold true. Under the hypothesis of Theorem 1, the operators*

$$L_0^{-1}\mathfrak{C} : H^+(D) \to H^+(D),\ \mathfrak{C}L_0^{-1} : H^-(D) \to H^-(D),\ \iota L_0^{-1}\iota'\mathfrak{C}_0 : L^2(D) \to L^2(D)$$

are compact and their orders are finite:

$$ord\,(\mathfrak{C}L_0^{-1}) = ord\,(L_0^{-1}\mathfrak{C}) = ord\,(\iota L_0^{-1}\iota'\mathfrak{C}_0) = n/(2\rho + 1).$$

Moreover, the operators $L_0^{-1}\mathfrak{C}$ and $\mathfrak{C}L_0^{-1}$ are selfadjoint. Besides, the operators have the same systems of eigenvalues $\{\mu_\nu\}$, the system $\{b_\nu^{(+)}\}$ of eigenvectors of the *operator* $L_0^{-1}\mathfrak{C}$ *is complete in the spaces* $H^+(D)$, $L^2(D)$ *and* $H^-(D)$. *Moreover, the system* $\{b_\nu^{(+)}\}$ *is an orthonormal basis in* $H^+(D)$, *the system* $\{b_\nu^{(-)} = \mathfrak{C}b_\nu^{(+)}\}$ *of eigenvectors of the operator* $\mathfrak{C}L_0^{-1}$ *is an orthogonal basis in* $H^-(D)$, *the system* $\{b_\nu^{(0)} = \iota b_\nu^{(+)}\}$ *of eigenvectors of the operator* $\iota L_0^{-1}\iota'\mathfrak{C}_0$ *is an orthogonal basis in the space* $L_h^2(D)$ *and the system* $\{\sqrt{|a_0^{(2)}|}b_\nu^{(+)}\}$ *is an orthogonal basis in* $L^2(D)$. *If, in addition, (12) holds, then the operator* $\iota L_0^{-1}\iota'\mathfrak{C}_0$ *is selfadjoint in* $L_h^2(D)$.

Proof. The proof is standard for self-adjoint operator pencils (see [16] or even [21, Suppl. II, Introduction and P. 1, § 2] for Ordinary Differential Equations. □

Now we can use the famous Keldysh' Theorem on the weak perturbation of compact selfadjoint operators (see, [12], [10], [16], or elsewhere).

Corollary 2. *Let (9) hold true. Under the hypothesis of Theorem 1, for each compact operator* $\delta_c L : H^+(D) \to H^-(D)$ *we have*

1) *for any* $\varepsilon > 0$ *all characteristic values* λ_ν *(except for a finite number) of the operator pencil* $L(\lambda) = L_0 + \delta_c L + \lambda^2\mathfrak{C}$ *belong to the corners*

$$M_\varepsilon = \{|\arg(\lambda) - \pi/2| < \varepsilon\}, \quad M_{-\varepsilon} = \{|\arg(\lambda) + \pi/2| < \varepsilon\}$$

and $\lim_{\nu\to\infty} |\lambda_\nu| = +\infty$;

2) *the system of root vectors of the family* $L(\lambda) = L_0 + \delta_c L + \lambda^2\mathfrak{C}$ *is complete in the spaces* $H^+(D)$, $L^2(D)$ *and* $H^-(D)$.

Finally, we may apply the method of rays of minimal growth of the resolvent to obtain the completeness of root vectors in the case of more general perturbations.

Theorem 5. *Let either* Ψ *is given by the multiplication on a function* $\psi \in L^\infty(\partial D)$ *or* $\partial D \in C^\infty$ *and let* Ψ *is a pseudodifferential operator on* ∂D. *Under the hypothesis of Theorem 1, let also (9) and*

$$\Phi = \sup_{x,y\in\overline{D}} (\varphi_0(x) - \varphi_0(y)) < \pi(2\rho+1)/2n. \tag{29}$$

hold true. If $\varphi_0 \in C^{0,1}(\overline{D})$ *and*

$$\|\delta_s L\|^2 + (\max(0, -\cos((\pi(2\rho+1) - 2n\Phi)/4n)))^2 < 1,$$

then we have

1) *for any* $\varepsilon > 0$, *all characteristic values* λ_ν *(except for a finite number) of the family* $L(\lambda) = L_0 + \delta_s L + \delta_c L + \lambda^2 C$ *belong to the corners*

$$\{|\arg(\lambda) \pm \pi/2| < \pi(2\rho+1)/2n + \varepsilon\}$$

and $\lim_{\nu\to\infty} |\lambda_\nu| = +\infty$;

2) *the system of root vectors of the family* $L(\lambda) = L_0 + \delta_c L + \delta_s L + \lambda^2 C$ *is complete in the spaces* $H^+(D)$, $H^-(D)$ *and* $L^2(D)$.

Proof. As the operator $\gamma_0^2 C : H^+(D) \to H^-(D)$ is compact, the family $\tilde{L}(\tilde{\lambda}) = L_0 + \delta_s L + \tilde{\delta}_c L + \tilde{\lambda}^2 C$ with $\tilde{\delta}_c L = \delta_c L + \gamma_0^2 C$ and $\tilde{\lambda}^2 = \lambda^2 - \gamma_0^2$ satisfies conditions of Theorem 5, too. Moreover, the operator $\tilde{L}(0) = L(\gamma_0)$ is continuously invertible. Since the root functions and root vectors of the families $\tilde{L}(\tilde{\lambda})$ and $L(\lambda)$ have obvious relations, we can replace the family $L(\lambda)$ by the family $\tilde{L}(\tilde{\lambda})$. Hence we may consider that the operator $L(0)$ is continuously invertible. Now, according to Lemma 6, the proof of the theorem can be reduced to the investigation of the properties of one of the operators $L^{-1}(0)C$ and $L(0)C^{-1} = (L^{-1}(0)C)^{-1}$.

If $\varphi_0 \in C^{0,1}(\overline{D})$, then the multiplication on the function $e^{\sqrt{-1}\varphi_0} \in C^{0,1}(\overline{D})$ induces a bounded linear operator $\delta_C : H^+(D) \to H^+(D)$. Hence the operator $CL^{-1}(0)$ can be presented in the following form:

$$CL^{-1}(0) = (\mathfrak{C}L_0^{-1})\, L_0\, \delta_C\, L^{-1}(0).$$

It follows from Theorem 4 that the operator $\mathfrak{C}L_0^{-1}$ belongs to the Schatten class $\mathfrak{S}_{n/(2\rho+1)+\varepsilon}$ with any $\varepsilon > 0$, i.e. $CL^{-1}(0) \in \mathfrak{S}_{n/(2\rho+1)+\varepsilon}$ with any $\varepsilon > 0$, too (see [10, Ch. 2, §2]). Besides, estimates (14) and (29) imply that the angle between any two neighboring rays of minimal growth of the resolvent of the operator $L(0)C^{-1}$ is less than $\pi(2\rho+1)/2n$. Thus the statement of the theorem follows from the standard arguments with the use of the Phragmen-Lindelöf theorem which go back at least as far as [1] (see also [19] for the non-coercive case). □

Remark 1. *Actually, it follows from the reducing procedure of Lemma 6 that in Corollary 2 and Theorem 4 we should claim the multiple (double) completeness of root vectors related to the operator pencil $L(\lambda)$ (see [12], [22] and elsewhere).*

5. An example

Consider an instructive example. Let $n = 2$ and $A_0^{(2)}$ be a (2×2) matrix with real-valued entries of class $L^\infty(D)$ and

$$\tilde{A}(x, \partial, \lambda)V(x) = -\vartheta\Delta_2 I_2 V(x) - (\vartheta + \vartheta_1)\nabla_2 \mathrm{div}_2 V(x) + \lambda^2 A_0^{(2)}(x)V(x)$$

the Lamé type system, where $V(x) = (V_1(x), V_2(x))$ is an unknown vector, I_2 is the identity (2×2)-matrix, Δ_2 the Laplace operator, ∇_2 and div_2 are the gradient operator and the divergence operators in $\mathbb{R}^2$, respectively, and ϑ, ϑ_1 are the Lamé parameters. This operator plays an essential role in the two-dimensional Linear Elasticity Theory (see, for instance, [9]); the vector $V(x)$ represents the displacement of points of an elastic body. This operator can also be considered as part of a linearisation system of the stationary version of the two-dimensional Navier-Stokes type equations for a viscous compressible fluid with known pressure and unknown velocity vector $V(x)$ (see [14, §15]); in this case, the Lamé parameters represent viscosities. The system is strongly elliptic and formally selfadjoint non-negative if $\vartheta > 0$, $2\vartheta + \vartheta_1 > 0$. Let us consider a very special case where the first Lamé parameter ϑ_1 is negative and $\vartheta_1 = -\vartheta$. Then, $\tilde{A}(x, \partial, \lambda)$ reduces to

$$\tilde{A}(x, \partial, \lambda) = -\vartheta\Delta_2 I_2 + \lambda^2 A_0^{(2)}(x). \tag{30}$$

On the other hand,

$$-\Delta_2 I_2 V = \mathrm{rot}_2^* \mathrm{rot}_2 V + \mathrm{div}_2^* \mathrm{div}_2 V,$$

where $\mathrm{rot}_2 V = (\partial_1 V_2 - \partial_2 V_1)$ is the rotation operator in $\mathbb{R}^2$ and rot_2^*, div_2^* are the formal adjoint operators for rot_2, div_2, respectively. Assume now that the matrix $A_0^{(2)}(x)$ has the following form $A_0^{(2)}(x) = \alpha(x)U(x)$, where $\alpha(x) \in L^\infty(D)$ is a non-negative function and

$$U(x) = \begin{pmatrix} U_1(x) & -U_2(x) \\ U_2(x) & U_1(x) \end{pmatrix}$$

is an orthogonal matrix with entries $U_j \in L^\infty(D)$. Then, after the complexification

$$u(z) = V_1(z) + \sqrt{-1}V_2(z),\ z = x_1 + \sqrt{-1}x_2,$$

system (30) with real-valued coefficients reduces to the following equation with complex-valued coefficients

$$A(x, \partial, \lambda)u = 4\vartheta \overline{\partial}^* \overline{\partial}\, u + \lambda^2 a_0^{(2)}(x)u,$$

where $\overline{\partial} = 1/2(\frac{\partial}{\partial x_1} + \sqrt{-1}\frac{\partial}{\partial x_2})$ is the Cauchy-Riemann operator, $\overline{\partial}^* = -1/2(\frac{\partial}{\partial x_1} - \sqrt{-1}\frac{\partial}{\partial x_2})$ is its formal adjoint and

$$a_0^{(2)}(x) = \alpha(x)\left(U_1(x) + \sqrt{-1}U_2(x)\right).$$

Then, with a proper operator $\Psi : H^\rho(\partial D) \to L^2(\partial D)$, the Robin type operator B has the form

$$B = 2\vartheta\,(\nu_1 - \sqrt{-1}\nu_2)\bar{\partial} + \Psi^*\Psi,$$

where (ν_1, ν_2) is the unit normal vector field to ∂D. The boundary operators

$$\frac{\partial}{\partial \nu} = \nu_1 \partial_1 + \nu_2 \partial_2,\ \bar{\partial}_\nu = (\nu_1 - \sqrt{-1}\nu_2)\bar{\partial} = \frac{1}{2}\left(\frac{\partial}{\partial \nu} + \sqrt{-1}(\nu_1\partial_2 - \nu_2\partial_1)\right)$$

are known as the normal derivative and the complex normal derivative with respect to ∂D, respectively. Thus, we obtain a mixed problem of the type considered above:

$$\begin{cases} \left(-\vartheta\Delta_2 + \lambda^2 a_0^{(2)}\right) u(z) = f \text{ in } D, \\ \left(2\vartheta\bar{\partial}_\nu + \Psi^*\Psi\right) u(z) = 0 \text{ at } \partial D. \end{cases} \tag{31}$$

Note that the usual boundary conditions for the Navier-Stokes equations or the Lamé type operator are formulated by using the boundary stress tensor σ. In our particular case, the tensor has the following components:

$$\sigma_{i,j} = \vartheta\left(\delta_{i,j}\frac{\partial}{\partial \nu} + \nu_j\frac{\partial}{\partial x_i} - \nu_i\frac{\partial}{\partial x_j}\right),\ 1 \le i, j \le 2. \tag{32}$$

Hence, with the tangential operator $\partial_{\tau_0} = \left((\nu(x)\mathrm{div}_2)^{\mathrm{T}} - \nu(\mathrm{x})\mathrm{div}_2\right)$, we have

$$\sigma = \vartheta\left(\frac{\partial}{\partial \nu}I_2 + \partial_\tau\right) = \vartheta\,(\tilde{\sigma} + 2\partial_{\tau_0}), \tag{33}$$

where the boundary tensor $\tilde{\sigma}$ corresponds to the boundary operator $2\bar{\partial}_\nu$ after the decomplexification of the mixed problem (31), i.e., in the matrix form (31) reads as

$$\begin{cases} \left(-\vartheta\Delta_2 I_2 + \lambda^2 A_0^{(2)}\right) V(x) = F \text{ in } D, \\ ((\sigma - 2\vartheta\partial_{\tau_0}) + \Psi^*\Psi I_2)\, V(x) = 0 \text{ at } \partial D. \end{cases}$$

In Elasticity Theory, the boundary tensor $\tilde{\sigma} = \vartheta^{-1}\sigma - 2\partial_{\tau_0}$ was discovered in [6].

We continue with the mixed problem (31). The corresponding scalar product of the space $H^+(D)$ related to the mixed problem has the form

$$(u, v)_+ = 4\vartheta\,(\bar{\partial}u, \bar{\partial}v)_{L^2(D)} + (\Psi u, \Psi v)_{L^2(\partial D)}.$$

Then, Theorem 1 grants the embedding of the space $H^+(D)$ into the Sobolev-Slobodetskii space $H^s(D)$. However, for $0 < \rho < 1/2$, each holomorphic function $u \in H^{\rho+1/2}(D)$ belongs to $H^+(D)$ but there is no reason for it to belong to $H^1(D)$, i.e., the embedding is sharp. For $\rho = 0$, the embedding described in Theorem 1 is sharp, too, but the arguments are more subtle (see [18] or [19]). In particular, the Shapiro-Lopatinskii conditions are violated on the smooth part of ∂D.

In some cases we can obtain reasonable formulas for solutions to the problem. Let D be the unit circle $\mathbb{B}$ around the origin in $\mathbb{C}$ and $S = \emptyset$. We pass to polar coordinates $z = r\, e^{\sqrt{-1}\phi}$ in $\mathbb{R}^2$, where $r = |x|$ and $\phi \in [0, 2\pi]$, and set

$$\vartheta = 1,\ \Psi^*\Psi\ = 2\left(1 - \frac{\partial^2}{\partial^2\phi}\right)^{\rho/2} \qquad a_0^{(2)}(z) = |z|^{2d},\ d \geq 0,$$

($-\frac{\partial^2}{\partial^2\phi}$ being the Laplace-Beltrami operator on $\partial\mathbb{B}$). Then, $a_0^{(2)} \in C^{0,2d}(\overline{D})$ if $0 < d \leq 1/2$. Besides,

$$\frac{\partial}{\partial\nu} = r\partial_r,\ \overline{\partial}_\nu = \bar{z}\,\bar{\partial} = \frac{1}{2}\left(r\partial_r + \sqrt{-1}\,\partial_\phi\right). \tag{34}$$

Consider the Sturm-Liouville problem for the ordinary differential equation with respect to the variable r in the interval $(0, 1)$,

$$\left(r\partial_r^2 + \partial_r - k^2 r^{-1} + \mu^2 r^{2d+1}\right) g = 0 \text{ in } (0, 1), \tag{35}$$

$$g \text{ is bounded at } 0, \tag{36}$$

$$\left(r\partial_r - k + (1{+}k^2)^{\rho/2}\right) g = 0 \text{ at } r = 1, \tag{37}$$

see [21, Suppl. II, Introduction and P. 1, § 2]. Actually, as we have seen above, μ are non-negative real numbers (with $\mu^2 = -\lambda^2$) and then (35) is a particular case of the Bessel equation. Its (real-valued) solution $g(r)$ is a Bessel function defined on $(0, +\infty)$, and the space of all solutions is two-dimensional. For example, if $\lambda^2 = 0$ and $d = 0$, then $g(r) = \alpha r^k + \beta r^{-k}$ with arbitrary constants α and β is a general solution to (35). In the general case, the space of solutions to (35) contains a one-dimensional subspace $\{\alpha g_k(r, \mu) = \alpha \mathcal{J}_{\frac{|k|}{d+1}}\left(\frac{\mu r^{d+1}}{d+1}\right)\}$ of functions bounded at the point $r = 0$, where $\mathcal{J}_p(t)$ are Bessel functions, cf. (see, for instance, [21]). As usual, for each $k \in \mathbb{Z}$ the proper system of eigenvalues $\{\mu_k^{(\nu)}\}_{\nu\in\mathbb{N}}$ can be found as solutions to the transcendental equation

$$\frac{\mu}{d+1}\mathcal{J}'_{\frac{|k|}{d+1}}\left(\frac{\mu}{d+1}\right)+\left((1+k^2)^{\rho/2}-k\right)\mathcal{J}_{\frac{|k|}{d+1}}\left(\frac{\mu}{d+1}\right)=0$$

induced by (37) with $g_k(\cdot,\mu)$ instead of g. For any $k \in \mathbb{Z}$, fix a non-trivial solution $g_k^{(\nu)}(r)$ of problem (35) corresponding to an eigenvalue $\mu_k^{(\nu)}$. This system is an orthogonal basis in the weighted Lebesgue space $L^2_d(0,1)$ with the scalar product

$$h_d(g,f)=\int_0^1 r^{2d+1}g(r)f(r)\,dr,$$

see [21, Suppl. II, Introduction and P. 1, § 2]. Then the function

$$u_k^{(\nu)}(z)=g_k^{(\nu)}(r)e^{\sqrt{-1}k\phi}$$

satisfies

$$\begin{cases}\left(-\Delta_2+(\lambda_k^{(\nu)})^2|z|^{2d}\right)u_k^{(\nu)}(z)=0 \text{ in } \mathbb{C}\\ \left(\bar{\partial}_\nu+(1-\frac{\partial^2}{\partial^2\phi})^{\rho/2}\right)u_\nu^{(k)}(z)=0 \text{ at } \partial\mathbb{B}\end{cases}, \tag{38}$$

where $(\lambda_k^{(\nu)})^2=-(\mu_k^{(\nu)})^2$ Indeed, by (35) and Fubini's theorem we conclude that this equality holds in $\mathbb{C}\setminus\{0\}$ (here we used the fact that $u_\nu^{(k)}$ is bounded at the origin). On the other hand, the boundary condition (38) follows immediately from (34) and (37). By the construction, the system $\{u_\nu^{(k)}\}_{k\in\mathbb{Z},\nu\in\mathbb{N}}$ consists of eigenfunctions of the family $L(\lambda)=L_0+\lambda^2C$. Obviously, it coincides with the system of all eigenvectors constructed in Theorem 4 if it is complete in the space $L^2_h(\mathbb{B})$ with the scalar product

$$h(u,v)=\int_D |z|^{2d}u(z)\overline{v}(z)dx.$$

But $\{h_k\}_{k\in\mathbb{Z}}$ is an orthogonal basis in $L^2(\partial\mathbb{B})$ and $\{g_k^{(\nu)}\}_{k\in\mathbb{Z}_+,\nu\in\mathbb{N}}$ is an orthogonal basis in $L^2_d(0,1)$ and hence Fubini's Theorem implies that the system is an orthogonal basis in the space $L^2_h(\mathbb{B})$.

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