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Solutions of generalized fractional kinetic equations involving Aleph functions

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Abstract. In view of the usefulness and great importance of the kinetic equation in certain astrophysical problems, the authors develop a new and further generalized form of the fractional kinetic equation in terms of the Aleph-function by using the Sumudu transform. This new generalization can be used for the computation of the change of chemical composition in stars like the sun. The manifold generality of the Aleph-function is discussed in terms of the solution of the above fractional kinetic equation. The main results, being of general nature, are shown to be some unification and extension of many known results given, for example, by Saxena *et al.* [23, 25, 31], Saxena and Kalla [22], Chaurasia and Kumar [6], Dutta *et al.* [8], etc.

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1. Introduction and preliminaries

Throughout this paper, let $\mathbb{N}$, $\mathbb{R}$ and $\mathbb{C}$ be the sets of positive integers, real numbers and complex numbers, respectively, and $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$. The Aleph-function is defined by means of Mellin-Barnes type integral in the following manner [36, 37] (see also [19, 28, 30, 32]):

$$\aleph\left[z\right] = \aleph^{m,n}_{p_i,q_i,\tau_i;r}\left[z\left|\begin{matrix}(a_j,A_j)_{1,n},\ldots,[\tau_j(a_{ji},A_{ji})]_{n+1,p_i;r}\\(b_j,B_j)_{1,m},\ldots,[\tau_j(b_{ji},B_{ji})]_{m+1,q_i;r}\end{matrix}\right.\right] := \frac{1}{2\pi i}\int_L \Omega^{m,n}_{p_i,q_i,\tau_i;r}(s)\, z^{-s} ds, \quad (1)$$

where $z \in \mathbb{C} \setminus \{0\}$, $i = \sqrt{-1}$ and

$$\Omega^{m,n}_{p_i,q_i,\tau_i;r}(s) = \frac{\prod_{j=1}^{m}\Gamma(b_j + B_j s)\,.\prod_{j=1}^{n}\Gamma(1 - a_j - A_j s)}{\sum_{i=1}^{r}\tau_i \prod_{j=m+1}^{q_i}\Gamma(1 - b_{ji} - B_{ji}s)\,.\prod_{j=n+1}^{p_i}\Gamma(a_{ji} + A_{ji}s)}. \quad (2)$$

The integration path $L = L_{i\gamma\infty}\,(\gamma \in \mathbb{R})$ extends from $\gamma - i\infty$ to $\gamma + i\infty$. The poles of the gamma functions $\Gamma(1 - a_j - A_j s)$, $j, n \in \mathbb{N}, 1 \le j \le n$ do not coincide with

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those of $\Gamma(b_j + B_j s)$, $j, m \in \mathbb{N}; 1 \leq j \leq m$. The parameters p_i, q_i are non-negative integers satisfying the condition $0 \leq n \leq p_i, 1 \leq m \leq q_i, \tau_i > 0$ for $1 \leq i \leq r$. The parameters $A_j, B_j, A_{ji}, B_{ji} > 0$ and $a_j, b_j, a_{ji}, b_{ji} \in \mathbb{C}$. The empty product in (2) is interpreted as unity. For the existence conditions and further details of the Aleph-function, one can refer to [26, 27, 29].

In the literature, there are numerous integral transforms which are widely used in physics and astronomy as well as in engineering. Integral transforms are also used to solve some differential equations in a rather efficient way.

In the early 90's the Watugala [38, 39] introduced a new integral transform, the so-called Sumudu transform, and applied it further to the solution of an ordinary differential equation in control engineering problems. For further details and fundamental properties of the Sumudu transform, see [2, 3, 4, 5]. Let A be the class of exponentially bounded functions $f : \mathbb{R} \to \mathbb{R}$, that is,

$$|f(t)| < \begin{cases} Me^{-t/\tau_1}, & t \leq 0, \\ Me^{t/\tau_2}, & t \geq 0, \end{cases}$$

where M, τ_1 and τ_2 are some positive real constants. The Sumudu transform defined on the set A is given by the following formula

$$G(u) = S[f(t); u] = \int_0^\infty e^{-t} f(ut)\, dt \quad \text{for} \quad u \in (-\tau_1, \tau_2). \tag{3}$$

The Sumudu transform given in (3) can also be derived directly from the Fourier integral. The Sumudu transform has been shown to be the theoretical dual of the Laplace transform. It is interesting to compare the Sumudu transform (3) with the well-known Laplace transform (see, e.g., [33]) defined by

$$F(p) = \pounds[f(t)] = \int_0^\infty e^{-pt} f(t)\, dt, \quad \Re(p) > 0. \tag{4}$$

The Sumudu transform of the $\aleph$-function is given by Saxena, Ram and Kumar [31] in the following manner:

$$S\left[t^{\lambda-1}\aleph^{m,n}_{p_i,q_i,\tau_i;r}\left[z\,t^{\sigma}\,\middle|\,\begin{matrix}(a_j, A_j)_{1,n}, \ldots, [\tau_j(a_{ji}, A_{ji})]_{n+1,p_i;r} \\ (b_j, B_j)_{1,m}, \ldots, [\tau_j(b_{ji}, B_{ji})]_{m+1,q_i;r}\end{matrix}\right]\right]$$
$$= u^{\lambda-1}\aleph^{m,n+1}_{p_i+1,q_i,\tau_i;r}\left[zu^{\sigma}\,\middle|\,\begin{matrix}(1-\lambda, \sigma), (a_j, A_j)_{1,n}, \ldots, [\tau_j(a_{ji}, A_{ji})]_{n+1,p_i;r} \\ (b_j, B_j)_{1,m}, \ldots, [\tau_j(b_{ji}, B_{ji})]_{m+1,q_i;r}\end{matrix}\right], \tag{5}$$

where $\lambda, u, z \in \mathbb{C}, \Re(u) > 0, \sigma > 0$, and

$$\Re(\lambda) + \sigma \min_{1 \leq j \leq m} \frac{\Re(b_j)}{B_j} > 0, \quad |\arg z| < \frac{\pi\zeta_l}{2}, \qquad \zeta_l > 0,$$

and

$$\zeta_l = \sum_{j=1}^{m} b_j - \sum_{j=1}^{n} a_j + \tau_l\left(\sum_{j=m+1}^{q_l} b_{jl} - \sum_{j=n+1}^{p_l} a_{jl}\right) + \frac{1}{2}(p_l - q_l), \qquad 1 \leq l \leq r. \tag{6}$$

From (5), it is clear that

$$S^{-1}\left[u^{\lambda}\aleph_{p_i,q_i,\tau_i;r}^{m,n}\left[zu^{\sigma}\left|\begin{array}{c}(a_j,A_j)_{1,n},\ldots,[\tau_j(a_{ji},A_{ji})]_{n+1,p_i;r}\\(b_j,B_j)_{1,m},\ldots,[\tau_j(b_{ji},B_{ji})]_{m+1,q_i;r}\end{array}\right.\right]\right]$$
$$=t^{\lambda-1}\aleph_{p_i,q_i+1,\tau_i;r}^{m,n}\left[zt^{\sigma}\left|\begin{array}{c}(a_j,A_j)_{1,n},\ldots,[\tau_j(a_{ji},A_{ji})]_{n+1,p_i;r}\\(1-\lambda,\sigma),(b_j,B_j)_{1,m},\ldots,[\tau_j(b_{ji},B_{ji})]_{m+1,q_i;r}\end{array}\right.\right],\quad(7)$$

where

$$\Re(\lambda)+\sigma\min_{1\le j\le n}\frac{1-\Re(a_j)}{A_j}>0,\quad|\arg z|<\frac{\pi\zeta_l}{2},$$

and ζ_l is given in (6).

2. Fractional kinetic equations

Recently, a remarkable interest has been developed in the study of the solution of fractional kinetic equations due to their importance in astrophysics and mathematical physics. The kinetic equations of fractional order have been successfully used to determine certain physical phenomena governing diffusion in porous media, reaction and relaxation processes in complex systems etc. Therefore, a large body of research in the solution of these equations has been published in the literature.

Haubold and Mathai [11] established a functional differential equation between the rate of change of reaction, the destruction rate and the production rate as follows:

$$\frac{dN}{dt}=-\delta(N_t)+p(N_t),\quad(8)$$

where $N=N(t)$ is the rate of reaction, $\delta(N_t)=:\delta$ is the rate of destruction, $p=p(N)$ is the rate of production and N_t denotes the function defined by $N_t(t^*)=N(t-t^*),t^*>0$.

A special case of (8), when spatial fluctuations or homogeneities in the quantity $N(t)$ are neglected, has been investigated and was given by the following equation (see [11]; see also [15]):

$$\frac{dN_i}{dt}=-c_iN_i(t),\quad(9)$$

where the initial condition $N_i(t=0)=N_0$ is the number of density of species i at time $t=0,c_i>0$. If we decline the index i and integrate the standard kinetic equation (9), we have

$$N(t)-N_0=-c_0\,{}_0D_t^{-1}N(t),\quad(10)$$

where ${}_0D_t^{-1}$ is the standard fractional integral operator.

Haubold and Mathai [11] gave the fractional generalization of the standard kinetic equation (9) as

$$N(t)-N_0=-c_0^{\nu}\,{}_0D_t^{-\nu}N(t),\quad(11)$$

where ${}_0D_t^{-\nu}$ is the Riemann-Liouville fractional integral operator (see, e.g., [21]) defined as

$${}_0D_t^{-\nu}f(t)=\frac{1}{\Gamma(\nu)}\int_0^t(t-u)^{\nu-1}f(u)\,du,\quad t>0,\Re(\nu)>0.\quad(12)$$

The solution of the fractional equation (11) is given by (see [11])

$$N(t) = N_0 \sum_{k=0}^{\infty} \frac{(-1)^k}{\Gamma(\nu k + 1)} (c_0 t)^{\nu k}. \tag{13}$$

By applying the convolution theorem for the Sumudu transform [2, 4, 5], (12) can be written in the following form:

$$S\left\{{}_0D_t^{-\nu} f(t)\right\} = S\left\{\frac{t^{\nu-1}}{\Gamma(\nu)}\right\}.S\{f(t)\} = u^{\nu} G(u), \tag{14}$$

where $G(u)$ is given in (3).

It is easy to see that the Sumudu transform of the function $f(t) = t^{\lambda}$ is given by

$$S[f(t)] = \int_0^{\infty} (tu)^{\lambda} e^{-t} dt = u^{\lambda} \Gamma(\lambda + 1), \quad \Re(\lambda) > -1. \tag{15}$$

Hilfer [12, 13] have investigated fractional kinetic equations in order to determine and deduce certain physical phenomena that govern such processes as diffusion in porous media, reaction and relaxation in complex systems, anomalous diffusion, and so on.

Saxena et al. [31] presented a very interesting and new generalized form of the fractional kinetic equation in terms of the Mittag-Leffler function and the G-function by making use of the Sumudu transform.

Further, Saxena, Mathai and Haubold [23] investigated three generalizations of the fractional kinetic equation in terms of the Mittag-Leffler functions, which extended the work of Haubold and Mathai [11]. Saxena, Mathai and Haubold [25] also developed the solutions for fractional kinetic equations associated with the generalized Mittag-Leffler function. Recently, Gupta et al. [10] investigated further computable extensions of the generalized fractional kinetic equation in terms of a generalized Lauricella confluent hypergeometric function by using the Sumudu transform technique.

Fractional kinetic equations have also been studied by many authors, for example, Saichev and Zaslavsky [20], Saxena et al. [23, 24, 25], Saxena and Kalla [22], Chaurasia and Pandey [7], Gupta and Sharma [9], whose importance is given in view of solutions of certain physical problems.

In this paper, we introduce and investigate further computable extensions of the generalized fractional kinetic equation. The fractional kinetic equation and its solution, discussed in terms of the Aleph-function, are written in compact and easily computable form as in the next section.

3. Solution of generalized fractional kinetic equations

Theorem 1. *If $\lambda, \nu, \sigma, c_0, h > 0$, $\Re(u) > 0$ with $|u| < c_0^{-1}$, $c_0 \neq h, \tau_i > 0, i = 1, \ldots, r$, the solution of the generalized fractional kinetic equation*

$$N(t) - N_0 t^{\lambda-1} \aleph_{p_i,q_i,\tau_i;r}^{m,n}\left[ht^{\sigma}\left|\begin{matrix}(a_j,A_j)_{1,n},\ldots,[\tau_j(a_{ji},A_{ji})]_{n+1,p_i;r}\\(b_j,B_j)_{1,m},\ldots,[\tau_j(b_{ji},B_{ji})]_{m+1,q_i;r}\end{matrix}\right.\right] = -c_0^{\nu}\, {}_0D_t^{-\nu} N(t) \tag{16}$$

is given by the following formula:

$$N(t) = N_0\, t^{\lambda-2} \sum_{k=0}^{\infty} \left(-c_0^{\nu}\, t^{\nu}\right)^k \times \aleph_{p_i+1,q_i+1,\tau_i;r}^{m,n+1}\left[h\, t^{\sigma}\left|\begin{matrix}(1-\lambda\,,\sigma),(a_j,A_j)_{1,n},\ldots,[\tau_j(a_{ji},A_{ji})]_{n+1,p_i;r}\\(b_j,B_j)_{1,m},\ldots,[\tau_j(b_{ji},B_{ji})]_{m+1,q_i;r},(2-\lambda-\nu k,\sigma)\end{matrix}\right.\right]. \tag{17}$$

Proof. Applying the Sumudu transform to both sides of (16) and using (5) and (14), we get

$$\begin{aligned} S\{N(t)\} - N_0 S\left\{t^{\lambda-1}\aleph_{p_i,q_i,\tau_i;r}^{m,n}\left[ht^{\sigma}\left|\begin{matrix}(a_j,A_j)_{1,n},\ldots,[\tau_j(a_{ji},A_{ji})]_{n+1,p_i;r}\\(b_j,B_j)_{1,m},\ldots,[\tau_j(b_{ji},B_{ji})]_{m+1,q_i;r}\end{matrix}\right.\right]\right\} \\ = -c_0^{\nu}\left\{{}_0D_t^{-\nu}N(t)\right\}, \\ N^*(u) = N_0 u^{\lambda-1}\aleph_{p_i+1,q_i,\tau_i;r}^{m,n+1}\left[hu^{\sigma}\left|\begin{matrix}(1-\lambda,\sigma),(a_j,A_j)_{1,n},\ldots,[\tau_j(a_{ji},A_{ji})]_{n+1,p_i;r}\\(b_j,B_j)_{1,m},\ldots,[\tau_j(b_{ji},B_{ji})]_{m+1,q_i;r}\end{matrix}\right.\right] \\ -c_0^{\nu}u^{\nu}N^*(u), \end{aligned}$$

where $N^*(u) = S\{N(t);u\}$ and $S\left\{t^{\mu-1}\right\} = u^{\mu-1}\Gamma(\mu)$. Thus we have

$$N^*(u) = N_0\frac{u^{\lambda-1}}{(1+(c_0u)^{\nu})}\aleph_{p_i+1,q_i,\tau_i;r}^{m,n+1}\left[hu^{\sigma}\left|\begin{matrix}(1-\lambda,\sigma),(a_j,A_j)_{1,n},\ldots,[\tau_j(a_{ji},A_{ji})]_{n+1,p_i;r}\\(b_j,B_j)_{1,m},\ldots,[\tau_j(b_{ji},B_{ji})]_{m+1,q_i;r}\end{matrix}\right.\right]. \tag{18}$$

Using the relation $S^{-1}\{u^{\nu}\} = \frac{t^{\nu-1}}{\Gamma(\nu)}$, $\Re(\nu) > 0, \Re(u) > 0$, and taking into account of (7), we have

$$N(t) = N_0 t^{\lambda-2}\sum_{k=0}^{\infty}\frac{(-1)^k(1)_k(c_0t)^{\nu k}}{k!} \times \aleph_{p_i+1,q_i+1,\tau_i;r}^{m,n+1}\left[ht^{\sigma}\left|\begin{matrix}(1-\lambda,\sigma),(a_j,A_j)_{1,n},\ldots,[\tau_j(a_{ji},A_{ji})]_{n+1,p_i;r}\\(b_j,B_j)_{1,m},\ldots,[\tau_j(b_{ji},B_{ji})]_{m+1,q_i;r},(2-\lambda-\nu k,\sigma)\end{matrix}\right.\right].$$

This completes the proof of Theorem 1. □

Theorem 2. *Assume that* $\lambda > 0$, $\nu > 0$, $\sigma > 0$, $\omega > 0$, $k \in \mathbb{N}$, $\tau_i > 0$, $i = 1,\ldots,r$, $c_0 > 0$, $\Re(u) > 0$ *with* $|u| < c_0^{-1}$. *Then the solution of a generalized fractional kinetic equation*

$$\begin{aligned} N(t) - N_0 t^{\lambda-1}\aleph_{p_i,q_i,\tau_i;r}^{m,n}\left[\omega t^{\sigma}\left|\begin{matrix}(a_j,A_j)_{1,n},\ldots,[\tau_j(a_{ji},A_{ji})]_{n+1,p_i;r}\\(b_j,B_j)_{1,m},\ldots,[\tau_j(b_{ji},B_{ji})]_{m+1,q_i;r}\end{matrix}\right.\right] \\ = -\left[\sum_{\varepsilon=1}^{k}\binom{k}{\varepsilon}c_0^{\nu\varepsilon}\,{}_0D_t^{-\nu\varepsilon}\right]N(t) \end{aligned} \tag{19}$$

is given by the following formula:

$$N(t) = N_0 t^{\lambda-2}\sum_{\ell=0}^{\infty}\frac{(k)_{\ell}\left(-c_0^{\nu}t^{\nu}\right)^{\ell}}{\ell!} \times \aleph_{p_i+1,q_i+1,\tau_i;r}^{m,n+1}\left[\omega t^{\sigma}\left|\begin{matrix}(1-\lambda,\sigma),(a_j,A_j)_{1,n},\ldots,[\tau_j(a_{ji},A_{ji})]_{n+1,p_i;r}\\(b_j,B_j)_{1,m},\ldots,[\tau_j(b_{ji},B_{ji})]_{m+1,q_i;r},(2-\lambda-\nu\ell,\sigma)\end{matrix}\right.\right]. \tag{20}$$

Proof. Taking the Sumudu transform on both sides of (19) and using (5) and (14), we find that

$$\begin{aligned}&(1+c_0^{\nu}u^{\nu})^k N^*(u)\\&\quad= N_0 u^{\lambda-1}\aleph^{m,n+1}_{p_i+1,q_i,\tau_i;r}\left[\omega u^{\sigma}\left|\begin{matrix}(1-\lambda,\sigma),(a_j,A_j)_{1,n},\ldots,[\tau_j(a_{ji},A_{ji})]_{n+1,p_i;r}\\(b_j,B_j)_{1,m},\ldots,[\tau_j(b_{ji},B_{ji})]_{m+1,q_i;r}\end{matrix}\right.\right],\end{aligned}$$

which, upon solving for $N^*(u)$, yields

$$N^*(u)=\frac{N_0u^{\lambda-1}}{(1+(c_0u)^{\nu})^k}\aleph^{m,n+1}_{p_i+1,q_i,\tau_i;r}\left[\omega u^{\sigma}\left|\begin{matrix}(1-\lambda,\sigma),(a_j,A_j)_{1,n},\ldots,[\tau_j(a_{ji},A_{ji})]_{n+1,p_i;r}\\(b_j,B_j)_{1,m},\ldots,[\tau_j(b_{ji},B_{ji})]_{m+1,q_i;r}\end{matrix}\right.\right]. \tag{21}$$

By applying the generalized binomial formula

$$(1-x)^{-\gamma}=\sum_{n=0}^{\infty}\frac{(\gamma)_n}{n!}x^n \quad (|x|<1) \tag{22}$$

to a factor in (21), we find that, for $|u|<c_0^{-1}$,

$$\begin{aligned}N^*(u)=N_0\sum_{\ell=0}^{\infty}&\frac{(k)_{\ell}(-c_0^{\nu}u^{\nu})^{\ell}}{\ell!}u^{\lambda-1}\\&\times\aleph^{m,n+1}_{p_i+1,q_i,\tau_i;r}\left[\omega u^{\sigma}\left|\begin{matrix}(1-\lambda,\sigma),(a_j,A_j)_{1,n},\ldots,[\tau_j(a_{ji},A_{ji})]_{n+1,p_i;r}\\(b_j,B_j)_{1,m},\ldots,[\tau_j(b_{ji},B_{ji})]_{m+1,q_i;r}\end{matrix}\right.\right].\end{aligned} \tag{23}$$

If we now take the inverse Sumudu transform of (23), the result (20) readily follows. This completes the proof of Theorem 2. □

Remark 1. *If we set* $k=1$ *and* $\omega=h$ *in* Theorem 2*, then we get the result* (17)*, which is derived in* Theorem 1*.*

The generalized M-series is defined as follows (see [35]): For $z,\alpha,\beta\in\mathbb{C},\Re(\alpha)>0$,

$$\begin{aligned}&{}_p\overset{\alpha,\beta}{M}_q(a_1,\ldots,a_p;b_1,\ldots,b_q;z)\\&\quad:=\sum_{r=0}^{\infty}\frac{(a_1)_r\ldots(a_p)_r}{(b_1)_r\ldots(b_q)_r}\frac{z^r}{\Gamma(r\alpha+\beta)} &(24)\\&\quad=\frac{\Gamma(b_1)\ldots\Gamma(b_q)}{\Gamma(a_1)\ldots\Gamma(a_p)}\,{}_{p+1}\psi_{q+1}\left[\begin{matrix}(a_1,1),\ldots,(a_p,1),(1,1)\\(b_1,1),\ldots,(b_q,1),(\beta,\alpha)\end{matrix};z\right], &(25)\end{aligned}$$

where the last relationship exhibits the fact that the so-called generalized M-series is in fact a special case of the Fox-Wright function ${}_p\psi_q(z), p,q\in\mathbb{N}_0$ (see [14]; see also Mathai et al. [17]).

When $r=1,\tau_i=1,\sigma=\nu,p_1=p,q_1=q$ and h is replaced by h^{ν}, applying Fox's H-function [17], we obtain a (presumably) new result in terms of M-series asserted by the following theorem (see also [8]).

Theorem 3. *If $\lambda, \nu, c_0, h > 0, \Re(u) > 0, c_0 \neq h$, then the solution of a generalized fractional kinetic equation*

$$N(t) - N_0 t^{\lambda-1} \, {}_p\overset{\nu,\lambda}{M}_q (a_1, \ldots, a_p; b_1, \ldots, b_q; -h^\nu t^\nu) = -c_0^\nu \, {}_0D_t^{-\nu} N(t) \tag{26}$$

is given by the following relation:

$$N(t) = N_0 t^{\lambda-2} \sum_{k=0}^{\infty} (-c_0 t)^{\nu k} \, {}_p\overset{\nu,\lambda+\nu k-1}{M}_q (a_1, \ldots, a_p; b_1, \ldots, b_q; -h^\nu t^\nu), \tag{27}$$

where ${}_p\overset{\nu,\lambda}{M}_q$ is the generalized M-series.

Proof. Applying the Sumudu transform to both sides of (26), we get

$$N^*(u) = N_0 S \left\{ t^{\lambda-1} \sum_{r=0}^{\infty} \frac{(a_1)_r \ldots (a_p)_r}{(b_1)_r \ldots (b_q)_r} \frac{(-h^\nu t^\nu)^r}{\Gamma(\nu r + \lambda)} \right\} - c_0^\nu u^\nu N^*(u),$$

where $N^*(u) = S\{N(t); u\}$, and by using $S\left\{t^{\mu-1}\right\} = u^{\mu-1}\Gamma(\mu)$, we obtain

$$N^*(u) = \frac{N_0 u^{\lambda-1}}{(1 + c_0^\nu u^\nu)} \sum_{r=0}^{\infty} \frac{(a_1)_r \ldots (a_p)_r (-h^\nu u^\nu)^r}{(b_1)_r \ldots (b_q)_r},$$

or

$$N^*(u) = N_0 u^{\lambda-1} \sum_{k=0}^{\infty} (-1)^k c_0^{\nu k} u^{\nu k} \sum_{r=0}^{\infty} \frac{(a_1)_r \ldots (a_p)_r (-h^\nu)^r u^{\nu r}}{(b_1)_r \ldots (b_q)_r}. \tag{28}$$

Now, taking the inverse Sumudu transform on both sides of (28), we obtain the desired result (27). This completes the proof of Theorem 3. □

4. Special cases

Here we consider some interesting special cases of the results given in Section 3.

If we put $\tau_i = 1$, $1 \leq i \leq r$ in (16), we obtain the following result in terms of the I-function (see, e.g., [34]).

Corollary 1. *If $\lambda, \nu, \sigma, c_0, h > 0, \Re(u) > 0, c_0 \neq h$, then the solution of the following generalized fractional kinetic equation*

$$N(t) - N_0 t^{\lambda-1} I^{m,n}_{p_i,q_i;r} \left[ht^\sigma \left| \begin{matrix} (a_j, A_j)_{1,n}, \ldots, (a_{ji}, A_{ji})_{n+1,p_i;r} \\ (b_j, B_j)_{1,m}, \ldots, (b_{ji}, B_{ji})_{m+1,q_i;r} \end{matrix} \right. \right] = -c_0^\nu \, {}_0D_t^{-\nu} N(t) \tag{29}$$

is given by

$$N(t) = N_0 t^{\lambda-2} \sum_{k=0}^{\infty} (-c_0^\nu t^\nu)^k I^{m,n+1}_{p_i+1,q_i+1,;r} \left[ht^\sigma \left| \begin{matrix} (1-\lambda,\sigma), (a_j, A_j)_{1,n}, \ldots, (a_{ji}, A_{ji})_{n+1,p_i;r} \\ (b_j, B_j)_{1,m}, \ldots, (b_{ji}, B_{ji})_{m+1,q_i;r}, (2-\lambda-\nu k, \sigma) \end{matrix} \right. \right]. \tag{30}$$

If we take $\tau_i = 1$, $1 \leq i \leq r$ and set $r = 1$ in (16), then we get the following result in terms of the H-function (see, e.g., [17]).

Corollary 2. *If $\lambda, \nu, \sigma, c_0, h > 0, \Re(u) > 0, c_0 \neq h$, then the solution of the generalized fractional kinetic equation*

$$N(t) - N_0 t^{\lambda-1} H_{p,q}^{m,n}\left[h\, t^{\sigma} \left| \begin{matrix} (b_q, B_q) \\ (a_p, A_p) \end{matrix} \right.\right] = -c_0^{\nu}\, {}_0D_t^{-\nu} N(t) \tag{31}$$

is given by

$$N(t) = N_0 t^{\lambda-2} \sum_{k=0}^{\infty} \left(-c_0^{\nu} t^{\nu}\right)^k H_{p+1,q+1}^{m,n+1}\left[h t^{\sigma} \left| \begin{matrix} (1-\lambda, \sigma), (a_p, A_p) \\ (b_q, B_q), (2-\lambda-\nu k, \sigma) \end{matrix} \right.\right]. \tag{32}$$

The generalized Mittag-Leffler function introduced by Prabhakar [18] and defined by the series representation as follows:

$$E_{\nu,\lambda}^{\gamma}(z) = \sum_{n=0}^{\infty} \frac{(\gamma)_n z^n}{\Gamma(\nu n + \lambda)\, n!}, \quad \nu, \lambda, \gamma \in \mathbb{C}, \Re(\nu) > 0, \Re(\lambda) > 0. \tag{33}$$

In Theorem 3, if we set $p = 1 = q, b_1 = 1, a_1 = \gamma$, and $c = h$, then we arrive at the result given by Saxena, Ram and Kumar [31] as follows.

Corollary 3. *If $\nu > 0, c_0 > 0, \lambda > 0, \gamma > 0, \Re(u) > 0$, then the solution of the following equation*

$$N(t) - N_0 t^{\lambda-1} E_{\nu,\lambda}^{\gamma}\left(-c_0^{\nu} t^{\nu}\right) = -c_0^{\nu}\, {}_0D_t^{-\nu} N(t) \tag{34}$$

is given by

$$N(t) = N_0 t^{\lambda-2} E_{\nu,\lambda-1}^{\gamma+1}\left(-c_0^{\nu} t^{\nu}\right), \tag{35}$$

where $E_{\nu,\lambda}^{\gamma}(z)$ is the generalized Mittag-Leffler function.

If we take $p = 0 = q$ in and $c = h$ in (26), we obtain another known result given by Saxena, Ram and Kumar [31].

Corollary 4. *If $\nu > 0, c_0 > 0, \lambda > 0, \Re(u) > 0$, then the solution of the following generalized fractional kinetic equation*

$$N(t) - N_0 t^{\lambda-1} E_{\nu,\lambda}\left(-c_0^{\nu} t^{\nu}\right) = -c_0^{\nu}\, {}_0D_t^{-\nu} N(t) \tag{36}$$

is given by

$$N(t) = \frac{N_0 t^{\lambda-2}}{\nu}\left[E_{\nu,\lambda-2}\left(-c_0^{\nu} t^{\nu}\right) + (2+\nu-\lambda) E_{\nu,\lambda-1}\left(-c_0^{\nu} t^{\nu}\right)\right], \tag{37}$$

where $E_{\nu,\lambda}(z)$ is the Wiman function [40] (*also known as the Agarwal function* [1]).

If we set $p = 0 = q$ and $c_0 \neq h$ in (26), then we obtain another known result given by Saxena, Ram and Kumar [31].

Corollary 5. *If* $\nu > 0, c_0 > 0, h > 0, \lambda > 0, \Re(u) > |h|^{\nu/\alpha}, c_0 \neq h$, *then the solution of the equation*

$$N(t) - N_0 t^{\lambda-1} E_{\nu,\lambda}(-h^\nu t^\nu) = -c_0^\nu \, {}_0D_t^{-\nu} N(t) \tag{38}$$

is given by

$$N(t) = \frac{N_0}{c_0^\nu - h^\nu} t^{\lambda-\nu-2} \left[E_{\nu,\lambda-\nu-1}(-h^\nu t^\nu) - E_{\nu,\lambda-\nu-1}(-c_0^\nu t^\nu) \right]. \tag{39}$$

Furthermore, if $h \to 0$ in (38), we find that the solution of the following equation

$$N(t) - \frac{N_0 t^{\lambda-1}}{\Gamma(\lambda)} = -c_0^\nu \, {}_0D_t^{-\nu} N(t) \tag{40}$$

is given by

$$N(t) = \frac{N_0 t^{\lambda-\nu-2}}{c_0^\nu} \left[\frac{1}{\Gamma(\lambda-\nu-1)} - E_{\nu,\lambda-\nu-1}(-c_0^\nu t^\nu) \right]. \tag{41}$$

If we take $\nu = 1 = \lambda$ in (26), then the M-series reduces to the generalized hypergeometric function ${}_pF_q$ (see [14, 16]) and we have the following interesting result:

Corollary 6. *If* $c_0, h > 0, \Re(u) > 0, c_0 \neq h$, *then the solution of the generalized fractional kinetic equation*

$$N(t) - N_0 \, {}_pF_q\left((a_j)_1^p; (b_j)_1^q; -ht\right) = -c_0 \, {}_0D_t^{-1} N(t), \tag{42}$$

there holds the relation

$$N(t) = N_0 t^{-1} \sum_{k=0}^{\infty} (-c_0 t)^k \, {}_p\overset{1,k}{M}_q(a_1, \ldots, a_p; b_1, \ldots, b_q; -ht). \tag{43}$$

5. Concluding remarks

It is further noted that a number of other special cases of our main results as illustrated in Section 4 can also be obtained. In this paper, we have studied a new fractional generalization of the standard kinetic equation and presented their solutions. It is not difficult to obtain several further analogous fractional kinetic equations and their solutions as those exhibited here by Theorem 3 and its Corollaries. Moreover, in view of close relationships of the $\aleph$-function and the M-series with other special functions, it does not seem difficult to construct various known and new fractional kinetic equations.

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