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Finite p-groups whose absolute central automorphisms are inner*

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Abstract. Nasrabadi and Farimani [Indag. Math. (N. S.) **26**(2015), 137–141] have given necessary and sufficient conditions for a finite autonilpotent p-group of class 2 such that all absolute central automorphisms are inner. We generalize this result of Nasrabadi and Farimani to arbitrary finite p-groups.

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1. Introduction

Let G be a finitely generated group and let G' and $Z(G)$ denote the commutator subgroup and the center of G, respectively. Let $\mathrm{Aut}(G)$ denote the full automorphism group and $\mathrm{Inn}(G)$ the inner automorphism group of G. For $g \in G$ and $\alpha \in \mathrm{Aut}(G)$, the element $[g, \alpha] = g^{-1}\alpha(g)$ is called the autocommutator of g and α. Inductively, define

$$[g, \alpha_1, \alpha_2, \ldots, \alpha_n] = [[g, \alpha_1, \alpha_2, \ldots, \alpha_{n-1}], \alpha_n],$$

where $\alpha_i \in \mathrm{Aut}(G)$. Hegarty [4] defined the absolute center $L(G)$ of G as

$$L(G) = \{g \in G \,|\, [g, \alpha] = 1, \text{ for all } \alpha \in \mathrm{Aut}(G)\}.$$

Let $L_1(G) = L(G)$, and for $n \geq 2$, define $L_n(G)$ inductively as

$$L_n(G) = \{g \in G \,|\, [g, \alpha_1, \alpha_2, \ldots, \alpha_n] = 1 \text{ for all } \alpha_1, \alpha_2, \ldots, \alpha_n \in \mathrm{Aut}(G)\}.$$

The autocommutator subgroup G^* of G is defined as

$$G^* = \langle [g, \alpha] \,|\, g \in G, \alpha \in \mathrm{Aut}(G)\rangle.$$

It is easy to see that $L_n(G) \leq Z_n(G)$, the nth term of the upper central series of G, for all $n \geq 1$ and $G' \leq G^*$. An automorphism α of G is called an absolute central automorphism if it induces the identity automorphism on $G/L(G)$; or equivalently, $g^{-1}\alpha(g) \in L(G)$ for all $g \in G$. Let $\mathrm{Aut}_l(G)$ denote the group of all absolute central

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automorphisms of G. A group G is called autonilpotent of class at most n if $L_n(G) = G$ for some natural number n. Observe that if G is autonilpotent of class 2, then $G^* \leq L(G)$. Nasrabadi and Farimani [5, Main Theorem] proved that if G is a finite autonilpotent p-group of class 2, then $\mathrm{Aut}_l(G) = \mathrm{Inn}(G)$ if and only if $L(G) = Z(G)$ and $Z(G)$ is cyclic. In Propositions 1 and 2, we give necessary and sufficient conditions for a finitely generated group G with $G' \leq L(G)$ such that $\mathrm{Aut}_l(G) \simeq \mathrm{Inn}(G)$ and, as a consequence, obtain the necessary and sufficient conditions on a finite p-group G such that $\mathrm{Aut}_l(G) = \mathrm{Inn}(G)$.

By $\mathrm{Hom}(G, A)$ we denote the group of all homomorphisms of G into an abelian group A. The torsion rank and torsion-free rank of G are denoted as $d(G)$ and $\rho(G)$, respectively. By $\exp(G)$ we denote the exponent of the torsion part of G, and by $\pi(G)$ the set of primes dividing the order of the torsion part of G. By X^n we denote the direct product of n-copies of a group X and by C_p a cyclic group of order p. The Sylow p-subgroup of the torsion part of G is denoted as G_p. All other unexplained notations, if any, are standard. The following well known results will be used very frequently without further reference.

Lemma 1. *Let U, V and W be abelian groups. Then*

(i) $\mathrm{Hom}(U \times V, W) \simeq \mathrm{Hom}(U, W) \times \mathrm{Hom}(V, W)$,

(ii) $\mathrm{Hom}(U, V \times W) \simeq \mathrm{Hom}(U, V) \times \mathrm{Hom}(U, W)$,

(iii) $\mathrm{Hom}(C_m, C_n) \simeq C_d$*, where d is the g.c.d. of m and n,*

(iv) if U is torsion-free of rank m, then $\mathrm{Hom}(U, V) \simeq V^m$*, and*

(v) if U is torsion and V is torsion-free, then $\mathrm{Hom}(U, V) = 1$.

2. Main results

Let G be a finitely generated non-abelian group such that $G' \leq L(G)$. Suppose that $\pi(G/Z(G)) = \{p_1, p_2, \ldots, p_d, p'_1, p'_2, \ldots, p'_{d'}\}$, $\pi(G/L(G)) = \{p_1, p_2, \ldots, p_e\}$ and $\pi(L(G)) = \{q_1, q_2, \ldots, q_f\}$. Let X, Y, Z be respective torsion parts and a, b, c respective torsion-free ranks of $G/Z(G), G/L(G)$ and $L(G)$. Let $X_{p_i} \simeq \prod_{j=1}^{l_i} C_{p_i^{\alpha_{ij}}}$, $X_{p'_i} \simeq \prod_{j=1}^{l'_i} C_{{p'_i}^{\alpha'_{ij}}}$, $Y_{p_i} \simeq \prod_{j=1}^{m_i} C_{p_i^{\beta_{ij}}}$ and $Z_{q_i} \simeq \prod_{j=1}^{n_i} C_{q_i^{\gamma_{ij}}}$, where for each i, l_i, l'_i, m_i, n_i, $\alpha_{ij} \geq \alpha_{i(j+1)}$, $\alpha'_{ij} \geq \alpha'_{i(j+1)}$, $\beta_{ij} \geq \beta_{i(j+1)}$ and $\gamma_{ij} \geq \gamma_{i(j+1)}$ are positive integers denoting the Sylow subgroups of X, Y and Z, respectively. Then

$$G/Z(G) \simeq X \times \mathbb{Z}^a \simeq \prod_{i=1}^{d} X_{p_i} \times \prod_{i=1}^{d'} X_{p'_i} \times \mathbb{Z}^a \simeq \prod_{i=1}^{d} \prod_{j=1}^{l_i} C_{p_i^{\alpha_{ij}}} \times \prod_{i=1}^{d'} \prod_{j=1}^{l'_i} C_{{p'_i}^{\alpha'_{ij}}} \times \mathbb{Z}^a,$$

$$G/L(G) \simeq Y \times \mathbb{Z}^b \simeq \prod_{i=1}^{e} Y_{p_i} \times \mathbb{Z}^b \simeq \prod_{i=1}^{e} \prod_{j=1}^{m_i} C_{p_i^{\beta_{ij}}} \times \mathbb{Z}^b$$

and

$$L(G) \simeq Z \times \mathbb{Z}^c \simeq \prod_{i=1}^{f} Z_{q_i} \times \mathbb{Z}^c \simeq \prod_{i=1}^{f} \prod_{j=1}^{n_i} C_{q_i^{\gamma_{ij}}} \times \mathbb{Z}^c.$$

Since $G/Z(G)$ is a quotient group of $G/L(G)$, it follows from [2, Section 25] that $d(X)+a \leq d(Y)+b$, $a \leq b$, $d \leq e$, $l_i \leq m_i$ and $\alpha_{ij} \leq \beta_{ij}$ for all $i, 1 \leq i \leq d$ and for all $j, 1 \leq j \leq l_i$.

The following lemma is a little but useful modification of arguments of Alperin [1, Lemma 3] and Fournelle [3, Section 2].

Lemma 2. *Let G be any group and Y a central subgroup of G contained in a normal subgroup X of G. Then the group of all automorphisms of G that induce the identity on both X and G/Y is isomorphic to* $\mathrm{Hom}(G/X, Y)$.

Proposition 1. *Let G be a finitely generated non-abelian torsion or torsion-free group such that $G' \leq L(G)$. Then $\mathrm{Aut}_l(G) \simeq \mathrm{Inn}(G)$ if and only if one of the following conditions holds:*

(i) G is torsion-free, $L(G)$ is infinite cyclic and $\rho(G/L(G)) = \rho(G/Z(G))$.

(ii) G is finite, $L(G) \simeq C_{\prod_{i=1}^{d} p_i^{\gamma_{i1}}} \times \prod_{i=d+1}^{f} Z_{q_i}$, $q_i \neq p_i$ for $d+1 \leq i \leq f$, and for $1 \leq i \leq d$, either $(G/L(G))_{p_i} = (G/Z(G))_{p_i}$ or $d((G/L(G))_{p_i}) = d((G/Z(G))_{p_i})$, $\alpha_{ij} = \gamma_{i1}$ for $1 \leq j \leq r_i$ and $\alpha_{ij} = \beta_{ij}$ for $r_i+1 \leq j \leq l_i$, where r_i is the largest integer between 1 and l_i such that $\beta_{ir_i} > \gamma_{i1}$.

Proof. Since $L(G)$ is a central subgroup fixed by all automorphisms, therefore $\mathrm{Aut}_l(G) \simeq \mathrm{Hom}(G/L(G), L(G))$ by Lemma 2. It is not very hard to see that if any of the two conditions is satisfied, then $\mathrm{Aut}_l(G) \simeq \mathrm{Inn}(G)$.

Conversely, suppose that $\mathrm{Aut}_l(G) \simeq \mathrm{Inn}(G)$. Then $\mathrm{Hom}(G/L(G), L(G)) \simeq G/Z(G)$ and thus

$$\mathrm{Hom}(Y \times \mathbb{Z}^b, Z \times \mathbb{Z}^c) \simeq X \times \mathbb{Z}^a. \tag{1}$$

First, assume that G is torsion-free. Then $L(G)$ is torsion-free and $G/Z(G)$ is also torsion-free by Lemma 1(i) & (v) and equation (1). As $\mathrm{Hom}(Y \times \mathbb{Z}^b, \mathbb{Z}^c) \simeq \mathbb{Z}^a$ by (1), we have $bc = a$. Since $a \leq b$, $c = 1$ and $a = b$. It follows that $L(G)$ is infinite cyclic and $\rho(G/L(G)) = \rho(G/Z(G))$.

Next, assume that G is torsion. Then G is finite, because both $L(G)$ and $G/L(G)$, being torsion and abelian, are finite. Thus $\mathrm{Hom}(Y, Z) \simeq X$ by (1). Since $d(X_{p_i}) \leq d(Y_{p_i})$, $q_i = p_i$ and $n_i = 1$ for $1 \leq i \leq d$ and $q_i \neq p_i$ for $i > d$. Thus $L(G) \simeq \prod_{i=1}^{d} C_{p_i^{\gamma_{i1}}} \times \prod_{i=d+1}^{f} Z_{q_i}$ and therefore

$$\begin{aligned}
\mathrm{Hom}(Y, Z) &\simeq \mathrm{Hom}(\prod_{i=1}^{e}\prod_{j=1}^{m_i} C_{p_i^{\beta_{ij}}}, \prod_{i=1}^{d} C_{p_i^{\gamma_{i1}}} \times \prod_{i=d+1}^{f}\prod_{j=1}^{n_i} C_{q_i^{\gamma_{ij}}}) \\
&\simeq \mathrm{Hom}(\prod_{i=1}^{d}\prod_{j=1}^{m_i} C_{p_i^{\beta_{ij}}}, \prod_{i=1}^{d} C_{p_i^{\gamma_{i1}}}) \\
&\simeq \prod_{i=1}^{d} \mathrm{Hom}(\prod_{j=1}^{m_i} C_{p_i^{\beta_{ij}}}, C_{p_i^{\gamma_{i1}}}).
\end{aligned}$$

Since $X \simeq \prod_{i=1}^{d} \prod_{j=1}^{l_i} C_{p_i^{\alpha_{ij}}}$, thus it follows that for $1 \le i \le d$,

$$\mathrm{Hom}(\prod_{j=1}^{m_i} C_{p_i^{\beta_{ij}}}, C_{p_i^{\gamma_{i1}}}) \simeq \prod_{j=1}^{l_i} C_{p_i^{\alpha_{ij}}},$$

and hence $l_i = m_i$. We thus have

$$\mathrm{Hom}(\prod_{j=1}^{l_i} C_{p_i^{\beta_{ij}}}, C_{p_i^{\gamma_{i1}}}) \simeq \prod_{j=1}^{l_i} C_{p_i^{\alpha_{ij}}} \tag{2}$$

for $1 \le i \le d$. Two cases now arise here. Either $\exp(Y_{p_i}) \le \exp(Z_{p_i})$ or $\exp(Y_{p_i}) > \exp(Z_{p_i})$. If $\exp(Y_{p_i}) \le \exp(Z_{p_i})$, then $\beta_{ij} \le \gamma_{i1}$ for each j and thus

$$\mathrm{Hom}(\prod_{j=1}^{l_i} C_{p_i^{\beta_{ij}}}, C_{p_i^{\gamma_{i1}}}) \simeq \prod_{j=1}^{l_i} C_{p_i^{\beta_{ij}}}.$$

It therefore follows from (2) that $\alpha_{ij} = \beta_{ij}$ for each j, and thus $(G/Z(G))_{p_i} = (G/L(G))_{p_i}$. And, if $\exp(Y_{p_i}) > \exp(Z_{p_i})$, then there exists the largest integer r_i between 1 and l_i such that $\beta_{ir_i} > \gamma_{i1}$ and $\beta_{ij} \le \gamma_{i1}$ for each $j, r_i + 1 \le j \le l_i$. Then $\mathrm{Hom}(\prod_{j=1}^{l_i} C_{p_i^{\beta_{ij}}}, C_{p_i^{\gamma_{i1}}}) \simeq \prod_{j=1}^{r_i} C_{p_i^{\gamma_{i1}}} \times \prod_{j=r_i+1}^{l_i} C_{p_i^{\beta_{ij}}}$ and hence, by (2), $\alpha_{ij} = \gamma_{i1}$ for $1 \le j \le r_i$ and $\alpha_{ij} = \beta_{ij}$ for $r_i + 1 \le j \le l_i$. $\square$

Proposition 2. *Let G be a finitely generated non-abelian infinite mixed group such that $G' \le L(G)$. Then $\mathrm{Aut}_l(G) \simeq \mathrm{Inn}(G)$ if and only if one of the following conditions holds:*

(i) $L(G)$ is infinite cyclic, $G/Z(G)$ is torsion-free and $\rho(G/Z(G)) = \rho(G/L(G))$.

(ii) $L(G)$ is finite cyclic and (a) $G/L(G)$ is mixed and $G/Z(G) \simeq \mathrm{Hom}(Y, Z) \times Z^b$ or (b) $G/L(G)$ is torsion-free and $G/Z(G) \simeq Z^b$.

(iii) $L(G) \simeq C_{\prod_{i=1}^{d} p_i^{\gamma_{i1}}} \times \prod_{i=d+1}^{f} Z_{q_i} \times \mathbb{Z}^c$ (c is either 1 or arbitrary), $q_i \ne p_i$ for $d+1 \le i \le f$, both $G/Z(G)$ and $G/L(G)$ are finite, and for $1 \le i \le d$, either $(G/L(G))_{p_i} = (G/Z(G))_{p_i}$ or $d((G/L(G))_{p_i}) = d((G/Z(G))_{p_i})$, $\alpha_{ij} = \gamma_{i1}$ for $1 \le j \le r_i$ and $\alpha_{ij} = \beta_{ij}$ for $r_i + 1 \le j \le l_i$, where r_i is the largest integer between 1 and l_i such that $\beta_{ir_i} > \gamma_{i1}$.

Proof. It is easy to see that if any of the three conditions holds, then $\mathrm{Aut}_l(G) \simeq \mathrm{Inn}(G)$. For the converse part, we proceed according to the structure of $L(G)$. First, suppose that $L(G)$ is torsion-free. Then $G/Z(G)$ is torsion-free and $G/L(G)$ is either mixed or torsion-free by Lemma 1(v) and equation (1). In both cases, it is easy to see from (1) that $c = 1$ and $a = b$. Thus $L(G)$ is infinite cyclic and $\rho(G/Z(G)) = \rho(G/L(G))$.

Next, suppose that $L(G)$ is torsion. If $G/L(G)$ is torsion, then G is finite, which is not so. If $G/L(G)$ is mixed, then $\mathrm{Hom}(Y \times \mathbb{Z}^b, Z) \simeq X \times \mathbb{Z}^a$ by (1) and hence $a = 0$. It follows that $d(X) \le d(Y) + b$. On the other hand, from (1), we have that

$d(X) = d(\mathrm{Hom}(Y, Z)) + bd(Z)$. Hence $d(Z) = 1$. Thus $L(G)$ is finite cyclic and $G/Z(G) \simeq \mathrm{Hom}(Y, Z) \times Z^b$. Finallly, if $G/L(G)$ is torsion-free, then, by equation (1), $\mathrm{Hom}(\mathbb{Z}^b, Z) \simeq X \times \mathbb{Z}^a$ and thus $a = 0$ and $X \simeq Z^b$. Since $d(X) + a \leq d(Y) + b$, $d(X) \leq b$ and hence $Z = L(G)$ is cyclic.

Finally, we suppose that $L(G)$ is a mixed group. First, assume that $G/L(G)$ is torsion-free. Then $\mathrm{Hom}(\mathbb{Z}^b, Z \times \mathbb{Z}^c) \simeq X \times \mathbb{Z}^a$ by (1) and hence $b\,c = a$, implying that $a = b$ and $c = 1$. As $d(X) + a \leq b$, $d(X) = 0$ and hence $d(Z) = 0$. Thus $L(G)$ is infinite cyclic, $G/Z(G)$ is torsion-free and $\rho(G/Z(G)) = \rho(G/L(G))$. Next, we assume that $G/L(G)$ is a mixed group. Then $\mathrm{Hom}(Y, Z) \times Z^b \simeq X$ and $\mathbb{Z}^{bc} \simeq \mathbb{Z}^a$, again implying that $a = b$ and $c = 1$. Thus $d(X) \leq d(Y)$ and hence $a = b = 0$. Therefore (1) reduces to $\mathrm{Hom}(Y, Z) \simeq X$. Now, by proceeding as in Proposition 1(ii), we can prove that $L(G) \simeq C_{\prod_{i=1}^{d} p_i^{\gamma_{i1}}} \times \prod_{i=d+1}^{f} Z_{q_i} \times \mathbb{Z}$, and for $1 \leq i \leq d$, either $(G/L(G))_{p_i} = (G/Z(G))_{p_i}$ or $d((G/L(G))_{p_i}) = d((G/Z(G))_{p_i})$, $\alpha_{ij} = \gamma_{i1}$ for $1 \leq j \leq r_i$ and $\alpha_{ij} = \beta_{ij}$ for $r_i + 1 \leq j \leq l_i$, where r_i is the largest integer between 1 and l_i such that $\beta_{ir_i} > \gamma_{i1}$. Finally assume that $G/L(G)$ is torsion. Then $\mathrm{Hom}(Y, Z \times \mathbb{Z}^c) \simeq X \times \mathbb{Z}^a$ implying $a = 0$, c arbitrary and $\mathrm{Hom}(Y, Z) \simeq X$. Again, proceeding as in Proposition 1(ii), we can prove that $L(G) \simeq C_{\prod_{i=1}^{d} p_i^{\gamma_{i1}}} \times \prod_{i=d+1}^{f} Z_{q_i} \times \mathbb{Z}^c$, and for $1 \leq i \leq d$, either $(G/L(G))_{p_i} = (G/Z(G))_{p_i}$ or $d((G/L(G))_{p_i}) = d((G/Z(G))_{p_i})$, $\alpha_{ij} = \gamma_{i1}$ for $1 \leq j \leq r_i$ and $\alpha_{ij} = \beta_{ij}$ for $r_i + 1 \leq j \leq l_i$, where r_i is the largest integer between 1 and l_i such that $\beta_{ir_i} > \gamma_{i1}$. ☐

Let G be a finite p-group such that $G' \leq L(G)$. Let $G/Z(G) \simeq \prod_{i=1}^{r} C_{p^{\alpha_i}}$, $G/L(G) \simeq \prod_{i=1}^{s} C_{p^{\beta_j}}$ and $L(G) \simeq \prod_{i=1}^{t} C_{p^{\gamma_i}}$, where $\alpha_i \geq \alpha_{i+1}$, $\beta_i \geq \beta_{i+1}$ and $\gamma_i \geq \gamma_{i+1}$ are positive integers. Since $G/Z(G)$ is a quotient group of $G/L(G)$, $r \leq s$ and $\alpha_i \leq \beta_i$ for $1 \leq i \leq r$.

Corollary 1. *Let G be a finite non-abelian p-group. Then $\mathrm{Aut}_l(G) = \mathrm{Inn}(G)$ if and only if $G' \leq L(G)$, $L(G)$ is cyclic and either $L(G) = Z(G)$ or $d(G/L(G)) = d(G/Z(G))$, $\alpha_i = \gamma_1$ for $1 \leq i \leq k$ and $\alpha_i = \beta_i$ for $k + 1 \leq i \leq r$, where k is the largest integer such that $\beta_k > \gamma_1$.*

Proof. Observe that if $\mathrm{Aut}_l(G) = \mathrm{Inn}(G)$, then for any commutator $[a, b] \in G'$, $[a, b] = a^{-1} I_b(a) \in L(G)$, where I_b is the inner automorphism of G induced by b, and thus $G' \leq L(G)$. The result now follows from Proposition 1(ii). ☐

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