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Uniqueness and value distribution for q-shifts of meromorphic functions*

Hong Yan Xu[1,†], Kai Liu[2] and Ting Bin Cao[2]

[1] *Department of Informatics and Engineering, Jingdezhen Ceramic Institute, Jingdezhen, 333 403 Jiangxi, P. R. China*
[2] *Department of Mathematics, Nanchang University, Nanchang, 330 031 Jiangxi, P. R. China*

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Abstract. In this paper, we deal with value distribution for q-shift polynomials of transcendental meromorphic functions with zero order and obtain some results which improve the previous theorems given by Liu and Qi [18]. In addition, we investigate value sharing for q-shift polynomials of transcendental entire functions with zero order and obtain some results which extend the recent theorem given by Liu, Liu and Cao [17].
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1. Introduction and main results

In this paper, meromorphic function f will always mean meromorphic in the complex plane $\mathbb{C}$. We assume that the reader is familiar with the fundamental results and standard notations of Nevanlinna value distribution theory, such as the proximity function $m(r,f)$, the counting function $N(r,f)$, the characteristic function $T(r,f)$, the first and second main theorems, the lemma on logarithmic derivatives and so on, for details about Nevalinna theory, see Hayman [10], Yi and Yang [24]. For a meromorphic function f, $S(r,f)$ denotes any quantity satisfying $S(r,f) = o(T(r,f))$ for all r outside a possible exceptional set of the finite logarithmic measure, $\mathbb{S}(f)$ denotes the family of all meromorphic functions α such that $T(r,\alpha) = S(r,f) = o(T(r,f))$, where $r \to \infty$ outside of a possible exceptional set of the finite logarithmic measure. For convenience, we agree that $\mathbb{S}(f)$ includes all constant functions and $\widehat{\mathbb{S}} := \mathbb{S}(f) \cup \{\infty\}$. In addition, by $S_1(r,f)$ we denote any quantity satisfying $S_1(r,f) = o(T(r,f))$ for all r on a set of logarithmic density 1.

Let $f(z)$ and $g(z)$ be two nonconstant meromorphic functions. If for some $a \in \mathbb{C} \cup \{\infty\}$, the zeros of $f(z)-a$ and $g(z)-a$ (if $a=\infty$, zeros of $f(z)-a$ and $g(z)-a$ are the poles of $f(z)$ and $g(z)$, respectively) coincide in locations and multiplicities,

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†Corresponding author. *Email addresses:* `xhyhhh@126.com` (H. Y. Xu), `liukai418@126.com` (K. Liu), `tbcao@ncu.edu.cn` (T. B. Cao)

we say that $f(z)$ and $g(z)$ share the value a CM (counting multiplicities) and if they coincide in locations only, we say that $f(z)$ and $g(z)$ share a IM (ignoring multiplicities).

In recent years, there has been an increasing interest in studying difference equations, the difference product and the q-difference in the complex plane $\mathbb{C}$, a number of papers (including [4, 7, 9, 12, 15, 16, 19, 21, 22, 25]) have focused on the uniqueness of difference analogues of Nevanlinna theory. Halburd and Korhonen [7] established a difference analogue of the Logarithmic Derivative Lemma, and then applied it to prove a number of results on meromorphic solutions of complex difference equations. Afterwards, Barnett, Halburd, Korhonen and Morgan [2] also established an analogue of the Logarithmic Derivative Lemma on q-difference operators.

Liu, Liu and Cao [17], Chen, Huang and Zheng [3], and Luo and Lin [20] studied zeros distributions of difference polynomials of meromorphic functions and obtained the following results:

Theorem 1 (see [17], Theorem 1.2). *Let f be a transcendental meromorphic function of finite order and c a nonzero complex constant. If $n \geq 6$, then the difference polynomial $f(z)^n f(z+c) - \alpha(z)$ has infinitely many zeros, where $\alpha(z) \in \mathbb{S}(f)$.*

Theorem 2 (see [17], Corollary 1.3). *Let $f(z)$ be a transcendental entire function of finite order and c a nonzero complex constant. If $f(z)$ has the Borel exceptional value 0, then $H(z) = f(z)f(z+c)$ takes every nonzero value $a \in \mathbb{C}$ infinitely often.*

Theorem 3 (see [20], Theorem 1). *Let f be a transcendental entire function of finite order σ and c be a nonzero complex constant, Let $P(z) = a_n z^n + a_{n-1} z^{n-1} + \cdots + a_1 z + a_0$ be a nonzero polynomial, where $a_0, a_1, \ldots, a_n (\neq 0)$ are complex constants, and let m be the number of the distinct zeros of $P(z)$. Then for $n > m$, $P(f)f(z+c) = a(z)$ has infinitely many solutions, where $a(z) \in \mathbb{S}(f)\backslash\{0\}$.*

For the q-difference of meromorphic functions, Zhang and Korhonen [23] studied value distribution of q-difference polynomials of meromorphic functions and obtained the following result.

Theorem 4 (see [23], Theorem 4.1). *Let f be a transcendental meromorphic (resp. entire) function of zero order and q non-zero complex constant. Then for $n \geq 6$ (resp. $n \geq 2$), $f(z)^n f(qz)$ assumes every non-zero value $a \in \mathbb{C}$ infinitely often.*

Recently, Liu and Qi [18] firstly investigated value distributions for a q-shift of the meromorphic function and obtained the following result.

Theorem 5 (see [18], Theorem 3.6). *Let f be a zero-order transcendental meromorphic function, $n \geq 6, q \in \mathbb{C} \setminus \{0\}$, $\eta \in \mathbb{C}$, and $R(z)$ a rational function. Then the q-shift difference polynomial $f(z)^n f(qz+\eta) - R(z)$ has infinitely many zeros.*

A natural question is what can we get about the zeros of $P(f)f(qz+\eta) = a(z)$ and $P(f)[f(qz+\eta) - f(z)] = a(z)$, where $P(f), a(z)$ are stated as in Theorem 3 and q, η are stated as in Theorem 5? Corresponding to this question, we get the following theorems which are the improvements of Theorems 4 and 5.

Theorem 6. *Let f be a zero-order transcendental meromorphic (resp. entire) function, $q \in \mathbb{C}\setminus\{0\}$, $\eta \in \mathbb{C}$. Then for $n > m+4$ (resp. $n > m$), $P(f)f(qz+\eta) = a(z)$ has infinitely many solutions, where $a(z) \in \mathbb{S}(f)\setminus\{0\}$, $P(f)$and m are stated in Theorem 3.*

Theorem 7. *Let f be a zero-order transcendental meromorphic (resp. entire) function, $q \in \mathbb{C} \setminus \{0\}$, $\eta \in \mathbb{C}$. Then for $n > m + 6$ (resp. $n > m + 2$), $P(f)[f(qz + \eta) - f(z)] = a(z)$ has infinitely many solutions, where $a(z) \in \mathbb{S}(f)\setminus\{0\}$, $P(f)$ and m are stated in Theorem 3.*

For the uniqueness of the difference and the q-difference of meromorphic functions, some results were obtained (see [11, 12, 15, 17, 20, 25, 26]). Here, we only state some recent theorems as follows.

Theorem 8 (see [26], Theorem 5.1)**.** *Let $f(z)$ and $g(z)$ be two transcendental meromorphic (resp. entire) functions of zero order. Suppose that q is a non-zero complex constant and n is an integer satisfying $n \geq 8$ (resp. $n \geq 4$). If $f(z)^n f(qz)$ and $g(z)^n g(qz)$ share $1, \infty$ CM, then $f(z) \equiv tg(z)$ for $t^{n+1} = 1$.*

Theorem 9 (see [26], Theorem 5.2)**.** *Let $f(z)$ and $g(z)$ be two transcendental entire functions of zero order. Suppose that q is a non-zero complex constant and $n \geq 6$ is an integer. If $f(z)^n(f(z) - 1)f(qz)$ and $g(z)^n(g(z) - 1)g(qz)$ share 1 CM, then $f(z) \equiv g(z)$.*

Theorem 10 (see [20], Theorem 2)**.** *Let f and g be transcendental entire functions of finite order, c a nonzero complex constant, let $P(z)$ be stated as in Theorem 3, and let $n > 2\Gamma_0 + 1$ be an integer, where $\Gamma_0 = m_1 + 2m_2$, m_1 is the number of the simple zero of $P(z)$ and m_2 is the number of multiple zeros of $P(z)$. If $P(f)f(z+c)$ and $P(g)g(z + c)$ share 1 CM, then one of the following results holds:*

(i) *$f \equiv tg$ for a constant t such that $t^d = 1$, where $d = GCD\{\lambda_0, \lambda_1, \cdots, \lambda_n\}$ and*

$$\lambda_i = \begin{cases} i+1, & a_i \neq 0, \\ n+1, & a_i = 0, \end{cases} \qquad i = 0, 1, 2, \ldots, n;$$

(ii) *f and g satisfy the algebraic equation $R(f, g) \equiv 0$, where $R(\omega_1, \omega_2) = P(\omega_1)\omega_1(z+c) - P(\omega_2)\omega_2(z + c)$;*

(iii) *$f(z) = e^{\alpha(z)}, g(z) = e^{\beta(z)}$, where $\alpha(z)$ and $\beta(z)$ are two polynomials, b is a constant satisfying $\alpha + \beta \equiv b$ and $a_n^2 e^{(n+1)b} = 1$.*

In this paper, we will investigate the uniqueness problem of q-shifts of entire functions and obtain the following results.

Theorem 11. *Let f and g be transcendental entire functions of zero order, and let $q \in \mathbb{C} \setminus \{0\}$, $\eta \in \mathbb{C}$, $P(f), \Gamma_0, d$ be stated as in Theorem 10. If $P(f)f(qz + \eta)$ and $P(g)g(qz + \eta)$ share 1 CM and $n > 2\Gamma_0 + 1$, then one of the following cases holds:*

(i) *$f \equiv tg$ for a constant t such that $t^d = 1$;*

(ii) f *and* g *satisfy the algebraic equation* $R(f,g) \equiv 0$*, where* $R(\omega_1,\omega_2) = P(\omega_1)\omega_1(qz+\eta) - P(\omega_2)\omega_2(qz+\eta)$*;*

(iii) $fg \equiv \mu$*, where* μ *is a complex constant satisfying* $a_n^2\mu^{n+1} \equiv 1$*.*

To state the other theorem, we can explain some notations and definitions as follows.

Definition 1 (see [13, 14])**.** *Let* l *be a nonnegative integer or infinity. For* $a \in \mathbb{C} \cup \{\infty\}$*, by* $E_l(a;f)$ *we denote the set of all* a*-points of* f *where an* a*-point of multiplicity* k *is counted* k *times if* $k \le l$ *and* $l+1$ *times if* $k > l$*. If* $E_l(a;f) = E_l(a;g)$*, we say that* f, g *share the value* a *with weight* l*.*

Theorem 12. *Under the assumptions of Theorem 11, if* $E_l(1; P(f)f(qz+\eta)) = E_l(1; P(g)g(qz+\eta))$ *and* l, n, m *are integers satisfying one of the following conditions:*

(I) $l = 2$*,* $n > 2\Gamma_0 + m + 2 - \lambda$*;*

(II) $l = 1$*,* $n > 2\Gamma_0 + 2m + 3 - 2\lambda$*;*

(III) $l = 0$*,* $n > 2\Gamma_0 + 3m + 4 - 3\lambda$*;*

(IV) $l \ge 3$*,* $n > 2\Gamma_0 + 1$*.*

Then the conclusions of Theorem 11 hold, where $\lambda = \min\{\Theta(0,f), \Theta(0,g)\}$ *and* m *is stated as in Theorem 3.*

2. Some lemmas

In what follows, we explain some definitions and notations which are used in this paper. For $a \in \mathbb{C} \cup \infty$, we define

$$\Theta(a,f) = 1 - \limsup_{r\to\infty} \frac{\overline{N}(r, \frac{1}{f-a})}{T(r,f)}.$$

For $a \in \mathbb{C} \cup \infty$ and k a positive integer, by $\overline{N}_{(k}(r, \frac{1}{f-a})$ we denote the counting function of those a-points of f whose multiplicities are not less than k in counting the a-points of f we ignore the multiplicities (see [10]) and $N_k(r, \frac{1}{f-a}) = \overline{N}(r, \frac{1}{f-a}) + \overline{N}_{(2}(r, \frac{1}{f-a}) + \cdots + \overline{N}_{(k}(r, \frac{1}{f-a})$.

Definition 2 (see [1])**.** *When* f *and* g *share* 1 *IM, we denote by* $\overline{N}_L(r, \frac{1}{f-1})$ *the counting function of the* 1*-points of* f *whose multiplicities are greater than* 1*-points of* g*, where each zero is counted only once; similarly, we have* $\overline{N}_L(r, \frac{1}{g-1})$*. Let* z_0 *be a zero of* $f-1$ *of multiplicity* p *and a zero of* $g-1$ *of multiplicity* q*, by* $N_{11}(r, \frac{1}{f-1})$ *we also denote the counting function of those* 1*-points of* f *where* $p = q = 1$*.*

Lemma 1 (see [6])**.** *Let* f *and* g *be two meromorphic functions. If* f *and* g *share 1 CM, then one of the following three cases holds:*

(i)

$$T(r,f)+T(r,g)\le 2N_2(r,f)+2N_2(r,g)+2N_2\left(r,\frac{1}{f}\right)+2N_2\left(r,\frac{1}{g}\right)$$
$$+S(r,f)+S(r,g);$$

(ii) $f\equiv g$;

(iii) $f\cdot g=1$.

Lemma 2 (see [5]). *Let f and g be two meromorphic functions, and let l be a positive integer. If $E_l(1;f)=E_l(1;g)$, then one of the following cases must occur:*

(i)

$$T(r,f)+T(r,g)\le N_2(r,f)+N_2(r,g)+N_2\left(r,\frac{1}{f}\right)+N_2\left(r,\frac{1}{g}\right)+\overline{N}\left(r,\frac{1}{f-1}\right)$$
$$+\overline{N}\left(r,\frac{1}{g-1}\right)-N_{11}\left(r,\frac{1}{f-1}\right)+\overline{N}_{(l+1}\left(r,\frac{1}{f-1}\right)$$
$$+\overline{N}_{(l+1}\left(r,\frac{1}{g-1}\right)+S(r,f)+S(r,g);$$

(ii) $f=\frac{(b+1)g+(a-b-1)}{bg+(a-b)}$, *where $a(\neq 0), b$ are two constants.*

Lemma 3 (see [5]). *Let f and g be two meromorphic functions. If f and g share 1 IM, then one of the following cases must occur:*

(i)

$$T(r,f)+T(r,g)\le 2\left[N_2(r,f)+N_2\left(r,\frac{1}{f}\right)+N_2(r,g)+N_2\left(r,\frac{1}{g}\right)\right]$$
$$+3\overline{N}_L\left(r,\frac{1}{f-1}\right)+3\overline{N}_L\left(r,\frac{1}{g-1}\right)$$
$$+S(r,f)+S(r,g);$$

(ii) $f=\frac{(b+1)g+(a-b-1)}{bg+(a-b)}$, *where $a(\neq 0), b$ are two constants.*

Lemma 4 (see [24]). *Let f be a nonconstant meromorphic function and $P(f)=a_0+a_1f+a_2f^2+\cdots+a_nf^n$, where $a_0,a_1,a_2,\cdots,a_n$ are constants and $a_n\neq 0$. Then*

$$T(r,P(f))=nT(r,f)+S(r,f).$$

Lemma 5 (see [8], Lemma 2.1). *Let f be a non-constant meromorphic function, $s>0$, $\alpha<1$, and let $F\subset\mathbb{R}_+$ be the set of all r such that $T(r,f)\le\alpha T(r+s,f)$. If the logarithmic measure of F is infinite, that is, $\int_F\frac{dt}{t}=\infty$, then f is of infinite order of growth.*

Lemma 6 (see [4], Theorem 2.1). *Let $f(z)$ be a meromorphic function of finite order ρ and c a non-zero complex constant. Then, for each $\varepsilon>0$, we have*

$$T(r,f(z+c))=T(r,f(z))+O\left(r^{\rho-1+\varepsilon}\right)+O(\log r).$$

Lemma 7 (see [26], Theorem 1.1 and Theorem 1.3)**.** *Let $f(z)$ be a transcendental meromorphic function of zero order and q a nonzero complex constant. Then*

$$T(r, f(qz)) = (1 + o(1))T(r, f(z))$$

and

$$N(r, f(qz)) = (1 + o(1))N(r, f(z)),$$

on a set of logarithmic density 1.

Remark 1. *Under the assumptions of Lemma 7, from the definition of $S_1(r, f)$ we have*

$$T(r, f(qz)) = T(r, f(z)) + S_1(r, f),$$
$$N(r, f(qz)) = N(r, f(z)) + S_1(r, f).$$

Lemma 8. *Let $f(z)$ be a transcendental meromorphic function of zero order and q, η two nonzero complex constants. Then*

$$\begin{aligned} T(r, f(qz+\eta)) &= T(r, f(z)) + S_1(r, f),\\ N\left(r, \frac{1}{f(qz+\eta)}\right) &\le N\left(r, \frac{1}{f}\right) + S_1(r, f),\\ N(r, f(qz+\eta)) &\le N(r, f) + S_1(r, f),\\ \overline{N}\left(r, \frac{1}{f(qz+\eta)}\right) &\le \overline{N}\left(r, \frac{1}{f}\right) + S_1(r, f),\\ \overline{N}(r, f(qz+\eta)) &\le \overline{N}(r, f) + S_1(r, f). \end{aligned}$$

Proof. From Lemma 6 and Lemma 7, we easily get the first equality of this lemma.

Next, the idea of the proof of other inequalities of this lemma are from [12, 25]. From Lemma 7, we have

$$\begin{aligned} N\left(r, \frac{1}{f(qz+\eta)}\right) &= N\left(r, \frac{1}{f\left(z+\frac{\eta}{q}\right)}\right) + S_1\left(r, f\left(z+\frac{\eta}{q}\right)\right) \\ &\le N\left(r+\left|\frac{\eta}{q}\right|, \frac{1}{f}\right) + S_1\left(r, f\left(z+\frac{\eta}{q}\right)\right). \end{aligned} \tag{1}$$

By Lemma 6 and f is meromorphic function of zero order, we have

$$S_1\left(r, f\left(z+\frac{\eta}{q}\right)\right) = S_1(r, f).$$

And by Lemma 5, we have

$$N\left(r+\left|\frac{\eta}{q}\right|, \frac{1}{f}\right) \le N\left(r, \frac{1}{f}\right) + S(r, f), \tag{2}$$

outside of a possible exceptional set with the finite logarithmic measure.

From (1),(2) and $N\left(r, \frac{1}{f}\right) \leq N\left(r+\left|\frac{\eta}{q}\right|, \frac{1}{f}\right)$, we have

$$N\left(r+\left|\frac{\eta}{q}\right|, \frac{1}{f}\right) = N\left(r, \frac{1}{f}\right) + S(r,f)$$

and

$$N\left(r, \frac{1}{f(qz+\eta)}\right) \leq N\left(r, \frac{1}{f}\right) + S_1(r,f).$$

Similarly, we can get

$$\overline{N}\left(r, \frac{1}{f(qz+\eta)}\right) \leq \overline{N}\left(r, \frac{1}{f}\right) + S_1(r,f).$$

Set $h(z) = \frac{1}{f(z)}$, then $h(qz+\eta) = \frac{1}{f(qz+\eta)}$. From above, we can prove other inequalities. □

Lemma 9 (see [18], Theorem 2.1)**.** *Let $f(z)$ be a nonconstant zero-order meromorphic func- tion and $q \in \mathbb{C}\setminus\{0\}$. Then*

$$m\left(r, \frac{f(qz+\eta)}{f(z)}\right) = S(r,f),$$

on a set of logarithmic density 1.

Lemma 10. *Let f be a transcendental meromorphic function of zero order, $q(\neq 0), \eta$ complex constants, and let $P(z)$ be stated as in Theorem 3. Then we have*

$$(n-1)T(r,f) + S_1(r,f) \leq T(r, P(f)f(qz+\eta)) \leq (n+1)T(r,f) + S_1(r,f). \quad (3)$$

If f is a transcendental entire function of zero order, we have

$$T(r, P(f)f(qz+\eta)) = T(r, P(f)f) + S_1(r,f) = (n+1)T(r,f) + S_1(r,f). \quad (4)$$

Proof. Set $F(z) = P(f)f(qz+\eta)$. If f is a transcendental entire function of zero order, from Lemma 9 and Lemma 4, we have

$$\begin{aligned}
T(r, F(z)) = m(r, F(z)) &\leq m(r, P(f)f(z)) + m\left(r, \frac{f(qz+\eta)}{f(z)}\right) \\
&\leq m(r, P(f)f(z)) + S_1(r,f) = T(r, P(f)f(z)) + S_1(r,f) \\
&= (n+1)T(r,f) + S_1(r,f).
\end{aligned}$$

On the other hand, from Lemma 9, we have

$$\begin{aligned}
(n+1)T(r,f) = T(r, P(f)f(z)) + S(r,f) &= m(r, P(f)f(z)) + S(r,f) \\
&\leq m(r, F(z)) + m\left(r, \frac{f(z)}{f(qz+\eta)}\right) \\
&= T(r, F(z)) + S_1(r,f).
\end{aligned}$$

Thus, we can get (4).

If f is a meromorphic function of zero order, from Lemma 8 and Lemma 4, we have

$$T(r,P(f)f(qz+\eta)) \le T(r,P(f)) + T(r,f(qz+\eta)) \le (n+1)T(r,f) + S_1(r,f).$$

On the other hand, from Lemma 9 and Lemma 4, we have

$$\begin{aligned}(n+1)T(r,f) &= T(r,P(f)f) + S(r,f) = m(r,P(f)f) + N(r,P(f)f) + S(r,f)\\ &\le m\left(r, F(z)\frac{f(z)}{f(qz+\eta)}\right) + N\left(r, F(z)\frac{f(z)}{f(qz+\eta)}\right) + S(r,f)\\ &\le T(r,F(z)) + 2T(r,f) + S_1(r,f).\end{aligned}$$

Thus, we can get (3). □

Using the same method as in Lemma 10, we can easily get the following lemma.

Lemma 11. *Let f be a transcendental meromorphic function of zero order, $q(\neq 0), \eta$ complex constants, and let $P(z)$ be stated as in Theorem 3. Then we have*

$$T(r,P(f)[f(qz+\eta)-f(z)]) \ge (n-1)T(r,f) + S_1(r,f).$$

If f is a transcendental entire function of zero order, we have

$$T(r,P(f)[f(qz+\eta)-f(z)]) \ge nT(r,f) + S_1(r,f).$$

Lemma 12. *Let $f(z)$ and $g(z)$ be transcendental entire functions of zero order, $P(z)$ be stated as in Theorem 3. If $n \ge 2$, and*

$$P(f)f(qz+\eta)P(g)g(qz+\eta) \equiv t, \tag{5}$$

where $q(\neq 0), \eta, t(\neq 0)$ are complex constants, then we have $fg = \mu$, where $a_n^2\mu^{n+1} = t$.

Proof. Suppose that the roots of $P(z) = 0$ are $b_1, b_2, \ldots, b_m$ with multiplicities $l_1, l_2, \ldots, l_m$. Then we have $l_1 + l_2 + \cdots + l_m = n$. From (5), we have

$$(f-b_1)^{l_1}(f-b_2)^{l_2}\cdots(f-b_m)^{l_m} f(qz+\eta)(g-b_1)^{l_1}(g-b_2)^{l_2}\cdots(g-b_m)^{l_m} g(qz+\eta) \equiv t. \tag{6}$$

Since f, g are nonconstant entire functions, from (6), we can deduce that $b_1 = b_2 = \cdots = b_m = 0$. If fact, from (6), we can get that $b_1, b_2, \ldots, b_m$ are Picard exceptional values. If $m \ge 2$ and $b_j \neq 0 (j = 1, 2, \ldots, m)$, by Picard's theorem of the entire function, we can get that Picard's exceptional values of f are at least three. Thus, we can get a contradiction. Hence, $m = 1$ and $l_1 = n$, that is, there exists a complex constant γ satisfying $P(f) = a_n(f-\gamma)^n$ and $P(g) = a_n(g-\gamma)^n$. Then

$$a_n(f-\gamma)^n f(qz+\eta) a_n(g-\gamma)^n g(qz+\eta) \equiv t. \tag{7}$$

Since f, g are transcendental entire functions, by Picard's theorem, we can get that $f - \gamma = 0$ and $g - \gamma = 0$ do not have zeros. Then, we obtain that $f(z) = e^{\alpha(z)} +$

γ, $g(z) = e^{\beta(z)} + \gamma$, where $\alpha(z), \beta(z)$ are two nonconstant functions. From (7), we get that $f(qz+\eta) \neq 0$ and $g(qz+\eta) \neq 0$. Thus, we can get $\gamma = 0$, that is,

$$a_n^2 f(z)^n f(qz+\eta) g(z)^n g(qz+\eta) \equiv t. \tag{8}$$

Set $M(z) = f(z)g(z)$. If $M(z)$ is nonconstant, from (8), we have

$$a_n^2 M(z)^n M(qz+\eta) \equiv t,$$

that is,

$$a_n^2 M(z)^n \equiv \frac{t}{M(qz+\eta)}. \tag{9}$$

Since f, g are transcendental entire functions of zero order, from (9), Lemma 4, Lemma 8 and $n \geq 2$, we can get a contradiction.

Thus, $M(z)$ is a constant. From (9), we can get $f(z)g(z) \equiv \mu$, where μ is a complex constant satisfying $a_n^2 \mu^{n+1} \equiv t$.

Therefore, the proof of Lemma 12 is complete. □

3. Proofs of Theorems 6 and 7

3.1. The proof of Theorem 6

Proof. **Case 1.** If f is a transcendental meromorphic function of zero order, we first suppose that $P(f)f(qz+\eta) = a(z)$ has finitely many solutions. From Lemma 10, we have $S(r, P(f)f(qz+\eta)) = S(r,f)$. By the Second Fundamental Theorem, Lemma 8 and the definition of m, we have

$$\begin{aligned} T(r, P(f)f(qz+\eta)) &\leq \overline{N}(r, P(f)f(qz+\eta)) + \overline{N}\left(r, \frac{1}{P(f)f(qz+c)}\right) \\ &\quad + \overline{N}\left(r, \frac{1}{P(f)f(qz+c) - a(z)}\right) + S(r,f) \\ &\leq 2\overline{N}(r,f) + \overline{N}\left(r, \frac{1}{P(f)}\right) + \overline{N}\left(r, \frac{1}{f(qz+c)}\right) + S(r,f) \\ &\leq (m+3)T(r,f) + S_1(r,f). \end{aligned} \tag{10}$$

From Lemma 10 and (10), we have

$$(n-1)T(r,f) \leq (m+3)T(r,f) + S_1(r,f),$$

that is,

$$(n-m-4)T(r,f) \leq S_1(r,f). \tag{11}$$

Since $n > m+4$ and f is a transcendental meromorphic function, we can get a contradiction. Thus, $P(f)f(qz+\eta) = a(z)$ has infinitely many solutions when f ia a transcendental meromorphic function of zero order.

Case 2. If f is a transcendental entire function, we suppose that $P(f)f(qz+\eta) = a(z)$ has finitely many solutions. By using the same argument as in Case 1 and (4), we have

$$(n+1)T(r,f) \leq (m+1)T(r,f) + S_1(r,f).$$

Since $n > m$ and f is transcendental, we can get a contradiction.

Thus, we can get the conclusions of Theorem 6. □

3.2. The Proof of Theorem 7

Proof. Similarly to the proof of Theorem 6, and using Lemma 12, we can easily prove Theorem 7. □

4. Proofs of Theorems 11 and 12

In this section, set $F(z) = P(f)f(qz+\eta)$ and $G(z) = P(g)g(qz+\eta)$.

4.1. The proof of Theorem 11

Proof. From the assumptions of Theorem 11, we have that $F(z), G(z)$ share 1 CM. Then, the following three cases will be considered.

Case 1. Suppose that $F(z), G(z)$ satisfy Lemma 1(i). Since $f(z), g(z)$ are entire functions of zero order, from Lemma 10, we have $S(r,F) = S(r,f)$, $S(r,G) = S(r,g)$. Then, from Lemma 1(i) and Lemma 8, we have

$$\begin{aligned} T(r,F(z)) + T(r,G(z)) &\leq 2N_2\left(r,\frac{1}{F}\right) + 2N_2\left(r,\frac{1}{G}\right) + S(r,f) + S(r,g) \qquad (12)\\ &\leq 2N_2\left(r,\frac{1}{P(f)}\right) + 2N_2\left(r,\frac{1}{f(qz+\eta)}\right) + 2N_2\left(r,\frac{1}{P(g)}\right)\\ &\quad + 2N_2\left(r,\frac{1}{g(qz+\eta)}\right) + S(r,f) + S(r,g)\\ &\leq 2\Gamma_0 T(r,f) + 2\Gamma_0 T(r,g) + 2N\left(r,\frac{1}{f(qz+\eta)}\right)\\ &\quad + 2N\left(r,\frac{1}{f(qz+\eta)}\right) + S_1(r,f) + S_1(r,g)\\ &\leq 2(\Gamma_0+1)[T(r,f)+T(r,g)] + S_1(r,f) + S_1(r,g). \end{aligned}$$

From Lemma 9 and (12), we have

$$(n+1)[T(r,f)+T(r,g)] \leq 2(\Gamma_0+1)[T(r,f)+T(r,g)] + S_1(r,f) + S_1(r,g),$$

that is,

$$(n-2\Gamma_0-1)[T(r,f)+T(r,g)] \leq S_1(r,f) + S_1(r,g). \qquad (13)$$

Since $n > 2\Gamma_0 + 1$ and f, g are transcendental functions, we can get a contradiction.

Case 2. If $F(z) \equiv G(z)$, that is,

$$P(f)f(qz+\eta) \equiv P(g)g(qz+\eta). \qquad (14)$$

Set $h = \frac{f}{g}$. If h is not a constant, from (14), we can get that f and g satisfy the algebraic equation $R(f,g) \equiv 0$, where $R(\omega_1,\omega_2) = P(\omega_1)\omega_1(qz+c) - P(\omega_2)\omega_2(qz+c)$.

If h is a constant. Substituting $f = gh$ into (14), we can get

$$g(qz+\eta)[a_n g^n(h^{n+1}-1) + a_{n-1}g^{n-1}(h^n-1) + \cdots + a_0(h-1)] \equiv 0, \qquad (15)$$

where $a_n(\neq 0), a_{n-1}, \ldots, a_0$ are constants.

Since g is a transcendental entire function, we have $g(qz+\eta) \not\equiv 0$. Then, from (15), we have

$$a_n g^n(h^{n+1}-1) + a_{n-1}g^{n-1}(h^n-1) + \cdots + a_0(h-1) \equiv 0, \qquad (16)$$

If $a_n \neq 0$ and $a_{n-1} = a_{n-2} = \cdots = a_0 = 0$, then from (16) and g being a transcendental function, we can get $h^{n+1} = 1$.

$a_n \neq 0$ and there exists $a_i \neq 0\,(i \in \{0,1,2,\ldots,n-1\})$. Suppose that $h^{n+1} \neq 1$, by Lemma 4 and (16), we have $T(r,g) = S(r,g)$ which is a contradiction with a transcendental function g. Then $h^{n+1} = 1$. Similarly to this discussion, we can get that $h^{j+1} = 1$, when $a_j \neq 0$ for some $j = 0, 1, \ldots, n$.

Thus, from the definition of d, we can get that $f \equiv tg$, where t is a constant such that $t^d = 1$, $d = GCD\{\lambda_0, \lambda_1, \cdots, \lambda_n\}$.

Case 3. If $F(z)G(z) \equiv 1$. From Lemma 12, we can get that $fg = \mu$ for a constant μ such that $a_n^2\mu^{n+1} \equiv 1$.

Thus, this completes the proof of Theorem 11. □

4.2. The proof of Theorem 12

From the assumptions of Theorem 12 , we have $E_l(1; F(z)) = E_l(1; G(z))$.

Proof. (I) $l = 2$. Since

$$\begin{aligned}
\overline{N}\left(r, \frac{1}{F-1}\right) + \overline{N}\left(r, \frac{1}{G-1}\right) - N_{11}\left(r, \frac{1}{F-1}\right) \qquad\qquad (17)\\
+ \frac{1}{2}\overline{N}_{(l+1}\left(r, \frac{1}{F-1}\right) + \frac{1}{2}\overline{N}_{(l+1}\left(r, \frac{1}{G-1}\right)\\
\leq \frac{1}{2}N\left(r, \frac{1}{F-1}\right) + \frac{1}{2}N\left(r, \frac{1}{G-1}\right)\\
\leq \frac{1}{2}T(r,F) + \frac{1}{2}T(r,G) + S(r,F) + S(r,G).
\end{aligned}$$

$$\begin{aligned}
\overline{N}_{(l+1}\left(r, \frac{1}{F-1}\right) &\leq \frac{1}{2}N(r, \frac{F}{F'}) = \frac{1}{2}N(r, \frac{F'}{F}) + S(r,F) \leq \frac{1}{2}\overline{N}\left(r, \frac{1}{F}\right) + S(r,F)\\
&\leq \frac{m}{2}T(r,f) + \frac{1}{2}\overline{N}\left(r, \frac{1}{f}\right) + S_1(r,f),
\end{aligned}$$

and

$$\overline{N}_{(l+1}\left(r, \frac{1}{G-1}\right) \leq \frac{m}{2}T(r,g) + \frac{1}{2}\overline{N}\left(r, \frac{1}{g}\right) + S_1(r,g).$$

Case 1. If $F(z), G(z)$ satisfy Lemma 2(i), from transcendental entire function $f(z), g(z)$ and (17), we have

$$T(r, F(z) + T(r, G(z)) \leq 2N_2\left(r, \frac{1}{F}\right) + 2N_2\left(r, \frac{1}{G}\right) + mT(r, f) + mT(r, g) + \overline{N}\left(r, \frac{1}{f}\right) + \overline{N}\left(r, \frac{1}{g}\right) + S_1(r, f) + S_1(r, g).$$

From Lemma 10 and $\lambda = \min\{\Theta(0, f), \Theta(0, g)\}$, for any $\varepsilon(0 < \varepsilon < n-2\Gamma_0-m-2+\lambda)$, we have

$$(n - 2\Gamma_0 - m - 2 + \lambda - \varepsilon)[T(r, f) + T(r, g)] \leq S_1(r, f) + S_1(r, g). \tag{18}$$

Since $n > 2\Gamma_0 + m + 2 - \lambda$ and f, g are transcendental functions, we can get a contradiction.

Case 2. If $F(z), G(z)$ satisfy Lemma 2(ii), that is,

$$F = \frac{(b + 1)G + (a - b - 1)}{bG + (a - b)}, \tag{19}$$

where $a(\neq 0), b$ are two constants.

We consider three cases as follows.

Subcase 2.1. $b \neq 0, -1$. If $a - b - 1 \neq 0$, then by (19) we know

$$\overline{N}\left(r, \frac{1}{G + \frac{a-b-1}{b+1}}\right) = \overline{N}\left(r, \frac{1}{F}\right).$$

Since f, g are entire functions of zero order, by the Second Fundamental Theorem and Lemma 7 and Lemma 8, we have

$$\begin{aligned} T(r, G) &\leq \overline{N}\left(r, \frac{1}{G}\right) + \overline{N}\left(r, \frac{1}{G + \frac{a-b-1}{b+1}}\right) + S(r, g) \\ &\leq \overline{N}\left(r, \frac{1}{G}\right) + \overline{N}\left(r, \frac{1}{F}\right) + S(r, g) \\ &\leq (m + 1)T(r, g) + mT(r, f) + \overline{N}\left(r, \frac{1}{f}\right) + S_1(r, f) + S_1(r, g). \end{aligned}$$

Then from Lemma 8, we have

$$(n - m)T(r, g) \leq mT(r, f) + \overline{N}\left(r, \frac{1}{f}\right) + S_1(r, f) + S_1(r, g).$$

Similarly, we have

$$(n - m)T(r, f) \leq mT(r, g) + \overline{N}\left(r, \frac{1}{g}\right) + S_1(r, f) + S_1(r, g).$$

From the above two inequalities, we have

$$(n - 2m - 1 + \lambda - \varepsilon)[T(r, f) + T(r, g)] \leq S_1(r, f) + S_1(r, g). \tag{20}$$

From the definitions of m and Γ_0, we have $m = m_1 + m_2$. Since $2\Gamma_0 + m + 2 - \lambda - (2m + 1 - \lambda) \geq 0$, that is, $n > 2\Gamma_0 + m + 2 - \lambda \geq 2m + 1 - \lambda$. From (20) and since f, g are transcendental, we can get a contradiction.

If $a - b - 1 = 0$, then by (19) we know $F = ((b+1)G)/(bG + 1)$. Since f, g are entire functions, we get that $-\frac{1}{b}$ is a Picard's exceptional value of $G(z)$. By the Second Fundamental Theorem, we have

$$T(r, G) \leq \overline{N}\left(r, \frac{1}{G}\right) + S(r, G) \leq (m+1)T(r, g) + S_1(r, g).$$

Then, from Lemma 10 and $n > 2\Gamma_0 + m + 2 - \lambda$, we know $T(r, g) \leq S_1(r, g)$, a contradiction.

Subcase 2.2. $b = -1$. Then (19) becomes $F = a/(a + 1 - G)$.

If $a + 1 \neq 0$, then $a + 1$ is a Picard's exceptional value of G. Similarly to the discussion in Subcase 2.1, we can deduce a contradiction again.

If $a + 1 = 0$, then $FG \equiv 1$, that is,

$$P(f)f(qz + \eta)P(g)g(qz + \eta) \equiv 1.$$

Since $n > 2\Gamma_0 + m + 2 - \lambda \geq 2$, by Lemma 12, we can get that $fg = \mu$ for a constant μ such that $a_n^2 \mu^{n+1} \equiv 1$.

Subcase 2.3. $b = 0$. Then (19) becomes $F = (G + a - 1)/a$.

If $a - 1 \neq 0$, then $\overline{N}\left(r, \frac{1}{G+a-1}\right) = \overline{N}\left(r, \frac{1}{F}\right)$. Similarly to the discussion in Subcase 2.1, we can deduce a contradiction again.

If $a - 1 = 0$, then $F \equiv G$, that is,

$$P(f)f(qz + \eta) \equiv P(g)g(qz + \eta).$$

Using the same argument as in the proof of Case 2 in Theorem 11, we can get that f, g satisfy Theorem 11(ii).

(II) $l = 1$. Since

$$\begin{aligned}\overline{N}\left(r, \frac{1}{F-1}\right) &+ \overline{N}\left(r, \frac{1}{G-1}\right) - N_{11}\left(r, \frac{1}{F-1}\right) \\ &\leq \frac{1}{2}N\left(r, \frac{1}{F-1}\right) + \frac{1}{2}N\left(r, \frac{1}{G-1}\right) \\ &\leq \frac{1}{2}T(r, F) + \frac{1}{2}T(r, G) + S(r, F) + S(r, G).\end{aligned} \tag{21}$$

From Lemma 8, we have

$$\begin{aligned}\overline{N}_{(2}\left(r, \frac{1}{F}\right) &\leq N\left(r, \frac{F}{F'}\right) = N\left(r, \frac{F'}{F}\right) + S(r, f) \leq \overline{N}\left(r, \frac{1}{F}\right) + S(r, f) \\ &\leq mT(r, f) + \overline{N}\left(r, \frac{1}{f}\right) + S_1(r, f),\end{aligned} \tag{22}$$

and

$$\overline{N}_{(2}\left(r, \frac{1}{G}\right) \leq mT(r, g) + \overline{N}\left(r, \frac{1}{g}\right) + S_1(r, g). \tag{23}$$

Case 1. If $F(z), G(z)$ satisfy Lemma 2(i), from f, g as entire functions and (21)-(23), we have

$$T(r,F)+T(r,G)\leq 2(\Gamma_0+m+1)[T(r,f)+T(r,g)]+2\overline{N}\left(r,\frac{1}{f}\right)+2\overline{N}\left(r,\frac{1}{g}\right)+S_1(r,f)+S_1(r,g).$$

From Lemma 10 and $\lambda=\min\{\Theta(0,f),\Theta(0,g)\}$, for any $\varepsilon(0<\varepsilon<n-2\Gamma_0-2m-3+2\lambda)$, we have

$$[n-2\Gamma_0-2m-3+2\lambda-\varepsilon]\,[T(r,f)+T(r,g)]\leq S_1(r,f)+S_1(r,g). \tag{24}$$

Since $n>2\Gamma_0+2m+3-2\lambda$, from (24) and since f, g are transcendental, we can get a contradiction.

Case 2. If $F(z), G(z)$ satisfy Lemma 2(ii). Similarly to the proof of Case 2 in (I), we can get the conclusions of Theorem 12.

(III) $l=0$, that is, $F(z), G(z)$ share 1 IM. From the definitions of $F(z), G(z)$, we have

$$\overline{N}_L\left(r,\frac{1}{F-1}\right)\leq N\left(r,\frac{F}{F'}\right)=N\left(r,\frac{F'}{F}\right)+S(r,F)\leq\overline{N}\left(r,\frac{1}{F}\right)+S(r,F) \tag{25}$$
$$\leq mT(r,f)+\overline{N}\left(r,\frac{1}{f}\right)+S_1(r,f);$$

similarly, we have

$$\overline{N}_L\left(r,\frac{1}{G-1}\right)\leq mT(r,g)+\overline{N}\left(r,\frac{1}{g}\right)+S_1(r,f). \tag{26}$$

Case 1. Suppose that $F(z), G(z)$ satisfy Lemma 3(i). From (25) and (26), we have

$$T(r,F(z))+T(r,G(z))\leq 2N_2\left(r,\frac{1}{F}\right)+2N_2\left(r,\frac{1}{G}\right)+3mT(r,f)+3mT(r,g)+3\overline{N}\left(r,\frac{1}{f}\right)+3\overline{N}\left(r,\frac{1}{g}\right)+S_1(r,f)+S_1(r,g).$$

From Lemma 10, for any $\varepsilon(0<\varepsilon<n-2\Gamma_0-3m-4+3\lambda)$, we can get

$$(n-2\Gamma_0-3m-4+3\lambda-\varepsilon)[T(r,f)+T(r,g)]\leq S_1(r,f)+S_1(r,g). \tag{27}$$

Since $n>2\Gamma_0+3m+4-3\lambda$, we can get a contradiction.

Case 2. Suppose that $F(z), G(z)$ satisfy Lemma 3(ii). Similarly to the proof of Case 2 in (I), we can easily get the conclusions of Theorem 12.

(IV) $l \geq 3$. Since

$$\begin{aligned}\overline{N}\left(r, \frac{1}{F(z)-1}\right) + \overline{N}\left(r, \frac{1}{G(z)-1}\right) + \overline{N}_{(l+1}\left(r, \frac{1}{F(z)-1}\right) \\ + \overline{N}_{(l+1}\left(r, \frac{1}{G(z)-1}\right) - N_{11}\left(r, \frac{1}{F(z)-1}\right) \\ \leq \frac{1}{2}N\left(r, \frac{1}{F(z)-1}\right) + \frac{1}{2}\left(r, \frac{1}{G(z)-1}\right) + S(r,F) + S(r,G) \\ \leq \frac{1}{2}T(r,F) + \frac{1}{2}T(r,G) + S(r,F) + S(r,G).\end{aligned}$$

Case 1. Suppose that $F(z), G(z)$ satisfy Lemma 2(i). From Lemmas 8 and 9, we have

$$(n+1)[T(r,f)+T(r,g)] \leq 2(\Gamma_0+1)[T(r,f)+T(r,g)] + S_1(r,f) + S_1(r,g),$$

that is,

$$(n-2\Gamma_0-1)[T(r,f)+T(r,g)] \leq +S_1(r,f) + S_1(r,g). \tag{28}$$

Since $n > 2\Gamma_0 + 1$ and f, g are transcendental functions, we can get a contradiction.

Case 2. Suppose that $F(z), G(z)$ satisfy Lemma 2(ii). Similarly to the proof of Case 2 in (I), we can easily get the conclusions of Theorem 12.

Thus, the proof of Theorem 12 is complete. □

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