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Periodic solutions for a class of differential equations with delays depending on state*

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Abstract. In this paper, we use Schauder and Banach fixed point theorems to study the existence, uniqueness and stability of periodic solutions of a class of iterative differential equations

$$x'(t) = \sum_{m=1}^{k} \sum_{l=1}^{\infty} C_{l,m}(t)(x^{[m]}(t))^l + G(t),$$

where $x^{[m]}(t)$ denotes the mth iterate of $x(t)$ for $m = 1, 2, \ldots, k$..
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1. Introduction

A delay differential equation of the form

$$x'(t) = f(t, x(t), x(t - \tau_1(t)), \ldots, x(t - \tau_k(t)))$$

has been discussed in [1] and [7]. In particular, the delay functions $\tau_j(z), j = 0, 1, \ldots, k$ that not only depend on an unknown function, but also state, $\tau_j(z, x(z))$, $j = 0, 1, \ldots, k$, have been much studied in the literatures. In [4], Cooke points out that it is highly desirable to establish the existence and stability properties of periodic solutions for equations of the form

$$x'(t) + ax(t - h(t, x(t))) = F(t),$$

in which lag $h(t, x(t))$ implicitly involves $x(t)$. Eder [5] considers the iterative functional differential equation

$$x'(t) = x^{[2]}(t)$$

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and obtains that every solution either vanishes identically or is strictly monotonic. Fečkan [6] studies the equation

$$x'(t) = f(x^{[2]}(t))$$

by obtaining an existence theorem for solutions satisfying $x(0) = 0$. Staněk [11] studies the global properties of solutions of the functional differential equation $x'(t) = x(t) + x(x(t))$, and shows that every solution either vanishes identically or is strictly monotonic. Later, in [9], Si, Li and Cheng consider the equation

$$x'(t) = x^{[m]}(t)$$

and establish sufficient conditions for the existence of analytic solutions. Si and Wang [10] discuss the smooth solutions of the equation

$$x'(t) = \lambda_1 x(t) + \lambda_2 x^{[2]}(t) + \ldots + \lambda_n x^{[n]}(t) + f(t).$$

Recently, by Schröder transformation, Liu and Si [8] consider the analytic solutions of the form

$$x'(t) = \sum_{m=0}^{k} \sum_{l=1}^{\infty} C_{l,m}(t)(x^{[m]}(t))^l + G(t),$$

where $x^{[i]}(t)$ denotes the ith iterate of $x(t), i = 1, 2, \ldots, n$. For some various properties of solutions for several delay functional differential equations, we refer the interested reader to [2, 3].

In this paper, we consider the existence of periodic solutions of

$$x'(t) = \sum_{m=1}^{k} \sum_{l=1}^{\infty} C_{l,m}(t)(x^{[m]}(t))^l + G(t). \tag{1}$$

We denote by $C(\mathbb{R}, \mathbb{R})$ the set of all real valued continuous functions from $\mathbb{R}$ into $\mathbb{R}$.

For $T > 0$, define

$$\mathcal{P}_T = \Big\{ x \in C(\mathbb{R}, \mathbb{R}) : x(t + T) = x(t),\ \forall t \in \mathbb{R} \Big\}.$$

Then $\mathcal{P}_T$ is a Banach space with the norm

$$\|x\| = \max_{t \in [0,T]} |x(t)| = \max_{t \in \mathbb{R}} |x(t)|.$$

For $P > 0, L \geq 0$, define the sets

$$\mathcal{P}_T(P, L) = \Big\{ x \in \mathcal{P}_T : \|x\| \leq P,\ |x(t_2) - x(t_1)| \leq L|t_2 - t_1|,\ \forall t_1, t_2 \in \mathbb{R} \Big\},$$
$$\mathcal{P}_T(P) = \Big\{ x \in \mathcal{P}_T : \|x\| \leq P \Big\},$$

which are closed convex and bounded subset of $\mathcal{P}_T$, and we wish to find T-periodic functions $x \in \mathcal{P}_T(P, L)$ satisfying (1).

2. Existence of periodic solutions

In this section, the existence of periodic solutions of Eq. (1) will be proved. Let us state the Schauder fixed point theorem, which will be used to prove our main theorem.

Theorem 1 (Schauder)**.** *Let Ω be a closed convex compact subset of a Banach space. Suppose that $A : \Omega \to \Omega$ is continuous. Then there exists $z \in \Omega$ with $z = Az$.*

Throughout this paper, we assume that all functions are continuous with respect to their arguments and the following condition holds.

(H) $C_{l,m} \in \mathcal{P}_T(P_{l,m})$ and $G \in \mathcal{P}_T(P_G)$ are given, where $P_{l,m}$ and P_G are subject to appropriate constraints which will be specified later, if necessary. In addition,

$$C_{1,1}(t) < 0 \tag{2}$$

for all $t \in \mathbb{R}$.

We begin with the following lemma.

Lemma 1. *For any $\varphi, \psi \in \mathcal{P}_T(P, L)$,*

$$||\varphi^{[n]} - \psi^{[n]}|| \le \sum_{j=0}^{n-1} L^j \|\varphi - \psi\|, \ n = 1, 2, \dots. \tag{3}$$

Proof. The result follows from the definition of $\mathcal{P}_T(P, L)$. □

Now we rewrite (1) as a fixed point equation.

Lemma 2. *$x \in \mathcal{P}_T$ is a solution of (1) if and only if*

$$\begin{aligned} x(t) &= \sum_{l=2}^{\infty} \int_t^{t+T} C_{l,1}(u)(x(u))^l \Delta(t,u) du \\ &\quad + \sum_{m=2}^{k} \sum_{l=1}^{\infty} \int_t^{t+T} C_{l,m}(u)(x^{[m]}(u))^l \Delta(t,u) du + \int_t^{t+T} G(u)\Delta(t,u) du, \end{aligned}$$

where

$$\Delta(t,u) = \frac{e^{\int_u^t C_{1,1}(s)ds}}{e^{-\int_0^T C_{1,1}(s)ds} - 1}. \tag{4}$$

Proof. The proof is well-known but we present it here for the reader's convenience. It is easy to see that Eq. (1) can be written in the form of

$$\begin{aligned} &x'(t)e^{-\int_a^t C_{1,1}(s)ds} - x(t)C_{1,1}(t)e^{-\int_a^t C_{1,1}(s)ds} \\ &\quad = \left[\sum_{l=2}^{\infty} C_{l,1}(t)(x(t))^l + \sum_{m=2}^{k} \sum_{l=1}^{\infty} C_{l,m}(t)(x^{[m]}(t))^l + G(t)\right] e^{-\int_a^t C_{1,1}(s)ds} \end{aligned}$$

for a fixed $a \in \mathbb{R}$. If $x \in \mathcal{P}_T$ is a solution of (1), then integrating the above equality from t to $t+T$, we obtain

$$
\begin{aligned}
& x(t+T)e^{-\int_a^{t+T} C_{1,1}(s)ds} - x(t)e^{-\int_a^t C_{1,1}(s)ds} \\
&\quad = \sum_{l=2}^{\infty} \int_t^{t+T} C_{l,1}(u)(x(u))^l e^{-\int_a^u C_{1,1}(s)ds} du \\
&\qquad + \sum_{m=2}^{k} \sum_{l=1}^{\infty} \int_t^{t+T} C_{l,m}(u)(x^{[m]}(u))^l e^{-\int_a^u C_{1,1}(s)ds} du \\
&\qquad + \int_t^{t+T} G(u) e^{-\int_a^u C_{1,1}(s)ds} du.
\end{aligned}
$$

Using the fact $x(t+T) = x(t)$, the above expression can be putted in the form

$$
\begin{aligned}
x(t) &= \sum_{l=2}^{\infty} \int_t^{t+T} C_{l,1}(u)(x(u))^l \frac{e^{\int_u^t C_{1,1}(s)ds}}{e^{-\int_0^T C_{1,1}(s)ds} - 1} du \\
&\quad + \sum_{m=2}^{k} \sum_{l=1}^{\infty} \int_t^{t+T} C_{l,m}(u)(x^{[m]}(u))^l \frac{e^{\int_u^t C_{1,1}(s)ds}}{e^{-\int_0^T C_{1,1}(s)ds} - 1} du \\
&\quad + \int_t^{t+T} G(u) \frac{e^{\int_u^t C_{1,1}(s)ds}}{e^{-\int_0^T C_{1,1}(s)ds} - 1} du.
\end{aligned}
$$

This completes the proof. □

Clearly, $\Delta(t,u) = \Delta(t+T, u+T)$ for all $(t,u) \in \mathbb{R}^2$, and for $(t,u) \in \mathbb{R}^2$ with $u \in [t, t+T]$, we derive

$$
0 < \Delta(t,u) \le \frac{e^{-\int_t^{t+T} C_{1,1}(s)ds}}{e^{-\int_0^T C_{1,1}(s)ds} - 1} = \frac{1}{1 - e^{\int_0^T C_{1,1}(s)ds}} = M. \tag{5}
$$

Now we will need to construct a mapping satisfying the hypotheses of Theorem 1. To this aim, we consider a map $A : \mathcal{P}_T(P, L) \to \mathcal{P}_T$ defined as follows:

$$
\begin{aligned}
(Ax)(t) &= \sum_{l=2}^{\infty} \int_t^{t+T} C_{l,1}(u)(x(u))^l \Delta(t,u) du \\
&\quad + \sum_{m=2}^{k} \sum_{l=1}^{\infty} \int_t^{t+T} C_{l,m}(u)(x^{[m]}(u))^l \Delta(t,u) du \\
&\quad + \int_t^{t+T} G(u)\Delta(t,u) du,
\end{aligned} \tag{6}
$$

where $\Delta(t,u)$ is defined as in (4).

Lemma 3. *Suppose* (H) *holds and*

$$
MT\Big(\sum_{l=2}^{\infty} l P_{l,1} P^{l-1} + \sum_{m=2}^{k} \sum_{l=1}^{\infty} \sum_{j=0}^{m-1} l L^j P_{l,m} P^{l-1} \Big) < \infty, \tag{7}
$$

then operator A is Lipschitz continuous.

Proof. Take $x, y \in \mathcal{P}_T(P, L), t \in \mathbb{R}$; then by (3) and (5), we have

$$\begin{aligned}
&|(Ax)(t) - (Ay)(t)| \\
&\quad \le \sum_{l=2}^{\infty} \left| \int_t^{t+T} C_{l,1}(u)\Delta(t,u)\Big((x(u))^l - (y(u))^l\Big) du \right| \\
&\qquad + \sum_{m=2}^{k} \sum_{l=1}^{\infty} \left| \int_t^{t+T} C_{l,m}(u)\Big((x^{[m]}(u))^l - (y^{[m]}(u))^l\Big)\Delta(t,u) du \right| \\
&\quad \le M \sum_{l=2}^{\infty} P_{l,1} \int_t^{t+T} \Big|(x(u))^l - (y(u))^l\Big| du \\
&\qquad + M \sum_{m=2}^{k} \sum_{l=1}^{\infty} P_{l,m} \int_t^{t+T} \Big|(x^{[m]}(u))^l - (y^{[m]}(u))^l\Big| du \\
&\quad \le MT \sum_{l=2}^{\infty} l P_{l,1} |\xi(u)|^{l-1} \|x - y\| \\
&\qquad + MT \sum_{m=2}^{k} \sum_{l=1}^{\infty} \sum_{j=0}^{m-1} L^j l P_{l,m} |\eta_m(u)|^{l-1} \|x - y\| \\
&\quad \le MT \Big(\sum_{l=2}^{\infty} l P_{l,1} P^{l-1} + \sum_{m=2}^{k} \sum_{l=1}^{\infty} \sum_{j=0}^{m-1} l L^j P_{l,m} P^{l-1} \Big) \|x - y\|,
\end{aligned}$$

where $\xi(u)$ is between $x(u)$ and $y(u)$, $\eta_m(u)$ are between $x^{[m]}(u)$ and $y^{[m]}(u)$. Thus

$$\|Ax - Ay\| \le MT \Big(\sum_{l=2}^{\infty} l P_{l,1} P^{l-1} + \sum_{m=2}^{k} \sum_{l=1}^{\infty} \sum_{j=0}^{m-1} l L^j P_{l,m} P^{l-1} \Big) \|x - y\|. \tag{8}$$

From (7), we proved that A is Lipschitz continuous. This completes the proof. □

Now, it is easy to see by the Arzela-Ascoli theorem that $\mathcal{P}_T(P, L)$ is compact. As a matter of fact, we have the following result.

Lemma 4. *It holds*

$$\mathcal{P}_T(P, L) = \Big\{ x \in \mathcal{P}_T : \|x\| \le P,\ |x(t_2) - x(t_1)| \le L|t_2 - t_1|,\ \forall t_1, t_2 \in [0, T] \Big\}. \tag{9}$$

Proof. Clearly,

$$\mathcal{P}_T(P, L) \subset \Big\{ x \in \mathcal{P}_T : \|x\| \le P,\ |x(s_2) - x(s_1)| \le L|s_2 - s_1|,\ \forall s_1, s_2 \in [0, T] \Big\}.$$

On the other hand, let $x \in \mathcal{P}_T$ with $|x(s_2) - x(s_1)| \le L|s_2 - s_1|$, $\forall s_1, s_2 \in [0, T]$. For any $t_2 > t_1$ and $t_1, t_2 \in \mathbb{R}$, there are uniquely determined integer numbers p_2 and p_1 such that

$$p_1 T \le t_1 < (p_1 + 1)T, \quad p_2 T \le t_2 < (p_2 + 1)T.$$

Then either $p_1 = p_2 = p$ and hence

$$|x(t_2) - x(t_1)| = |x(t_2 - pT) - x(t_1 - pT)| \le L(t_2 - t_1),$$

or $p_2 \ge p_1 + 1$ and hence

$$\begin{aligned}
&|x(t_2) - x(t_1)| \\
&\quad \le |x(t_2) - x(p_2T)| + |x(p_2T) - x((p_1+1)T)| + |x((p_1+1)T) - x(t_1)| \\
&\quad \le |x(t_2 - p_2T) - x(0)| + |x(T) - x(t_1 - p_1T)| \\
&\quad \le L(t_2 - p_2T) + L(T - t_1 + p_1T)| \\
&\quad = L(t_2 - t_1 + T(1 + p_1 - p_2)) \\
&\quad \le L(t_2 - t_1),
\end{aligned}$$

which gives

$$\Big\{x \in \mathcal{P}_T : \|x\| \le P,\ |x(t_2) - x(t_1)| \le L|t_2 - t_1|,\ \forall t_1, t_2 \in [0, T]\Big\} \subset \mathcal{P}_T(P, L).$$

This proves (9). □

Now, we are ready to prove the following existence result.

Theorem 2. *Suppose* (H), (7) *and the following inequalities hold*

$$MT\Big(\sum_{l=2}^{\infty} P_{l,1}P^l + \sum_{m=2}^{k}\sum_{l=2}^{\infty} P_{l,m}P^l + P_G\Big) \le (1 - MT\sum_{m=2}^{k} P_{1,m})P, \tag{10}$$

$$\Big(1 + \frac{TP_{1,1}e^{TP_{11}}}{e^{-\int_0^T C_{11}(s)ds} - 1}\Big)\Big(\sum_{l=2}^{\infty} P_{l,1}P^l + P_G + \sum_{m=2}^{k}\sum_{l=1}^{\infty} P_{l,m}P^l\Big) \le L; \tag{11}$$

then Eq. (1) *has a periodic solution in* $\mathcal{P}_T(P, L)$.

Proof. First, for any $x, y \in \mathcal{P}_T(P, L)$, by (10), we have

$$\begin{aligned}
|(Ax)(t)| &\le \sum_{l=2}^{\infty}\int_t^{t+T} |C_{l,1}(u)||x(u)|^l|\Delta(t,u)|du \\
&\quad + \sum_{m=2}^{k}\sum_{l=1}^{\infty}\int_t^{t+T} |C_{l,m}(u)||x^{[m]}(u)|^l|\Delta(t,u)|du \\
&\quad + \int_t^{t+T} |G(u)||\Delta(t,u)|du, \\
&\le MT\Big(\sum_{l=2}^{\infty} P_{l,1}P^l + \sum_{m=2}^{k}\sum_{l=1}^{\infty} P_{l,m}P^l + P_G\Big) \\
&\le P.
\end{aligned}$$

Next, assuming $T \geq t_2 \geq t_1 \geq 0$, by (11), we obtain

$$
\begin{aligned}
&\sum_{l=2}^{\infty}\left|\int_{t_2}^{t_2+T} C_{l,1}(u)(x(u))^l \Delta(t_2,u)du - \int_{t_1}^{t_1+T} C_{l,1}(u)(x(u))^l \Delta(t_1,u)du\right| \\
&\quad \leq \frac{1}{e^{-\int_0^T C_{1,1}(s)ds}-1}\sum_{l=2}^{\infty}\Bigg(\Big|\int_{t_2}^{t_1} C_{l,1}(u)(x(u))^l e^{\int_u^{t_2} C_{1,1}(s)ds}du \\
&\qquad + \int_{t_1+T}^{t_2+T} C_{l,1}(u)(x(u))^l e^{\int_u^{t_2} C_{1,1}(s)ds}du\Big| \\
&\qquad + \int_{t_1}^{t_1+T} |C_{l,1}(u)||x(u)|^l \Big|e^{\int_u^{t_2} C_{1,1}(s)ds} - e^{\int_u^{t_1} C_{1,1}(s)ds}\Big|du\Bigg) \\
&\quad \leq \frac{1}{e^{-\int_0^T C_{1,1}(s)ds}-1}\sum_{l=2}^{\infty}\Bigg(\int_{t_1}^{t_2} |C_{l,1}(u)||x(u)|^l e^{\int_u^{t_2} C_{1,1}(s)ds}\Big(e^{-\int_0^T C_{1,1}(s)ds}-1\Big)du \\
&\qquad + \int_{t_1}^{t_1+T} |C_{l,1}(u)||x(u)|^l e^{\int_u^{t_2} C_{1,1}(s)ds}\Big(e^{-\int_{t_1}^{t_2} C_{1,1}(s)ds}-1\Big)du\Bigg) \\
&\quad \leq \frac{1}{e^{-\int_0^T C_{1,1}(s)ds}-1}\sum_{l=2}^{\infty} P_{l,1}P^l\Big(\Big(e^{-\int_0^T C_{1,1}(s)ds}-1\Big)+TP_{1,1}e^{TP_{1,1}}\Big)(t_2-t_1) \quad (12)
\end{aligned}
$$

and

$$
\begin{aligned}
&\left|\int_{t_2}^{t_2+T} G(u)\Delta(t_2,u)du - \int_{t_1}^{t_1+T} G(u)\Delta(t_1,u)du\right| \\
&\quad \leq \frac{1}{e^{-\int_0^T C_{1,1}(s)ds}-1}\Bigg(\Big|\int_{t_2}^{t_1} G(u)e^{\int_u^{t_2} C_{1,1}(s)ds}du + \int_{t_1+T}^{t_2+T} G(u)e^{\int_u^{t_2} C_{1,1}(s)ds}du\Big| \\
&\qquad + \int_{t_1}^{t_1+T} |G(u)|\Big|e^{\int_u^{t_2} C_{1,1}(s)ds} - e^{\int_u^{t_1} C_{1,1}(s)ds}\Big|du\Bigg) \\
&\quad \leq \frac{1}{e^{-\int_0^T C_{1,1}(s)ds}-1}\Bigg(\int_{t_1}^{t_2} |G(u)|e^{\int_u^{t_2} C_{1,1}(s)ds}\Big(e^{-\int_0^T C_{1,1}(s)ds}-1\Big)du \\
&\qquad + \int_{t_1}^{t_1+T} |G(u)|e^{\int_u^{t_2} C_{1,1}(s)ds}\Big(e^{-\int_{t_1}^{t_2} C_{1,1}(s)ds}-1\Big)du\Bigg) \\
&\quad \leq \frac{1}{e^{-\int_0^T C_{1,1}(s)ds}-1}P_G\Big(\Big(e^{-\int_0^T C_{1,1}(s)ds}-1\Big)+TP_{1,1}e^{TP_{1,1}}\Big)(t_2-t_1), \quad (13)
\end{aligned}
$$

$$
\begin{aligned}
&\sum_{m=2}^{k}\sum_{l=1}^{\infty}\left|\int_{t_2}^{t_2+T} C_{l,m}(u)(x^{[m]}(u))^l \Delta(t_2,u)du - \int_{t_1}^{t_1+T} C_{l,m}(u)(x^{[m]}(u))^l \Delta(t_1,u)du\right| \\
&\quad \leq \frac{1}{e^{-\int_0^T C_{1,1}(s)ds}-1}\sum_{m=2}^{k}\sum_{l=1}^{\infty}\Bigg(\Big|\int_{t_2}^{t_1} C_{l,m}(u)(x^{[m]}(u))^l e^{\int_u^{t_2} C_{1,1}(s)ds}du
\end{aligned}
$$

$$
\begin{aligned}
&+\int_{t_1+T}^{t_2+T} C_{l,m}(u)(x^{[m]}(u))^l e^{\int_u^{t_2} C_{1,1}(s)ds} du\Big| \\
&+\int_{t_1}^{t_1+T} |C_{l,m}(u)||x^{[m]}(u)|^l \Big| e^{\int_u^{t_2} C_{1,1}(s)ds} - e^{\int_u^{t_1} C_{1,1}(s)ds}\Big| du\Big) \\
\le{}& \frac{1}{e^{-\int_0^T C_{1,1}(s)ds}-1}\sum_{m=2}^{k}\sum_{l=1}^{\infty}\Big(\int_{t_1}^{t_2} |C_{l,m}(u)||x^{[m]}(u)|^l e^{\int_u^{t_2} C_{1,1}(s)ds} \\
&\times\Big(e^{-\int_0^T C_{1,1}(s)ds}-1\Big)du \\
&+\int_{t_1}^{t_1+T} |C_{l,m}(u)||x^{[m]}(u)|^l e^{\int_u^{t_2} C_{1,1}(s)ds}\Big(e^{-\int_{t_1}^{t_2} C_{1,1}(s)ds}-1\Big)du\Big) \\
\le{}& \frac{1}{e^{-\int_0^T C_{1,1}(s)ds}-1} \\
&\times\sum_{m=2}^{k}\sum_{l=1}^{\infty} P_{l,m}P^l\Big(\Big(e^{-\int_0^T C_{1,1}(s)ds}-1\Big)+TP_{1,1}e^{TP_{1,1}}\Big)(t_2-t_1).
\end{aligned}
\tag{14}
$$

By (12), (13) and (14), we have

$$
\begin{aligned}
&\Big|(Ax)(t_2)-(Ax)(t_1)\Big| \\
&\le \sum_{l=2}^{\infty}\left|\int_{t_2}^{t_2+T} C_{l,1}(u)(x(u))^l\Delta(t_2,u)du - \int_{t_1}^{t_1+T} C_{l,1}(u)(x(u))^l\Delta(t_1,u)du\right| \\
&\quad+\left|\int_{t_2}^{t_2+T} G(u)\Delta(t_2,u)du - \int_{t_1}^{t_1+T} G(u)\Delta(t_1,u)du\right| \\
&\quad+\sum_{m=2}^{k}\sum_{l=1}^{\infty}\Big|\int_{t_2}^{t_2+T} C_{l,m}(u)(x^{[m]}(u))^l\Delta(t_2,u)du \\
&\quad-\int_{t_1}^{t_1+T} C_{l,m}(u)(x^{[m]}(u))^l\Delta(t_1,u)du\Big| \\
&\le \left(1+\frac{TP_{1,1}e^{TP_{1,1}}}{e^{-\int_0^T C_{1,1}(s)ds}-1}\right)\Big(\sum_{l=2}^{\infty}P_{l,1}P^l + P_G + \sum_{m=2}^{k}\sum_{l=1}^{\infty}P_{l,m}P^l\Big)|t_2-t_1| \\
&\le L|t_2-t_1|.
\end{aligned}
$$

Therefore by Lemma 4, we obtain $Ax \in \mathcal{P}_T(P,L)$. So by Lemma 3, we see that all conditions of Schauder's theorem are satisfied on $\mathcal{P}_T(P,L)$. Thus there exists a fixed point x in $\mathcal{P}_T(P,L)$ such that $x = Ax$, from Lemma 2, x is a T-periodic solution of Eq. (1). This completes the proof. □

3. Uniqueness and stability

In this section, uniqueness and stability of (1) will be proved. An example is also provided to illustrate that the assumptions of Theorem 2 do not self-contradict.

Theorem 3. *In addition to the assumption of Theorem* 2*, suppose that*

$$\Gamma = MT\Big(\sum_{l=2}^{\infty} lP_{l,1}P^{l-1} + \sum_{m=2}^{k}\sum_{l=1}^{\infty}\sum_{j=0}^{m-1} lL^{j}P_{l,m}P^{l-1}\Big) < 1; \tag{15}$$

then Eq. (1) *has a unique solution in* $\mathcal{P}_T(P, L)$*.*

Proof. We know from the proof of Theorem 2 that $A : \mathcal{P}_T(P, L) \to \mathcal{P}_T(P, L)$. Moreover, by (8), we get

$$\|A\varphi - A\psi\| \le \Gamma\|\varphi - \psi\|, \quad \varphi, \psi \in \mathcal{P}_T(P, L).$$

(15) means $\Gamma < 1$, so the fixed point must be unique by the Banach fixed point theorem. □

Theorem 4. *The unique solution obtained in Theorem* 3 *depends continuously on the given functions* $C_{l,m}(t)$ *and* $G(t)$*, for* $l = 1, \ldots, \infty, m = 1, \ldots, k$*.*

Proof. Let functions $C_{l,m}, \widetilde{C}_{l,m} \in \mathcal{P}_T(P_{l,m})$, $l = 1, \ldots, \infty, m = 1, \ldots, k$ and $G, \widetilde{G} \in \mathcal{P}_T(P_G)$ be given. Then we consider the corresponding constants M, $\widetilde{M}$ and operators A, $\widetilde{A}$ defined by (5) and (6), respectively. Assuming corresponding conditions (2), (10), (11) and (15), there are two unique corresponding functions $x(t)$ and $\widetilde{x}(t)$ in $\mathcal{P}_T(P, L)$ such that

$$x = Ax, \quad \widetilde{x} = \widetilde{A}\widetilde{x}.$$

Then we have

$$\|x - \widetilde{x}\| \le \|Ax - A\widetilde{x}\| + \|A\widetilde{x} - \widetilde{A}\widetilde{x}\| \le \Gamma\|x - \widetilde{x}\| + \|A\widetilde{x} - \widetilde{A}\widetilde{x}\|,$$

which implies

$$\|x - \widetilde{x}\| \le \frac{\|A\widetilde{x} - \widetilde{A}\widetilde{x}\|}{1 - \Gamma}. \tag{16}$$

Next, for $u \in [t, t+T]$, we note

$$\begin{aligned}
&|\Delta(t,u) - \widetilde{\Delta}(t,u)| \\
&\quad\le \frac{|e^{\int_u^t C_{1,1}(s)ds} - e^{\int_u^t \widetilde{C}_{1,1}(s)ds}|}{e^{-\int_0^T C_{1,1}(s)ds} - 1} \\
&\qquad + e^{\int_u^t \widetilde{C}_{1,1}(s)ds}\left|\frac{1}{e^{-\int_0^T C_{1,1}(s)ds} - 1} - \frac{1}{e^{-\int_0^T \widetilde{C}_{1,1}(s)ds} - 1}\right| \\
&\quad= \frac{e^{\int_u^t C_{1,1}(s)ds}|1 - e^{\int_u^t (\widetilde{C}_{1,1}(s) - C_{1,1}(s))ds}|}{e^{-\int_0^T C_{1,1}(s)ds} - 1}\left(1 + \frac{e^{\int_u^t \widetilde{C}_{1,1}(s)ds}}{e^{-\int_0^T \widetilde{C}_{1,1}(s)ds} - 1}\right) \\
&\quad\le (1 + \widetilde{M})MTe^{2TP_{1,1}}\|C_{1,1} - \widetilde{C}_{1,1}\|.
\end{aligned} \tag{17}$$

Then using (17), for $u \in [t, t+T]$, we have

$$\begin{aligned}
&\left| C_{l,m}(u)(\widetilde{x}^{[m]}(u))^l \Delta(t,u) - \widetilde{C}_{l,m}(u)(\widetilde{x}^{[m]}(u))^l \widetilde{\Delta}(t,u) \right| \\
&\qquad \le |C_{l,m}(u) - \widetilde{C}_{l,m}(u)||\widetilde{x}^{[m]}(u)|^l |\Delta(t,u)| + |\widetilde{C}_{l,m}(u)||\widetilde{x}^{[m]}(u)|^l |\Delta(t,u) - \widetilde{\Delta}(t,u)| \\
&\qquad \le (M + P_{l,m}(1+\widetilde{M})MTe^{2TP_{1,1}})P^l \|C_{l,m} - \widetilde{C}_{l,m}\|, \qquad (18)
\end{aligned}$$

and

$$\begin{aligned}
&\left| \widetilde{G}(u)\widetilde{\Delta}(t,u) - G(u)\Delta(t,u) \right| \\
&\qquad \le \Big(|\widetilde{G}(u) - G(u)||\widetilde{\Delta}(t,u)| + |G(u)||\widetilde{\Delta}(t,u) - \Delta(t,u)| \Big) \\
&\qquad \le \widetilde{M}\|\widetilde{G} - G\| + P_G(1+\widetilde{M})MTe^{2TP_{1,1}}\|C_{1,1} - \widetilde{C}_{1,1}\|. \qquad (19)
\end{aligned}$$

From (18) and (19), we have

$$\begin{aligned}
&|(A\widetilde{x})(t) - (\widetilde{A}\widetilde{x})(t)| \\
&\quad \le \sum_{l=2}^{\infty} \int_t^{t+T} \left| C_{l,1}(u)(\widetilde{x}(u))^l \Delta(t,u) - \widetilde{C}_{l,1}(u)(\widetilde{x}(u))^l \widetilde{\Delta}(t,u) \right| du \\
&\qquad + \sum_{m=2}^{k} \sum_{l=1}^{\infty} \int_t^{t+T} \left| C_{l,m}(u)(\widetilde{x}^{[m]}(u))^l \Delta(t,u) - \widetilde{C}_{l,m}(u)(\widetilde{x}^{[m]}(u))^l \widetilde{\Delta}(t,u) \right| du \\
&\qquad + \int_t^{t+T} \left| \widetilde{G}(u)\widetilde{\Delta}(t,u) - G(u)\Delta(t,u) \right| du \\
&\quad \le \Upsilon \max\left\{ \sup_{(m,l)\in\{1,\dots,k\}\times\{1,2,\dots\}} \|C_{l,m} - \widetilde{C}_{l,m}\|, \|\widetilde{G} - G\| \right\}
\end{aligned}$$

for

$$\begin{aligned}
\Upsilon = T\Bigg(&M \sum_{(m,l)\in\{1,\dots,k\}\times\{1,2,\dots\}\setminus\{(1,1)\}} (1 + P_{l,m}(1+\widetilde{M})Te^{2TP_{1,1}})P^l \\
&+ \widetilde{M} + P_G(1+\widetilde{M})MT^2 e^{2TP_{1,1}} \Bigg).
\end{aligned}$$

Consequently, by (16), we arrive at

$$\|x - \widetilde{x}\| \le \frac{\Upsilon}{1-\Gamma} \max\left\{ \sup_{(m,l)\in\{1,\dots,k\}\times\{1,2,\dots\}} \|C_{l,m} - \widetilde{C}_{l,m}\|, \|\widetilde{G} - G\| \right\}.$$

This completes the proof. □

4. Examples

Example 1. *First, we show that the conditions in Theorem 2 do not self-contradict. Consider the following equation:*

$$x'(t) = \sum_{l=1}^{\infty} \frac{\sin 12t - 2}{100^l}(x(t))^l + \sum_{l=1}^{\infty} \frac{\sin 12t}{100^l}(x^{[2]}(t))^l + \frac{1}{100}\sin 12t, \qquad (20)$$

where $C_{l,m}(t) = 0$ *for* $m \notin \{1,2\}$, $C_{l,1}(t) = \frac{\sin 12t-2}{100^l}, C_{l,2}(t) = \frac{\sin 12t}{100^l}$, *in particular,* $C_{1,1}(t) = \frac{\sin 12t-2}{100}$, $G(t) = \frac{1}{100}\sin 12t$. *Take* $T = \frac{\pi}{6}, P = 5, L = 1, P_{l,m} = 0$ *for* $m \geq 3$, $P_{l,1} = \frac{3}{100^l}, P_{l,2} = \frac{1}{100^l}, P_G = \frac{1}{100}$. *A simple calculation yields*

$$95.99 < M = \frac{1}{1 - e^{\int_0^T C_{1,1}(s)ds}} < 96, \quad 50.26 < MT < 50.27,$$

by D'Alembert's criterion, we know

$$\begin{aligned} &MT\Big(\sum_{l=2}^{\infty} lP_{l,1}P^{l-1} + \sum_{l=1}^{\infty}\sum_{j=0}^{1} lL^j P_{l,2}P^{l-1}\Big) < MT\left(\frac{1}{50} + \sum_{l=2}^{\infty}\frac{l}{20^l}\right) < \infty, \\ &MT\Big(\sum_{l=2}^{\infty} P_{l,1}P^l + \sum_{l=2}^{\infty} P_{l,2}P^l + P_G\Big) < 1.04 < 2.48 < (1 - MTP_{1,2})P, \\ &\Big(1 + \frac{TP_{1,1}e^{TP_{1,1}}}{e^{-\int_0^T C_{11}(s)ds} - 1}\Big)\Big(\sum_{l=2}^{\infty} P_{l,1}P^l + P_G + \sum_{l=1}^{\infty} P_{l,2}P^l\Big) < 0.18 < 1 = L; \end{aligned} \tag{21}$$

then (7), (10) and (11) are satisfied. By Theorem 2, Eq. (20) has a $\frac{\pi}{6}$*-periodic solution* f *such that* $|f(t)| \leq 5$, *and* $|f(t_2) - f(t_1)| \leq |t_2 - t_1|$, $\forall t_1, t_2 \in \mathbb{R}$.

Next, we apply Theorem 3.

Example 2. *If we take* $L = \frac{2}{5}$ *in Example 1, then using* $\frac{2}{5} < 1$ *and* $0.18 < \frac{2}{5}$, *computations of (21) give that (7), (10) and (11) are also satisfied for this case. Furthermore, we derive*

$$\begin{aligned} \Gamma &= MT\Big(\sum_{l=2}^{\infty} lP_{l,1}P^{l-1} + \sum_{l=1}^{\infty}\sum_{j=0}^{1} lL^j P_{l,2}P^{l-1}\Big) = MT\Big(\frac{3}{5}\sum_{l=2}^{\infty}\frac{l}{20^l} + \frac{7}{25}\sum_{l=1}^{\infty}\frac{l}{20^l}\Big) \\ &= MT\left(\frac{7}{500} + \frac{22}{25}\sum_{l=2}^{\infty}\frac{l}{20^l}\right) \leq MT\left(\frac{7}{500} + \frac{22}{25}\sum_{l=2}^{\infty}\left(\frac{3}{40}\right)^l\right) \\ &= \frac{179}{9250}MT < 0.973 < 1, \end{aligned}$$

since $\left(\frac{3}{2}\right)^l > l$ *for any* $l \geq 0$. *Consequently, condition (15) also holds. So the conditions for Theorem 4 are satisfied and thus Eq. (20) has a unique* $\frac{\pi}{6}$*-periodic solution* f *such that* $|f(t)| \leq 5$, $|f(t_2) - f(t_1)| \leq \frac{2}{5}|t_2 - t_1|$, $\forall t_1, t_2 \in \mathbb{R}$. *And this* $\frac{\pi}{6}$*-periodic solution depends continuously on the given functions* $C_{l,1}(t) = \frac{\sin 12t-2}{100^l}, C_{l,2}(t) = \frac{\sin 12t}{100^l}$ *and* $G(t) = \frac{\sin 12t}{100}$ *for* $l = 1, \ldots, \infty$.

We see that the solution of Example 2 satisfies the properties of a solution of Example 1 and we do not know if a solution of Example 1 is different from the one of Example 2. Thus, we discuss this in the next case.

Example 3. *We consider*

$$x'(t) = \sum_{l=1}^{\infty}\frac{\sin 12t - 2}{100^l}(x(t))^l + \sum_{l=1}^{\infty}\frac{\sin 12t}{100^l}(x^{[2]}(t))^l + \lambda\sin 12t, \tag{22}$$

when $\lambda > 0$ is a parameter. So $P_G = \lambda$. Next, we consider P and L as variables to be defined by λ. Then (7) is finite if and only if $P < 100$ and (10), (11) and (15) have the forms

$$MT\Big(\frac{P^2}{25(100-P)} + \lambda\Big) \leq \Big(1 - \frac{MT}{100}\Big)P, \tag{23}$$

$$\Big(1 + \frac{TP_{1,1}e^{TP_{1,1}}}{e^{-\int_0^T C_{11}(s)ds} - 1}\Big)\Big(\frac{P(3P+100)}{100(100-P)} + \lambda\Big) \leq L, \tag{24}$$

$$\Gamma = MT\Big(\frac{3(200-P)P}{100(100-P)^2} + \frac{100(1+L)}{(100-P)^2}\Big) < 1, \tag{25}$$

respectively. First, (23) is equivalent to

$$0 < \lambda \leq F(P) = P\Big(\frac{1}{MT} + \frac{4}{P-100} + \frac{3}{100}\Big) \doteq \frac{(0.0498956P - 0.989564)P}{P-100}. \tag{26}$$

We see that necessarily we need

$$0 < P < P_+ = \frac{100(100-MT)}{3MT+100} \doteq 19.8327.$$

Using Mathematica we get that the function $F(P)$ over the interval $[0, P_+]$ has a unique maximum $F(P_{max}) \doteq 0.0546313$ at $P_{max} \doteq 10.4638$ (see Figure 1).

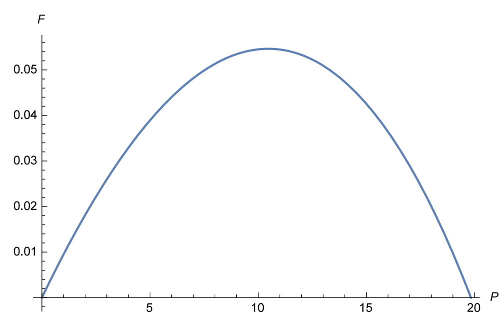

Figure 1: *The graph of $F(P)$ on $[0, P_+]$*

So in order to apply Theorem 2 to (22)*, we need to assume*

$$\lambda \leq F(P_{max}) \doteq 0.0546313. \tag{27}$$

When (27) *holds, then to find the smallest $P \in [0, P_+]$ satisfying* (26)*, we take P as the smallest (second) root $P(\lambda)$ of $\lambda = F(P)$ given by (see Figure 2)*

$$\begin{aligned} P(\lambda) &= \frac{50\left(M(\lambda T - T) - \sqrt{MT(((\lambda-14)\lambda+1)MT - 200(\lambda+1)) + 10000} + 100\right)}{3MT+100} \\ &\doteq 10.0209\lambda - 0.199373\sqrt{\lambda(2526.29\lambda - 45420.6) + 2473.84} + 9.91634. \end{aligned}$$

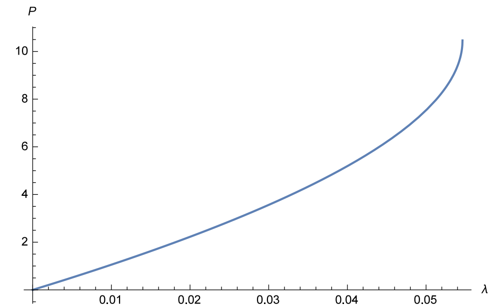

Figure 2: *The graph of* $P(\lambda)$ *on* $[0, F(P_{max})]$

Furthermore, the left-hand side of (24) *increases in* $P \in [0, P_+]$*, so the smallest* $L(\lambda)$ *satisfying* (24) *is as follows (see Figure 3)*

$$\begin{aligned} L(\lambda) &= \Big(1 + \frac{TP_{1,1}e^{TP_{1,1}}}{e^{-\int_0^T C_{11}(s)ds} - 1}\Big)\Big(\frac{P_\lambda(3P_\lambda + 100)}{100(100 - P_\lambda)} + \lambda\Big) \\ &\doteq 0.192837\lambda - 0.00383661\sqrt{\lambda(2526.29\lambda - 45420.6) + 2473.84} + 0.190824. \end{aligned}$$

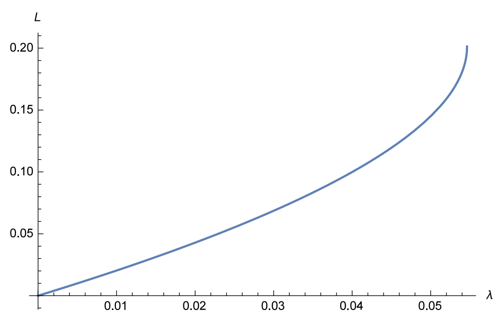

Figure 3: *The graph of* $L(\lambda)$ *on* $[0, F(P_{max})]$

Now, we see from (25) *that* Γ *increases with respect to* P *and* L*, so the smallest* $\Gamma(\lambda)$ *is determined by (see Figure 4)*

$$\begin{aligned} \Gamma(\lambda) &= MT\Big(\frac{3(200 - P(\lambda))P(\lambda)}{100(100 - P(\lambda))^2} + \frac{100(1 + L(\lambda))}{(100 - P(\lambda))^2}\Big) \\ &\doteq \Big(\lambda\Big(-7618.64\lambda + 151.578\sqrt{\lambda(2526.29\lambda - 45420.6) + 2473.84} + 161360\Big) \\ &\quad -1847.75\sqrt{\lambda(2526.29\lambda - 45420.6) + 2473.84} + 218351\Big) \\ &\quad \times \Big(-50.2623\lambda + \sqrt{\lambda(2526.29\lambda - 45420.6) + 2473.84} + 451.836\Big)^{-2}. \end{aligned}$$

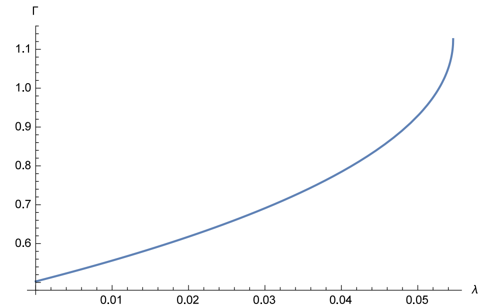

Figure 4: The graph of $\Gamma(\lambda)$ on $[0, F(P_{max})]$

By Figure 4, there is a unique solution $\lambda_1 \in [0, F(P_{max})]$ *of* $\Gamma(\lambda) = 1$ *with* $\lambda_1 \doteq 0.0527851$. *Summarizing we get:*

A: If $\lambda \in (0, \lambda_1)$, *then Theorem 3 gives a unique solution of* (22) *in* $\mathcal{P}_{\frac{\pi}{6}}(P(\lambda), L(\lambda))$.

B: If $\lambda \in [\lambda_1, F(P_{max})]$, *then Theorem 2 gives a solution of* (22) *in* $\mathcal{P}_{\frac{\pi}{6}}(P(\lambda), L(\lambda))$.

Applying this result to (20), *we have the case A with* $P(0.01) \doteq 1.05611$, $L(0.01) \doteq 0.0203232$ *and* $\Gamma(0.01) \doteq 0.556205 < 1$ *improving estimates of Examples 1 and 2.*

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