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Well-posedness for generalized mixed vector variational-like inequality problems in Banach space

Anurag Jayswal and Shalini Jha*

Department of Applied Mathematics, Indian Institute of Technology (ISM), Dhanbad-826 004, Jharkhand, India

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Abstract. In this article, we focus on studying about well-posedness of a generalized mixed vector variational-like inequality and optimization problems with the aforesaid inequality as a constraint. We establish a metric characterization of well-posedness in terms of an approximate solution set. Thereafter we prove sufficient conditions of generalized well-posedness by assuming the boundedness of an approximate solution set. We also prove that well-posedness of the considered optimization problems is closely related to that of generalized mixed vector variational-like inequality problems. Moreover, we present some examples to investigate the results established in this paper.

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Key words: generalized mixed vector variational-like inequality problems, well-posedness, relaxed η-α-P-monotonicity

1. Introduction

In the literature, there are several methods to solve an optimization problem. However, in many cases it is very costly to compute the solution using these methods and moreover, they may or may not guarantee that the solutions are exact. In such cases, well-posedness has a significant role which ensures the convergence of sequence of approximate solutions obtained through iterative techniques to the exact solution of the problem. The classical concept of well-posedness for a global minimization problem, which was first introduced by Tykhonov [18], requires the existence and uniqueness of minimizer and convergence of every minimizing sequence towards the unique minimizer. Thereafter, various authors generalized these concepts to problems with many minimizers (see, for example, [12, 13, 22]).

Variational inequality problems via set-valued mappings have been studied by several researchers (see, for example, [7, 11, 19]). Since variational inequality problems are closely related to mathematical programming problems under some mild conditions, consequently the concept of Tykhonov well-posedness has also been generalized to variational inequalities [1, 5, 8, 10, 12, 20] and thereafter to several other problems like equilibrium problems [6], fixed point problems [8], optimization

*Corresponding author. *Email addresses:* anurag_jais123@yahoo.com (A. Jayswal), jhashalini.rash89@gmail.com (S. Jha)

problems with equilibrium, variational inequality and mixed quasi variational-like inequality constraints [6, 13, 15].

Lignola and Morgan [15] proposed parametric well-posedness for optimization problems with variational inequality constraints by defining the approximating sequence. Moreover, in [14] Lignola discussed well-posedness, L-well-posednes and metric characterizations of well-posedness for quasi variational-inequality problems. Ceng and Yao [3] extended these concepts to derive the conditions under which the generalized mixed variational inequality problems are well-posed. Thereafter, Lin and Chuang [16] established well-posedness for variational inclusion, variational disclusion problems and optimization problems with variational inclusion, disclusion, and scalar equilibrium constraints in a generalized sense.

Recently, Fang et al. [9] extended the notion of well-posedness by perturbations to a mixed variational inequality problem in a Banach space. Very recently, Ceng et al. [2] explored the conditions of well-posedness for hemivariational inequality problems involving Clarke's generalized directional derivative under different types of monotonicity assumptions. Inspired by earlier research, we study well-posedness for generalized mixed vector variational-like inequality problems and constrained optimization problems involving a relaxed η-α-P-monotone operator.

Our paper is organized as follows: we recall some definitions and results useful in proving the main results in the paper in Section 2. In Section 3, we propose definitions of well-posedness, the approximating solution set and establish the metric characterizations of well-posedness for the generalized mixed vector variational-like inequality problems. Further, under suitable conditions we show that well-posedness is characterized in terms of the existence and uniqueness of solutions. We also establish these results in a generalized sense and establish sufficient conditions for generalized well-posedness in terms of boundedness of approximate solution sets. Thereafter, in Section 4, we establish well-posedness for optimization problems with generalized mixed vector variational-like inequality constraints. Moreover, we discuss several cases of well-posedness of optimization problems by assuming well-posedness of constraints in Section 5. Finally, we conclude our paper in Section 6.

2. Preliminaries

Suppose X and Y are two real Banach spaces. Let $K \subset X$ be a nonempty closed convex subset of X and $P \subset Y$ a closed convex proper cone with nonempty interior.

Throughout this paper, we shall use the following inequalities for all $x, y \in Y$:

(i) $x \leq_P y \Leftrightarrow y - x \in P$;

(ii) $x \nleq_P y \Leftrightarrow y - x \notin P$;

(iii) $x \leq_{P^o} y \Leftrightarrow y - x \in P^o$;

where P^o denotes the interior of P.

If "$\leq_P$" is a partial order, then $(Y, \leq_P)$ is called an ordered Banach space ordered by P.

Let $L(X, Y)$ denotes the space of all continuous linear mappings from X into Y and let $T : X \mapsto 2^{L(X,Y)}$ be a set-valued mapping. Suppose $A : L(X, Y) \mapsto L(X, Y)$

is a mapping and $f : K \times K \mapsto Y$, $\eta : X \times X \mapsto X$ are two bimappings. We consider the following generalized mixed vector variational-like inequality problem (GMVVLIP) [19]:

Find $x \in K$ and $u \in T(x)$ such that

$$\langle Au, \eta(y,x)\rangle + f(x,y) \nleq_{P^o} 0, \quad \forall y \in K.$$

Let $S = \{x \in K : \exists\ u \in T(x)$ such that $\langle Au, \eta(y,x)\rangle + f(x,y) \nleq_{P^o} 0,\ \forall y \in K\}$ denote the solution set of (GMVVLIP).

Now, we recall the following definitions and lemmas which will be used in the sequel of the paper.

Definition 1 (see [19]). *A mapping $\psi : K \mapsto Y$ is said to be P-convex if*

$$\psi(\mu x + (1-\mu)y) \leq_P \mu\psi(x) + (1-\mu)\psi(y), \quad \forall x, y \in K,\ \mu \in [0,1].$$

Definition 2. *A mapping $\psi : K \mapsto Y$ is said to be P-concave if*

$$\psi(\mu x + (1-\mu)y) \geq_P \mu\psi(x) + (1-\mu)\psi(y), \quad \forall x, y \in K,\ \mu \in [0,1].$$

Definition 3 (see [21]). *A set-valued mapping $T : K \mapsto 2^{L(X,Y)}$ is said to be monotone with respect to A if, for any $x, y \in K$,*

$$\langle Au - Av, x - y\rangle \geq_P 0, \quad \forall u \in T(x),\ v \in T(y).$$

Definition 4 (see [19]). *A set-valued mapping $T : K \mapsto 2^{L(X,Y)}$ is said to be relaxed η-α-P-monotone with respect to A if, for any $x, y \in K$,*

$$\langle Au - Av, \eta(x,y)\rangle - \alpha(x-y) \geq_P 0, \quad \forall\ u \in T(x),\ v \in T(y),$$

where $\alpha : X \mapsto Y$ is a mapping such that $\alpha(tz) = t^p\alpha(z), \forall t > 0,\ z \in X$ and $p > 1$ is constant.

Definition 5 (see [19]). *A bimapping $\zeta : X \times X \mapsto X$ is said to be affine with respect to the first argument if, for any $x_i \in K$ and $\lambda_i \geq 0,\ (1 \leq i \leq n)$ with $\sum_{i=1}^{n} \lambda_i = 1$ and for any $y \in K$,*

$$\zeta(\sum_{i=1}^{n} \lambda_i x_i, y) = \sum_{i=1}^{n} \lambda_i \zeta(x_i, y).$$

Lemma 1 (see [4]). *Let (Y,P) be an ordered Banach space with closed convex and pointed cone P and $P^o \neq \phi$. Then for all $x, y, z \in Y$, we have*

(i) $z \nleq_{P^o} x, \quad x \geq_P y \Rightarrow z \nleq_{P^o} y$;

(ii) $z \ngeq_{P^o} x, \quad x \leq_P y \Rightarrow z \ngeq_{P^o} y$.

Lemma 2 (see [17]). *Let $(X, \|.\|)$ be a normed linear space and H a Hausdorff metric on the collection $CB(X)$ of all nonempty, closed and bounded subsets of X induced by metric $d(u,v) = \|u - v\|$, which is defined by*

$$H(U,V) = \max\left\{\sup_{u\in U}\inf_{v\in V}\|u-v\|, \sup_{v\in V}\inf_{u\in U}\|u-v\|\right\}$$

for U and V in $CB(X)$. If U, V are compact sets in X, then for each $u \in U$ there exists $v \in V$ such that

$$\|u - v\| \leq H(U, V).$$

Definition 6. *A set-valued mapping $T : K \mapsto 2^{L(X,Y)}$ is said to be H-hemicontinuous if, for any $x, y \in K$ and $\nu \in (0,1)$, we have*

$$H(T(x + \nu(y - x)), T(x)) \to 0 \quad as\ \nu \to 0^+,$$

where H is the Hausdorff metric defined on $CB(L(X,Y))$ the closed and bounded subset of $L(X,Y)$.

Lemma 3 (see [19])**.** *Let K be closed and convex subset of a real Banach space X, Y be a real Banach space ordered by a nonempty closed convex pointed cone P with apex at the origin and $P^o \neq \phi$. Further, assume that $A : L(X,Y) \mapsto L(X,Y)$ is a continuous mapping and $T : K \mapsto 2^{L(X,Y)}$ be a nonempty compact set-valued mapping. Suppose that the following conditions hold:*

(*i*) *$f : K \times K \mapsto Y$ is a P-convex in the second argument with $f(x,x) = 0, \forall x \in K$;*

(*ii*) *$\eta : X \times X \mapsto X$ is an affine mapping in the first argument with $\eta(x,x) = 0$, $\forall x \in K$;*

(*iii*) *$T : K \mapsto 2^{L(X,Y)}$ is H-hemicontinuous and relaxed η-α-P-monotone with respect to A; then the following two problems are equivalent:*

(*a*) *there exist $x_o \in K$ and $u_o \in T(x_o)$ such that*

$$\langle Au_o, \eta(y, x_o)\rangle + f(x_o, y) \nleq_{P^o} 0, \quad \forall y \in K,$$

(*b*) *there exists $x_o \in K$ such that*

$$\langle Av, \eta(y, x_o)\rangle + f(x_o, y) - \alpha(y - x_o) \nleq_{P^o} 0, \quad \forall y \in K,\ v \in T(y).$$

3. Well-posedness of generalized mixed vector variational-like inequality problems via relaxed η-α-P-monotonicity

Lalita and Bhatia [12] established the results on well-posedness for variational inequality problems by defining generalized monotone operators. In this section, we establish well-posedness for generalized mixed vector variational-like inequality problems by means of the relaxed η-α-P-monotone operator. For this purpose, we will present the following definitions.

Definition 7. *A sequence $\{x_n\} \in K$ is said to be an approximating sequence for (GMVVLIP) if, there exist $u_n \in T(x_n)$ and a sequence of positive real numbers $\epsilon_n \to 0$ such that*

$$\langle Au_n, \eta(y, x_n)\rangle + f(x_n, y) + \epsilon_n e \nleq_{P^o} 0, \quad \forall y \in K,\ e \in int\ P.$$

Definition 8. *The generalized mixed vector variational-like inequality problem (GMVVLIP) is said to be well-posed if the following conditions hold:*

(i) *there exists a unique solution x_0 of (GMVVLIP),*

(ii) *every approximating sequence of (GMVVLIP) converges to x_0.*

Definition 9. *The generalized mixed vector variational-like inequality problem (GMVVLIP) is said to be well-posed in the generalized sense if the following conditions hold:*

(i) *the solution set S of (GMVVLIP) is nonempty,*

(ii) *every approximating sequence has a subsequence that converges to some point of S.*

To investigate well-posedness of (GMVVLIP), we consider the approximating solution set of (GMVVLIP) defined as:

$$S_\epsilon = \{x \in K : \exists\, u \in T(x) \text{ such that } \langle Au, \eta(y,x)\rangle + f(x,y) + \epsilon e \nleq_{P^o} 0,\ \forall y \in K,\ \epsilon \geq 0\}.$$

Remark 1. *It is obvious that $S = S_\epsilon$, when $\epsilon = 0$ and $S \subseteq S_\epsilon,\ \forall \epsilon > 0$.*

The diameter of the set A is denoted by diam A defined as

$$\text{diam } A = \sup_{a,b\in A} \|a - b\|.$$

Now, we present the metric characterization of well-posedness of (GMVVLIP) in terms of the approximating solution set.

Theorem 1. *Suppose all assumptions of Lemma 3 hold and $f(.,y),\ \eta(y,.)$ and α are continuous functions for all $y \in K$. Then (GMVVLIP) is well-posed if and only if*

$$S_\epsilon \neq \phi, \forall\ \epsilon > 0 \text{ and diam } S_\epsilon \to 0 \text{ as } \epsilon \to 0.$$

Proof. Suppose (GMVVLIP) is well-posed. Then it has a unique solution $x_0 \in S$. Since $S \subseteq S_\epsilon,\ \forall \epsilon > 0$, therefore, $S_\epsilon \neq \phi,\ \forall \epsilon > 0$. Suppose, contrary to the result, that diam $S_\epsilon \nrightarrow 0$ as $\epsilon \to 0$. Then there exist $r > 0$, a positive integer $m,\ \epsilon_n > 0$ with $\epsilon_n \to 0$ and $x_n,\ x'_n \in S_{\epsilon_n}$ such that

$$\|x_n - x'_n\| > r, \quad \forall n \geq m. \tag{1}$$

As $x_n, x'_n \in S_{\epsilon_n}$, hence there exist $u_n \in T(x_n)$ and $u'_n \in T(x'_n)$ such that

$$\langle Au_n, \eta(y,x_n)\rangle + f(x_n,y) + \epsilon_n e \nleq_{P^o} 0, \quad \forall y \in K,$$
$$\langle Au'_n, \eta(y,x'_n)\rangle + f(x'_n,y) + \epsilon_n e \nleq_{P^o} 0, \quad \forall y \in K.$$

Clearly, $\{x_n\}$ and $\{x'_n\}$ are approximating sequences of (GMVVLIP) which converge to x_0 because the problem is well-posed. Now,

$$\|x_n - x'_n\| = \|x_n - x_0 + x_0 - x'_n\| \leq \|x_n - x_0\| + \|x_0 - x'_n\| \leq \epsilon,$$

which contradicts (1), for some $\epsilon = r$.

Conversely, suppose $\{x_n\}$ is an approximating sequence of (GMVVLIP). Then there exist $u_n \in T(x_n)$ and a sequence of positive real numbers $\epsilon_n \to 0$ such that

$$\langle Au_n, \eta(y, x_n)\rangle + f(x_n, y) + \epsilon_n e \nleq_{P^o} 0, \quad \forall y \in K, \tag{2}$$

which implies that $x_n \in S_{\epsilon_n}$. Since diam $S_{\epsilon_n} \to 0$ as $\epsilon_n \to 0$, therefore $\{x_n\}$ is a Cauchy sequence which converges to some $x_0 \in K$ because K is closed. Now, T is relaxed η-α-P-monotone with respect to A on K; therefore by Definition 4, for any $y \in K$ and $u \in T(y)$, we have

$$\langle Au_n, \eta(y, x_n)\rangle + f(x_n, y) \leq_P \langle Au, \eta(y, x_n)\rangle + f(x_n, y) - \alpha(y - x_n). \tag{3}$$

By the continuity of $f,\ \eta$ and α, we have

$$\langle Au, \eta(y, x_0)\rangle + f(x_0, y) - \alpha(y - x_0) = \lim_{n\to\infty}\{\langle Au, \eta(y, x_n)\rangle + f(x_n, y) - \alpha(y - x_n)\},$$

which by using (3), yields

$$\langle Au, \eta(y, x_0)\rangle + f(x_0, y) - \alpha(y - x_0) \geq_P \lim_{n\to\infty}\{\langle Au_n, \eta(y, x_n)\rangle + f(x_n, y)\}. \tag{4}$$

Taking the limit in inequality (2), we have

$$\lim_{n\to\infty}\{\langle Au_n, \eta(y, x_n)\rangle + f(x_n, y)\} \nleq_{P^o} 0. \tag{5}$$

By combining (4) and (5) and using Lemma 1(ii), we obtain

$$\langle Au, \eta(y, x_0)\rangle + f(x_0, y) - \alpha(y - x_0) \nleq_{P^o} 0.$$

Thus, by Lemma 3, there exist $x_0 \in K$ and $u_0 \in T(x_0)$ such that

$$\langle Au_0, \eta(y, x_0)\rangle + f(x_0, y) \nleq_{P^o} 0, \quad \forall y \in K,$$

which implies that $x_0 \in S$. It remains to prove that x_0 is a unique solution of the problem (GMVVLIP). Contrarily, suppose x_1 and x_2 are two distinct solutions of (GMVVLIP). Then

$$0 < \|x_1 - x_2\| \leq \text{diam } S_\epsilon \to 0 \text{ as } \epsilon \to 0,$$

which is absurd. This completes the proof of the theorem. □

Corollary 1. *Suppose all the assumptions of Lemma 3 hold and $f(.,y),\ \eta(y,.)$ and α are continuous functions for all $y \in K$. Then the problem (GMVVLIP) is well-posed if and only if*

$$S \neq \phi \text{ and diam } S_\epsilon \to 0 \text{ as } \epsilon \to 0.$$

Proof. The proof follows lines similar to Theorem 1, hence it is omitted. □

In the next theorem, we show that well-posedness of (GMVVLIP) is equivalent to the existence and uniqueness of the solution.

Theorem 2. *Let K be a closed and convex subset of a real Banach space X, Y a real Banach space ordered by a nonempty closed convex pointed cone P with the apex at the origin and $P^o \neq \phi$. Further, assume that $A : L(X,Y) \mapsto L(X,Y)$ is a continuous mapping and let $T : K \mapsto 2^{L(X,Y)}$ be a nonempty compact set-valued mapping. Suppose that the following conditions hold:*

(i) $f : K \times K \mapsto Y$ *is P-convex in the second argument and P-concave in the first argument with* $f(x,x) = 0, \forall x \in K$;

(ii) $\eta : X \times X \mapsto X$ *is an affine mapping in the first and second argument with* $\eta(x,x) = 0,\ \forall x \in K$;

(iii) $T : K \mapsto 2^{L(X,Y)}$ *is H-hemicontinuous and relaxed η-α-P-monotone with respect to A;*

(iv) $f(.,y),\ \eta(y,.)$ *and α are continuous functions for all $y \in K$.*

Then (GMVVLIP) is well-posed if and only if it has a unique solution.

Proof. Suppose (GMVVLIP) is well-posed; then it has a unique solution. Conversely, let (GMVVLIP) have a unique solution x_0. Suppose, contrary to the result, that (GMVVLIP) is not well-posed. Then there exists an approximating sequence $\{x_n\}$ of (GMVVLIP) which does not converge to x_0. Since $\{x_n\}$ is an approximating sequence, then there exist $u_n \in T(x_n)$ and a sequence of positive real numbers $\epsilon_n \to 0$ such that

$$\langle Au_n, \eta(y, x_n)\rangle + f(x_n, y) + \epsilon_n e \nleq_{P^o} 0, \quad \forall y \in K. \tag{6}$$

Now, we show that $\{x_n\}$ is bounded. Suppose $\{x_n\}$ is not bounded. Then, without loss of generality, we can suppose that $\|x_n\| \to +\infty$ as $n \to +\infty$. Let $t_n = \frac{1}{\|x_n - x_0\|}$ and $w_n = x_0 + t_n(x_n - x_0)$. Without loss of generality, we can assume that $t_n \in (0,1)$ and $w_n \to w \neq x_0$.

By the hypothesis, T is relaxed η-α-P-monotone with respect to A; therefore, for any $x, y \in K$, we have

$$\langle Au - Au_0, \eta(y, x_0)\rangle - \alpha(y - x_0) \geq_P 0, \quad \forall u_0 \in T(x_0),\ u \in T(y),$$

which implies that

$$\langle Au_0, \eta(y, x_0)\rangle + f(x_0, y) \leq_P \langle Au, \eta(y, x_0)\rangle + f(x_0, y) - \alpha(y - x_0). \tag{7}$$

Since x_0 is a solution of (GMVVLIP), there exists $u_0 \in T(x_0)$ such that

$$\langle Au_0, \eta(y, x_0)\rangle + f(x_0, y) \nleq_{P^o} 0, \quad \forall y \in K. \tag{8}$$

By combining (7), (8) and using Lemma 1(ii), we get

$$\langle Au, \eta(y, x_0)\rangle + f(x_0, y) - \alpha(y - x_0) \nleq_{P^o} 0. \tag{9}$$

Further, f, η and α are continuous; therefore, we have

$$\langle Au, \eta(y, w)\rangle + f(w, y) - \alpha(y - w) = \lim_{n\to\infty} \{Au, \eta(y, w_n) + f(w_n, y) - \alpha(y - w_n)\}.$$

As η is affine in the second argument, f is P-concave in the first argument and using $w_n = x_0 + t_n(x_n - x_0)$, the above equation can be rewritten as

$$\langle Au, \eta(y, w)\rangle + f(w, y) - \alpha(y - w) \geq_P \langle Au, \eta(y, x_0)\rangle + f(x_0, y) - \alpha(y - x_0). \quad (10)$$

Using (9), (10) and Lemma 1(ii), we obtain

$$\langle Au, \eta(y, w)\rangle + f(w, y) - \alpha(y - w) \nleq_{P^o} 0.$$

Therefore, by Lemma 3, there exist $w \in K$ and $w_0 \in T(w)$ such that

$$\langle Aw_0, \eta(y, w)\rangle + f(w, y) \nleq_{P^o} 0, \quad \forall y \in K.$$

The above inequality implies that w is also a solution to (GMVVLIP), which contradicts the uniqueness of x_0. Hence, $\{x_n\}$ is a bounded sequence having a convergent subsequence $\{x_{n_k}\}$ converging to $\bar{x}$ (say) as $k \to \infty$. Again, from the definition of relaxed η-α-P-monotonicity, for any $x_{n_k}, y \in K$, we have

$$\langle Au - Au_{n_k}, \eta(y, x_{n_k})\rangle - \alpha(y - x_{n_k}) \geq_P 0, \quad \forall u_{n_k} \in T(x_{n_k}),\ u \in T(y).$$

The above inequality yields

$$\langle Au_{n_k}, \eta(y, x_{n_k})\rangle + f(x_{n_k}, y) \leq_P \langle Au, \eta(y, x_{n_k})\rangle + f(x_{n_k}, y) - \alpha(y - x_{n_k}). \quad (11)$$

Also, by the continuity of f, η and α, we have

$$\langle Au, \eta(y, \bar{x})\rangle + f(\bar{x}, y) - \alpha(y - \bar{x}) = \lim_{k\to\infty} \{\langle Au, \eta(y, x_{n_k})\rangle + f(x_{n_k}, y) - \alpha(y - x_{n_k})\},$$

which by using (11), becomes

$$\langle Au, \eta(y, \bar{x})\rangle + f(\bar{x}, y) - \alpha(y - \bar{x}) \geq_P \lim_{k\to\infty} \{\langle Au_{n_k}, \eta(y, x_{n_k})\rangle + f(x_{n_k}, y)\}. \quad (12)$$

On the behalf of (6), we can also write

$$\lim_{k\to\infty} \{\langle Au_{n_k}, \eta(y, x_{n_k})\rangle + f(x_{n_k}, y)\} \nleq_{P^o} 0. \quad (13)$$

From (12), (13) and Lemma 1(ii), we get

$$\langle Au, \eta(y, \bar{x})\rangle + f(\bar{x}, y) - \alpha(y - \bar{x}) \nleq_{P^o} 0.$$

Thus, by Lemma 3, there exist $\bar{x} \in K$ and $\bar{u} \in T(\bar{x})$ such that

$$\langle A\bar{u}, \eta(y, \bar{x})\rangle + f(\bar{x}, y) \nleq_{P^o} 0,$$

which shows that $\bar{x}$ is a solution to (GMVVLIP). Hence, $x_{n_k} \to \bar{x}$, i.e., $x_{n_k} \to x_0$ which implies that $x_n \to x_0$. This completes the proof of the theorem. □

Now, we give an example to verify Theorem 2.

Example 1. *Let $X = Y = \mathbb{R}$, $K = [0,1]$ and $P = [0,\infty)$. Let us define the mappings $T : K \mapsto 2^{L(X,Y)}$, $f : K \times K \mapsto Y$, $\eta : X \times X \mapsto X$, and $A : L(X,Y) \mapsto L(X,Y)$ as follows:*

$$\begin{aligned} T(x) &= \{u : \mathbb{R} \mapsto \mathbb{R} |\ u \textit{ is a continuous linear mapping such that } u(x) = -x\},\\ f(x,y) &= y - x,\\ \eta(x,y) &= \frac{1}{2}(y-x),\\ Av &= v, \quad \textit{and}\\ \alpha &= -x^2. \end{aligned}$$

(GMVVLIP1): Find $x \in K$ and $u \in T(x)$ such that

$$\langle u, \frac{1}{2}(x-y)\rangle + y - x \nleq_{P^o} 0, \quad \forall y \in K.$$

Clearly, $S = \{0\}$. It can be easily verified that T is relaxed η-α-P-monotone with respect to A and all conditions of Theorem 2 hold; therefore, (GMVVLIP1) is well-posed.

The following theorem shows that generalized well-posedness of (GMVVLIP) is equivalent to the non-emptiness of its solution set.

Theorem 3. *Suppose all assumptions of Lemma 3 hold. Further, assume that K is a compact set and $f(.,y)$, $\eta(y,.)$, α are continuous functions for all $y \in K$. Then (GMVVLIP) is well-posed in a generalized sense if and only if the solution set S is nonempty.*

Proof. Suppose (GMVVLIP) is well-posed. Then its solution set S is nonempty. Conversely, let $\{x_n\}$ be an approximating sequence of (GMVVLIP). Then there exist $u_n \in T(x_n)$ and a sequence of positive real numbers $\epsilon_n \to 0$ such that

$$\langle Au_n, \eta(y,x_n)\rangle + f(x_n,y) + \epsilon_n e \nleq_{P^o} 0, \quad \forall y \in K. \tag{14}$$

By the hypothesis, K is compact; therefore, $\{x_n\}$ has a subsequence $\{x_{n_k}\}$ converging to some point $x_0 \in K$. Since T is relaxed η-α-P-monotone with respect to A, by Definition 4, for any $x_{n_k}, y \in K$, we have

$$\langle Au - Au_{n_k}, \eta(y, x_{n_k})\rangle - \alpha(y - x_{n_k}) \geq_P 0, \quad \forall u_{n_k} \in T(x_{n_k}),\ u \in T(y),$$

which implies

$$\lim_{k\to\infty}\{\langle Au, \eta(y,x_{n_k})\rangle + f(x_{n_k},y) - \alpha(y - x_{n_k})\} \geq_P \lim_{k\to\infty}\{\langle Au_{n_k}, \eta(y,x_{n_k})\rangle + f(x_{n_k},y)\}.$$

By the hypothesis, η, f, α are continuous; therefore

$$\langle Au, \eta(y,x_0)\rangle + f(x_0,y) - \alpha(y-x_0) = \lim_{k\to\infty}\{\langle Au, \eta(y,x_{n_k})\rangle + f(x_{n_k},y) - \alpha(y - x_{n_k})\}.$$

Using the above inequality, we get

$$\langle Au, \eta(y,x_0)\rangle + f(x_0,y) - \alpha(y-x_0) \geq_P \lim_{k\to\infty}\{\langle Au_{n_k}, \eta(y,x_{n_k})\rangle + f(x_{n_k},y)\}. \tag{15}$$

On the behalf of (14), we can also write

$$\lim_{k\to\infty}\{\langle Au_{n_k},\eta(y,x_{n_k})\rangle + f(x_{n_k},y)\} \nleq_{P^o} 0. \tag{16}$$

By combining (15), (16) and using Lemma 1(ii), we get

$$\langle Au,\eta(y,x_0)\rangle + f(x_0,y) - \alpha(y-x_0) \nleq_{P^o} 0.$$

Thus, by Lemma 3, there exist $x_0 \in K$ and $u_0 \in T(x_0)$ such that

$$\langle Au_0,\eta(y,x_0)\rangle + f(x_0,y) \nleq_{P^o} 0,$$

which shows that $x_0 \in S$. This completes the proof of the theorem. □

Now, we give an example to illustrate the results established in Theorem 3.

Example 2. *Let $X = Y = \mathbb{R}^2$, $K = [0,1]\times[0,1]$ and $P = [0,\infty)\times[0,\infty)$. Let us define the mappings $T : K \mapsto 2^{L(X,Y)}$, $f : K\times K \mapsto Y$, $\eta : X\times X \mapsto X$, and $A : L(X,Y) \mapsto L(X,Y)$ as follows:*

$$\begin{aligned} T(x) &= \big\{w,z:\mathbb{R}^2\mapsto\mathbb{R}|\ w,z \ \text{ are continuous linear mappings such that} \\ &\qquad w(x_1,x_2)=x_1, z(x_1,x_2)=x_2\big\}, \\ f(x,y) &= y-x, \\ \eta(x,y) &= y-x, \\ Av &= -v, \quad \text{and} \\ \alpha &= 0. \end{aligned}$$

(GMVVLIP2): Find $x\in K$ and $u \in T(x)$ such that

$$\langle -u, x-y\rangle + y - x \nleq_{P^o} 0, \quad \forall y \in K.$$

Clearly, $S = [0,1]\times[0,1]$. It can be easily verified that T is relaxed η-α-P-monotone with respect to A and all conditions of Theorem 3 hold and $S \neq \phi$. Therefore, (GMVVLIP2) is well-posed in a generalized sense.

Now, we present sufficient conditions for well-posedness of (GMVVLIP) in a generalized sense.

Theorem 4. *Suppose all assumptions of Lemma 3 hold and $f(.,y)$, $\eta(y,.)$, α are continuous functions for all $y \in K$. If there exists some $\epsilon > 0$ such that $S_\epsilon \neq \phi$ and is bounded, then (GMVVLIP) is well-posed in a generalized sense.*

Proof. Let $\epsilon > 0$ such that $S_\epsilon \neq \phi$ and suppose $\{x_n\}$ is an approximating sequence of (GMVVLIP). Then there exist $u_n \in T(x_n)$ and a sequence of positive real numbers $\epsilon_n \to 0$ such that

$$\langle Au_n,\eta(y,x_n)\rangle + f(x_n,y) + \epsilon_n e \nleq_{P^o} 0, \quad \forall y \in K,$$

which implies that $x_n \in S_\epsilon$, $\forall n > m$. Therefore, $\{x_n\}$ is a bounded sequence that has a convergent subsequence $\{x_{n_k}\}$ converging to x_0 as $k\to\infty$. Following lines similar to the proof of Theorem 3, we get $x_0 \in S$. This completes the proof of the theorem. □

4. Well-posedness of optimization problems with generalized mixed vector variational-like inequality constraints (PGMVVLIC)

In this section, we study well-posedness of optimization problems and discuss the cases when the problem has either a unique solution or more than one solution.

Consider the optimization problem with generalized mixed vector variational-like inequality constraints:

$$\textbf{(PGMVVLIC)} \qquad \begin{array}{l} P-\text{minimize } \Phi(x) \\ \text{subject to } x \in S, \end{array}$$

where $\Phi : K \mapsto R$ and S is the solution set of (GMVVLIP).

Let ξ denote the solution set of (PGMVVLIC), i.e.,

$$\xi = \Big\{x \in K \mid \exists\; u \in T(x) \text{ such that } \Phi(x) \leq_P \inf_{y\in S} \Phi(y) \text{ and } \langle Au, \eta(y,x)\rangle + f(x,y) \nleq_{P^o} 0,\; \forall y \in K\Big\}.$$

Definition 10. *A sequence $\{x_n\} \in K$ is said to be an approximating sequence for (PGMVVLIC) if*

(*i*) $\lim_{n\to\infty} \sup\; \Phi(x_n) \leq_P \inf_{y\in S} \Phi(y)$,

(*ii*) *there exist $u_n \in T(x_n)$ and a sequence of positive real numbers $\epsilon_n \to 0$ such that*
$$\langle Au_n, \eta(y,x_n)\rangle + f(x_n,y) + \epsilon_n e \nleq_{P^o} 0, \quad \forall y \in K.$$

For $\delta, \epsilon \geq 0$, we denote the approximating solution set of (PGMVVLIC) by $\xi(\delta,\epsilon)$, i.e.,

$$\xi(\delta,\epsilon) = \Big\{x \in K \mid \exists\; u \in T(x) \text{ such that } \Phi(x) \leq_P \inf_{y\in S} \Phi(y) + \delta \text{ and } \langle Au, \eta(y,x)\rangle + f(x,y) + \epsilon e \nleq_{P^o} 0,\; \forall y \in K\Big\}.$$

Remark 2. *It is obvious that $\xi = \xi(\delta,\epsilon)$ when $(\delta,\epsilon) = (0,0)$ and $\xi \subseteq \xi(\delta,\epsilon)$, $\forall \delta, \epsilon > 0$.*

Now, we present the metric characterization of well-posedness of (PGMVVLIC) in terms of the approximating solution set.

Theorem 5. *Suppose all assumptions of Theorem 1 hold and Φ is lower semicontinuous. Then (PGMVVLIC) is well-posed if and only if*

$$\xi(\delta,\epsilon) \neq \phi, \forall\; \delta,\epsilon > 0 \textit{ and diam } \xi(\delta,\epsilon) \to 0 \textit{ as } (\delta,\epsilon) \to (0,0).$$

Proof. The necessary part directly follows from the proof of Theorem 1, so it is omitted. Conversely, suppose $\{x_n\}$ is an approximating sequence of (PGMVVLIC).

Then there exist $u_n \in T(x_n)$ and a sequence of positive real number $\epsilon_n \to 0$ such that

$$\lim_{n\to\infty} \sup \Phi(x_n) \ \leq_P \ \inf_{y\in S} \Phi(y), \tag{17}$$

$$\langle Au_n, \eta(y, x_n)\rangle + f(x_n, y) + \epsilon_n e \not\leq_{P^\circ} 0, \quad \forall y \in K, \tag{18}$$

which implies that $x_n \in \xi(\delta_n, \epsilon_n)$, for some $\delta_n \to 0$. Since diam $\xi(\delta, \epsilon) \to 0$ as $(\delta, \epsilon) \to (0, 0)$, $\{x_n\}$ is a Cauchy sequence converging to $x_0 \in K$ because K is closed. By the same argument as in Theorem 1, we get

$$\langle Au_0, \eta(y, x_0)\rangle + f(x_0, y) \not\leq_{P^\circ} 0, \forall u_0 \in T(x_0), y \in K. \tag{19}$$

Since Φ is lower semicontinuous,

$$\Phi(x_0) \leq_P \lim_{n\to\infty} \inf \Phi(x_n) \leq_P \lim_{n\to\infty} \sup \Phi(x_n).$$

By using (17), the above inequality reduces to

$$\Phi(x_0) \leq_P \inf_{y\in S} \Phi(y). \tag{20}$$

Thus, from (19) and (20), we conclude that x_0 solve (PGMVVLIC). The uniqueness of x_0 directly follows from the assumption diam $\xi(\delta, \epsilon) \to 0$ as $(\delta, \epsilon) \to (0, 0)$. This completes the proof of the theorem. □

Now, we give an example to illustrate the result established in Theorem 5.

Example 3. *Let $X = Y = \mathbb{R}$, $K = [0, 1]$ and $P = [0, \infty)$. Let us define the mappings $\Phi : K \mapsto R$, $T : K \mapsto 2^{L(X,Y)}$, $f : K \times K \mapsto Y$, $\eta : X \times X \mapsto X$, and $A : L(X, Y) \mapsto L(X, Y)$ as follows:*

$$\begin{aligned}
\Phi(x) &= |x^3|,\\
T(x) &= \{u : \mathbb{R} \mapsto \mathbb{R}|\ u \textit{ is a continuous linear mapping such that } u(x) = -x\},\\
f(x, y) &= y - x,\\
\eta(x, y) &= \frac{1}{2}(y - x),\\
Av &= v, \textit{ and}\\
\alpha &= -x^2.
\end{aligned}$$

(PGMVVLIC3) $\quad$ *$P -$ minimize $|x^3|$*
subject to $x \in S$,

where $S = \{x \in K|\ \exists\ u \in T(x)$ such that $\langle u, \frac{1}{2}(x - y)\rangle + y - x \not\leq_{P^\circ} 0,\ \forall y \in K\}$.
Clearly, $S = \{0\}$. Now,

$$\xi(\delta, \epsilon) = \Big\{x \in K|\ \exists\ u \in T(x) \textit{ such that } |x^3| \leq_P \delta \textit{ and } (y - x)(1 + \frac{x}{2}) + \epsilon \not\leq_{P^\circ} 0, \\ \forall y \in K\Big\}.$$

Here, diam $\xi(\delta, \epsilon) \to 0$ as $(\delta, \epsilon) \to (0, 0)$. Also, it can be easily verified that T is relaxed η-α-P-monotone with respect to A and all conditions of Theorem 5 hold. Therefore (PGMVVLIC3) is well-posed.

In the next theorem, we show that well-posedness of (PGMVVLIC) is equivalent to the existence and uniqueness of solutions.

Theorem 6. *Let all assumptions of Theorem 2 hold and let Φ be lower semicontinuous. Then the problem (PGMVVLIC) is well-posed if and only if it has a unique solution.*

Proof. The necessary condition is obvious. Conversely, let (PGMVVLIC) has a unique solution x_0. Then

$$\begin{aligned} \Phi(x_0) &= \inf_{y\in S}\Phi(y) \\ \langle Au_0, \eta(y,x_0)\rangle + f(x_0,y) &\nleq_{P^o} 0, \forall u_0 \in T(x_0), \quad \forall y \in K. \end{aligned}$$

Let $\{x_n\}$ be an approximating sequence. Then there exist $u_n \in T(x_n)$ and a sequence of positive real numbers $\epsilon_n \to 0$ such that

$$\begin{aligned} \lim_{n\to\infty}\sup\Phi(x_n) &\leq_P \inf_{y\in S}\Phi(y), \\ \langle Au_n, \eta(y,x_n)\rangle + f(x_n,y) + \epsilon_n e &\nleq_{P^o} 0, \quad \forall y \in K. \end{aligned}$$

Now, following lines similar to the proof of Theorem 2, we find out the sequence $\{x_n\}$ has a subsequence $\{x_{n_k}\}$ converging to $\bar{x}$, for any $\bar{x} \in K$ and

$$\langle A\bar{u}, \eta(y,\bar{x})\rangle + f(\bar{x},y) \nleq_{P^o} 0, \quad \forall \bar{u} \in T(\bar{x}),\ \forall y \in K. \tag{21}$$

Since Φ is lower semicontinuous, therefore,

$$\Phi(\bar{x}) \leq_P \lim_{k\to\infty}\inf\Phi(x_{n_k}) \leq_P \lim_{k\to\infty}\sup\Phi(x_{n_k}) \leq_P \inf_{y\in S}\Phi(y). \tag{22}$$

Thus, from (21) and (22), we conclude that $\bar{x} \in \xi$. This completes the proof of the theorem. □

Theorem 7. *Suppose all assumptions of Theorem 4 hold, Φ is lower semicontinuous and if there exists some $\epsilon > 0$ such that $\xi(\epsilon,\epsilon) \neq \phi$ and is bounded, then (PGMVVLIC) is well-posed in a generalized sense.*

Proof. Let $\epsilon > 0$ such that $\xi(\epsilon,\epsilon) \neq \phi$ and suppose $\{x_n\}$ is an approximating sequence of (GMVVLIP). Then

(i) $\lim_{n\to\infty}\sup\ \Phi(x_n) \leq_P \inf_{y\in S}\ \Phi(y)$,

(ii) there exist $u_n \in T(x_n)$ and a sequence of positive real numbers $\epsilon_n \to 0$ such that
$$\langle Au_n, \eta(y,x_n)\rangle + f(x_n,y) + \epsilon_n e \nleq_{P^o} 0, \quad \forall y \in K,\ \forall n \in N,$$

which implies that $x_n \in \xi(\epsilon,\epsilon)$, for all $n > m$. Therefore, $\{x_n\}$ is a bounded sequence that has a convergent subsequence $\{x_{n_k}\}$ converging to x_0 as $k \to \infty$. Following the lines similar to the proof of Theorem 6, we conclude that $x_0 \in \xi$. Hence, (PGMVVLIC) is well-posed in generalized sense. This completes the proof of the theorem. □

5. Well-posedness of optimization problems by using well-posedness of constraints

In this section, we derive well-posedness of (PGMVVLIC) by using well-posedness of (GMVVLIC).

Theorem 8. *Let K be a nonempty compact set and Φ lower semicontinuous. Suppose (PGMVVLIC) has a unique solution. If (GMVVLIP) is well-posed, then (PGMVVLIC) is well-posed.*

Proof. Let (PGMVVLIC) has a unique solution x_0. Consider $\{x_n\}$ as an approximating sequence for (PGMVVLIC). Then there exist $u_n \in T(x_n)$ and a sequence of positive real numbers $\epsilon_n \to 0$ such that

$$\lim_{n\to\infty} \sup \Phi(x_n) \ \leq_P \ \inf_{y\in S} \Phi(y)$$
$$\langle Au_n, \eta(y, x_n)\rangle + f(x_n, y) + \epsilon_n e \nleq_{P^o} 0, \quad \forall y \in K.$$

Since K is compact, therefore, $\{x_n\}$ has a subsequence $\{x_{n_k}\}$ converging to a $\bar{x}$ (say) as $k \to \infty$. Since (GMVVLIP) is well-posed, $\bar{x}$ solves (GMVVLIP), i.e.,

$$\langle A\bar{u}, \eta(y, \bar{x})\rangle + f(\bar{x}, y) \nleq_{P^o} 0, \quad \forall \bar{u} \in T(\bar{x}),\ \forall y \in K. \tag{23}$$

Now, Φ is lower semicontinuous; therefore, we have

$$\Phi(\bar{x}) \leq_P \lim_{k\to\infty} \inf \Phi(x_{n_k}) \leq_P \lim_{k\to\infty} \sup \Phi(x_{n_k}) \leq_P \inf_{y\in S} \Phi(y). \tag{24}$$

Thus, from (23) and (24) we conclude that $\bar{x}$ solves (PGMVVLIC). But (PGMVVLIC) has a unique solution x_0; therefore, $\bar{x} = x_0$ and $x_n \to x_0$. Hence, (PGMVVLIC) is well-posed. This completes the proof of the theorem. ☐

We can also prove the following results whose proof can be done by some minor modification in the proof of Theorem 8.

Theorem 9. *Suppose all assumptions of Theorem 8 hold. If (GMVVLIP) is well-posed in a generalized sense, then (PGMVVLIC) is well-posed.*

Theorem 10. *Let K be a nonempty compact set, Φ lower semicontinuous and $\xi \neq \phi$. If (GMVVLIP) is well-posed, then (PGMVVLIC) is well-posed in a generalized sense.*

Theorem 11. *Suppose all assumptions of Theorem 10 hold. If (GMVVLIP) is well-posed in a generalized sense, then (PGMVVLIC) is well-posed in a generalized sense.*

6. Conclusion

In this paper, we have investigated well-posedness for generalized mixed vector variational-like inequality problems and for optimization problems with generalized mixed vector variational-like inequality constraints having one solution or more than one solution. Further, in the forthcoming paper we will extend the idea of well-posedness by perturbations to the extended generalized mixed vector-variational like inequality problems over Banach spaces.

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