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Positive solutions for the system of higher order singular nonlinear boundary value problems

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Abstract. In this paper, by using Krasnosel'skii fixed point theorem and under suitable conditions, we present the existence of single and multiple positive solutions to the following systems

$$\begin{aligned}(-1)^m u^{(2m)} &= \lambda f(t,u(t),v(t)) = 0, \quad t\in[a,b],\\ (-1)^n v^{(2n)} &= \mu g(t,u(t),v(t)) = 0, \quad t\in[a,b],\\ u^{(2i)}(a) &= u^{(2i)}(b) = 0, \quad 0\le i\le m-1,\\ v^{(2j)}(a) &= v^{(2j)}(b) = 0, \quad 0\le j\le n-1,\end{aligned}$$

where $\lambda,\mu>0, m,n\in\mathbb{N}$. We derive two explicit eigenvalue intervals of λ and μ for the existence of at least one positive solution and the existence of at least two positive solutions for the above higher order two-point boundary value problem.

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Key words: positive solutions, nonlinear ordinary differential systems, singular boundary value problems, fixed point theorem

1. Introduction

In this paper, we consider the existence of single and multiple positive solutions to the following boundary value problem of nonlinear differential system

$$\begin{cases}(-1)^m u^{(2m)} = \lambda f(t,u(t),v(t)) = 0, \quad t\in[a,b]\\ (-1)^n v^{(2n)} = \mu g(t,u(t),v(t)) = 0, \quad t\in[a,b]\\ u^{(2i)}(a) = u^{(2i)}(b) = 0, \quad 0\le i\le m-1,\\ v^{(2j)}(a) = v^{(2j)}(b) = 0, \quad 0\le j\le n-1,\end{cases} \tag{1}$$

where $\lambda,\mu>0, m,n\in\mathbb{N}$, $f,g\in C[[a,b]\times[0,\infty)\times[0,\infty),[0,\infty)]$, and also f and g are allowed to be singular at $t=a$ or $t=b$. The following assumptions are made to establish our results.

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(H1) $f(t,u,v) \le p_1(t)q_1(t,u,v), g(t,u,v) \le p_2(t)q_2(t,u,v), (t,u,v) \in [a,b] \times [0,\infty) \times [0,\infty)$, where $q_i \in C[[a,b] \times [0,\infty) \times [0,\infty), [0,\infty)]$, and $p_i \in C[[a,b],[0,\infty)]$ satisfy

$$\int_a^b \frac{(s-a)(b-s)}{b-a} p_i(s)ds < +\infty, i = 1,2.$$

(H2) The limits

$$f_0 = \lim_{u+v\to 0} \min_{t\in[\frac{3a+b}{4},\frac{a+3b}{4}]} \frac{f(t,u,v)}{u+v}, \quad f_\infty = \lim_{u+v\to \infty} \min_{t\in[\frac{3a+b}{4},\frac{a+3b}{4}]} \frac{f(t,u,v)}{u+v},$$
$$g_0 = \lim_{u+v\to 0} \min_{t\in[\frac{3a+b}{4},\frac{a+3b}{4}]} \frac{g(t,u,v)}{u+v}, \quad g_\infty = \lim_{u+v\to \infty} \min_{t\in[\frac{3a+b}{4},\frac{a+3b}{4}]} \frac{g(t,u,v)}{u+v},$$
$$q_{i0} = \overline{\lim_{u+v\to 0}} \min_{t\in[\frac{3a+b}{4},\frac{a+3b}{4}]} \frac{q_i(t,u,v)}{u+v}, \quad q_{i\infty} = \overline{\lim_{u+v\to \infty}} \min_{t\in[\frac{3a+b}{4},\frac{a+3b}{4}]} \frac{q_i(t,u,v)}{u+v},$$

exist with $f_0, f_\infty, g_0, g_\infty, q_{i0}, q_{i\infty} \in [0,\infty)$, $i = 1,2$.

The aim of this paper is to establish some simple criteria for the existence of single and multiple solutions of the system (1) in explicit intervals for λ and μ. The rest of the paper is organized as follows. In Section 2, we present some preliminaries and lemmas that will be used to prove our main results. In Section 3, we discuss the existence of a single positive solution of the system (1). The intervals in which the parameters λ, μ can guarantee the existence of a solution are obtained. In Section 4, we study the existence conditions of at least two positive solutions of the system (1). Finally, in Section 5, we give an example as an application.

2. Preliminary results

In this section, we present some notation and lemmas that will be used to prove our results. Here we consider the Banach space $C[a,b] \times C[a,b]$ equipped with the standard norm

$$\|(u,v)\| = \|u\| + \|v\| = \max_{t\in[a,b]} |u(t)| + \max_{t\in[a,b]} |v(t)|, \ (u,v) \in C[a,b] \times C[a,b].$$

Let $G_n(t,s)$ be the Green's function of a homogeneous boundary value problem:

$$(-1)^n \omega^{(2n)}(t) = 0, \ t \in [a,b],$$
$$\omega^{(2i)}(a) = \omega^{(2i)}(b) = 0, \ 0 \le i \le n-1.$$

By induction, the Green's function $G_n(t,s)$ can be expressed as (see [1])

$$G_i(t,s) = \int_a^b G(t,u)G_{i-1}(u,s)du, \quad 2 \le i \le n, \tag{2}$$

where

$$G_1(t,s) = G(t,s) = \begin{cases} \frac{(t-a)(b-s)}{b-a}, & a \le t \le s \le b, \\ \frac{(s-a)(b-t)}{b-a}, & a \le s \le t \le b. \end{cases} \tag{3}$$

It is clear that

$$G_n(t,s) > 0, \quad (t,s) \in (a,b) \times (a,b). \tag{4}$$

Lemma 1. *For any $(t,s) \in [a,b] \times [a,b]$,*

$$G_n(t,s) \le \left(\frac{b-a}{6}\right)^{n-1} \frac{(s-a)(b-s)}{b-a}. \tag{5}$$

Proof. For $(t,s) \in [a,b] \times [a,b]$, it is clear from (3) that

$$G(t,s) \le \frac{(s-a)(b-s)}{b-a}. \tag{6}$$

i.e. (5) is true for $n = 1$. Assume that (5) holds for $n = k(\ge 1)$. Then, for $(t,s) \in [a,b] \times [a,b]$, it follows from (2), (4) and (6) that

$$\begin{aligned} G_{k+1}(t,s) &= \int_a^b G(t,u)G_k(u,s)du \\ &\le \int_a^b \frac{(u-a)(b-u)}{b-a} \left(\frac{b-a}{6}\right)^{k-1} \frac{(s-a)(b-s)}{b-a} du \\ &= \left(\frac{b-a}{6}\right)^{k} \frac{(s-a)(b-s)}{b-a}. \end{aligned}$$

Thus (5) is true for $n = k+1$. □

Lemma 2. *Let $\delta \in (a, \frac{a+b}{2})$, then for all $(t,s) \in [\delta, b-\delta] \times [a,b]$, we have*

$$G_n(t,s) \ge \theta_n(\delta) \frac{(s-a)(b-s)}{b-a} \ge \left(\frac{6}{b-a}\right)^{n-1} \theta_n(\delta) \max_{t\in[a,b]} G_n(t,s), \tag{7}$$

where $0 < \theta_n(\delta) < 1$ is a constant given by

$$\theta_n(\delta) = (\delta-a)^n \left(\frac{4\delta^3 - 6b\delta^2 + 6ab\delta - 3ab^2 + b^3}{6(b-a)}\right)^{n-1}.$$

Proof. For $(t,s) \in [\delta, b-\delta] \times [a,b]$, from (3) we find

$$\begin{aligned} G(t,s) &= \begin{cases} \frac{(t-a)(b-s)}{b-a}, & t \le s \\ \frac{(s-a)(b-t)}{b-a}, & s \le t \end{cases} \\ &\ge \begin{cases} \frac{(\delta-a)(b-s)}{b-a}, & t \le s \\ \frac{(s-a)(b-(b-\delta))}{b-a}, & s \le t \end{cases} \\ &\ge \frac{(\delta-a)(s-a)(b-s)}{b-a}. \end{aligned} \tag{8}$$

Hence (7) is true for $n = 1$. Suppose now that (7) holds for $n = k(\geq 1)$. Then, using (2), (4) and (8), we get for $(t, s) \in [\delta, b-\delta] \times [a, b]$,

$$\begin{aligned} G_{k+1}(t,s) &= \int_a^b G(t,u)G_k(u,s)du \\ &\geq \int_\delta^{b-\delta} G(t,u)G_k(u,s)du \\ &\geq \int_\delta^{b-\delta} \frac{(\delta-a)(u-a)(b-u)}{b-a}\theta_k(\delta)\frac{(s-a)(b-s)}{b-a}du \\ &= \theta_{k+1}(\delta)\frac{(s-a)(b-s)}{b-a}. \end{aligned}$$

So, (7) is true for $n = k+1$. $\square$

In Lemma 2, let

$$\begin{aligned} \sigma_m &= \left(\frac{6}{b-a}\right)^{m-1}\theta_m\left(\frac{3a+b}{4}\right) = \frac{(11b^3+27a^3-51ab^2+45a^2b)^{m-1}}{2^{6m-4}(b-a)^{m-2}}, \\ \sigma_n &= \left(\frac{6}{b-a}\right)^{n-1}\theta_n\left(\frac{3a+b}{4}\right) = \frac{(11b^3+27a^3-51ab^2+45a^2b)^{n-1}}{2^{6n-4}(b-a)^{n-2}}, \\ \sigma &= \min\{\sigma_m, \sigma_n\}. \end{aligned}$$

According to Lemma 1 and Lemma 2, one obviously has $0 < \sigma < 1$.

It is well known that the system (1) is equivalent to the equation

$$(u(t), v(t)) = \Big(\lambda \int_a^b G_m(t,s)f(s,u(s),v(s))ds,\ \mu \int_a^b G_n(t,s)g(s,u(s),v(s))ds\Big).$$

Under the conditions of (H1), we define the operators $A_\lambda, A_\mu : C[a,b] \times C[a,b] \to C[a,b]$ as

$$\begin{aligned} A_\lambda(u,v)(t) &= \lambda \int_a^b G_m(t,s)f(s,u(s),v(s))ds, \\ A_\mu(u,v)(t) &= \mu \int_a^b G_n(t,s)g(s,u(s),v(s))ds, \end{aligned}$$

and an operator $A : C[a,b] \times C[a,b] \to C[a,b] \times C[a,b]$ as

$$A(u,v) = \Big(A_\lambda(u,v), A_\mu(u,v)\Big),\ (u,v) \in C[a,b] \times C[a,b]. \tag{9}$$

It is clear that the existence of a positive solution to the system (1) is equivalent to the existence of a fixed point of A in $C[a,b] \times C[a,b]$.

We define a cone in $C[a,b] \times C[a,b]$ by

$$\kappa = \Big\{(u,v) : C[a,b]\times C[a,b] : u(t)\geq 0, v(t)\geq 0, \min_{\frac{3a+b}{4}\leq t\leq \frac{a+3b}{4}} (u(t)+v(t)) \geq \sigma\|(u,v)\|\Big\}.$$

Lemma 3. *$A : \kappa \to \kappa$ is completely continuous.*

Proof. Since the proof of the completely continuous is standard, we need only to prove $A(\kappa) \subset \kappa$.

In fact, for any $(t,s) \in [\frac{3a+b}{4}, \frac{a+3b}{4}] \times [a,b]$, we have

$$\begin{aligned}
&A_\lambda(u,v)(t) + A_\mu(u,v)(t)\\
&\qquad = \lambda \int_a^b G_m(t,s) f(s,u(s)v(s))ds + \mu \int_a^b G_n(t,s) g(s,u(s)v(s))ds\\
&\qquad \geq \lambda \theta_m \Big(\frac{3a+b}{4}\Big)\Big(\frac{6}{b-a}\Big)^{m-1} \max_{t\in[a,b]} \int_a^b G_m(t,s) f(s,u(s)v(s))ds\\
&\qquad\quad + \mu \theta_n \Big(\frac{3a+b}{4}\Big)\Big(\frac{6}{b-a}\Big)^{n-1} \max_{t\in[a,b]} \int_a^b G_n(t,s) g(s,u(s)v(s))ds\\
&\qquad = \sigma_m \|A_\lambda(u,v)\| + \sigma_n \|A_\mu(u,v)\| \geq \sigma \|A(u,v)\|,
\end{aligned}$$

hence,

$$\min_{t\in[\frac{3a+b}{4}, \frac{a+3b}{4}]} [A_\lambda(u,v)(t) + A_\mu(u,v)(t)] \geq \sigma \|A(u,v)\|.$$

Therefore, $A(\kappa) \subset \kappa$. □

3. Existence results

In this section, we discuss the existence of at least one positive solution to the system (1). We use the following notation for simplicity.

$$\begin{aligned}
A_1 =& \Big(\frac{b-a}{6}\Big)^{m-1} \int_a^b p_1(s) \frac{(s-a)(b-s)}{b-a} ds,\\
A_2 =& \Big(\frac{b-a}{6}\Big)^{n-1} \int_a^b p_2(s) \frac{(s-a)(b-s)}{b-a} ds,\\
B_1 =& \frac{11b^3 - 11a^3 + 33a^2b - 33ab^2}{96(b-a)} \theta_m \Big(\frac{3a+b}{4}\Big),\\
B_2 =& \frac{11b^3 - 11a^3 + 33a^2b - 33ab^2}{96(b-a)} \theta_n \Big(\frac{3a+b}{4}\Big).
\end{aligned}$$

Our approach is based on the following Krasnosel'skii fixed point theorem [12].

Lemma 4. *Let $\mathcal{B}$ be a Banach space and let $\mathcal{P} \subset \mathcal{B}$ be a cone in $\mathcal{B}$. Assume that Ω_1 and Ω_2 are open bounded subsets of $\mathcal{B}$ with $0 \in \Omega_1 \subset \overline{\Omega}_1 \subset \Omega_2$, and let*

$$T : \mathcal{P} \cap (\overline{\Omega}_2 \backslash \Omega_1) \ \to \mathcal{P}$$

be a completely continuous operator such that either

(i) $\|Tu\| \leq \|u\|,\ u \in \mathcal{P} \cap \partial\Omega_1,$ *and* $\|Tu\| \geq \|u\|,\ u \in \mathcal{P} \cap \partial\Omega_2,$ *or*

(ii) $\|Tu\| \geq \|u\|,\ u \in \mathcal{P} \cap \partial\Omega_1,$ *and* $\|Tu\| \leq \|u\|,\ u \in \mathcal{P} \cap \partial\Omega_2.$

Then, T has a fixed point in $\mathcal{P} \cap (\overline{\Omega}_2 \backslash \Omega_1)$.

Theorem 1. *Suppose* (H1), (H2) *hold, and* $0 < \alpha < 1$*, then we have the following results:*

(1) If $0 < q_{10}, f_\infty, q_{20}, g_\infty < \infty$, $A_1 q_{10} < \alpha\sigma B_1 f_\infty$*, then for each* $\lambda \in (\frac{1}{\sigma B_1 f_\infty}, \frac{\alpha}{A_1 q_{10}})$ *and* $\mu \in (0, \frac{1-\alpha}{A_2 q_{20}})$*, the system* (1) *has at least one positive solution.*

(2) If $0 < q_{10}, f_\infty, q_{20}, g_\infty < \infty$, $A_2 q_{20} < (1-\alpha)\sigma B_2 g_\infty$*, then for each* $\lambda \in (0, \frac{\alpha}{A_1 q_{10}})$ *and* $\mu \in (\frac{1}{\sigma B_2 g_\infty}, \frac{1-\alpha}{A_2 q_{20}})$*, the system* (1) *has at least one positive solution.*

Proof. We only prove case (1). The other case can be proved similarly. We construct the sets Ω_1 and Ω_2 in order to apply Lemma 4.

Let
$$\lambda \in \Big(\frac{1}{\sigma B_1 f_\infty}, \frac{\alpha}{A_1 q_{10}}\Big),\ \mu \in \Big(0, \frac{1-\alpha}{A_2 q_{20}}\Big),$$
and we choose $\epsilon > 0$ such that
$$\frac{1}{\sigma B_1 (f_\infty - \epsilon)} \le \lambda \le \frac{\alpha}{A_1(q_{10}+\epsilon)},\ 0 < \mu \le \frac{1-\alpha}{A_2(q_{20}+\epsilon)}.$$
By the definition of q_{10} and q_{20}, there exists $R_1 > 0$ such that
$$q_1(t,u,v) \le (q_{10}+\epsilon)(u+v),\ q_2(t,u,v) \le (q_{20}+\epsilon)(u+v), \quad \text{for } u+v \in [0, R_1].$$
Choosing $(u,v) \in \kappa$ with $\|(u,v)\| = R_1$, we have
$$\begin{aligned}
A_\lambda(u,v)(t) &= \lambda \int_a^b G_m(t,s) f(s,u(s),v(s))ds\\
&\le \lambda\Big(\frac{b-a}{6}\Big)^{m-1} \int_a^b \frac{(s-a)(b-s)}{b-a} p_1(s) q_1(s,u(s),v(s))ds\\
&\le \lambda\Big(\frac{b-a}{6}\Big)^{m-1} \int_a^b \frac{(s-a)(b-s)}{b-a} p_1(s)(q_{10}+\epsilon)(u+v)ds\\
&\le \lambda\Big(\frac{b-a}{6}\Big)^{m-1} (q_{10}+\epsilon)\|(u,v)\| \int_a^b \frac{(s-a)(b-s)}{b-a} p_1(s)ds\\
&\le \lambda A_1(q_{10}+\epsilon)\|(u,v)\| \le \alpha\|(u,v)\|,
\end{aligned}$$
$$\begin{aligned}
A_\mu(u,v)(t) &= \mu \int_a^b G_n(t,s) g(s,u(s),v(s))ds\\
&\le \mu\Big(\frac{b-a}{6}\Big)^{n-1} \int_a^b \frac{(s-a)(b-s)}{b-a} p_2(s) q_2(s,u(s),v(s))ds\\
&\le \mu\Big(\frac{b-a}{6}\Big)^{n-1} \int_a^b \frac{(s-a)(b-s)}{b-a} p_2(s)(q_{20}+\epsilon)(u+v)ds\\
&\le \mu\Big(\frac{b-a}{6}\Big)^{n-1} (q_{20}+\epsilon)\|(u,v)\| \int_a^b \frac{(s-a)(b-s)}{b-a} p_2(s)ds\\
&\le \mu A_2(q_{20}+\epsilon)\|(u,v)\|\\
&\le (1-\alpha)\|(u,v)\|,
\end{aligned}$$

then $\|A(u,v)\| \le \alpha\|(u,v)\| + (1-\alpha)\|(u,v)\| = \|(u,v)\|$. Consequently, if we set $\Omega_1 = \{(u,v) \in \kappa : \|(u,v)\| < R_1\}$, then

$$\|A(u,v)\| \le \|(u,v)\|, \quad \text{for all } (u,v) \in \kappa \cap \partial\Omega_1. \tag{10}$$

On the other hand, by the definition of f_∞, there exists $\overline{R}_2 > 0$, such that $f(t,u,v) \ge (f_\infty - \epsilon)(u+v)$, for all $u+v \in [\overline{R}_2, \infty)$. Let $R_2 = \max\{2R_1, \sigma^{-1}\overline{R}_2\}$ and $\Omega_2 = \{(u,v) \in \kappa : \|(u,v)\| < R_2\}$. If $(u,v) \in \kappa$ with $\|(u,v)\| = R_2$, then $\min_{t\in[\frac{3a+b}{4},\frac{a+3b}{4}]}(u+v) \ge \sigma\|(u,v)\| \ge \overline{R}_2$, thus we have

$$\begin{aligned}
A_\lambda(u,v)(t) &= \lambda\int_a^b G_m(t,s)f(s,u(s),v(s))ds \\
&\ge \lambda\theta_m\Big(\frac{3a+b}{4}\Big)\int_{\frac{3a+b}{4}}^{\frac{a+3b}{4}} \frac{(s-a)(b-s)}{b-a}(f_\infty-\epsilon)(u+v)ds \\
&\ge \lambda\theta_m\Big(\frac{3a+b}{4}\Big)(f_\infty-\epsilon)\sigma\|(u,v)\|\int_{\frac{3a+b}{4}}^{\frac{a+3b}{4}} \frac{(s-a)(b-s)}{b-a}ds \\
&= \frac{11b^3-11a^3+33a^2b-33ab^2}{96(b-a)}\lambda\theta_m\Big(\frac{3a+b}{4}\Big)(f_\infty-\epsilon)\sigma\|(u,v)\| \\
&= \lambda B_1(f_\infty-\epsilon)\sigma\|(u,v)\| \\
&\ge \|(u,v)\|, \quad \forall t \in \Big[\frac{3a+b}{4}, \frac{a+3b}{4}\Big],
\end{aligned}$$

then

$$\|A(u,v)\| \ge \|A_\lambda(u,v)\| \ge \|(u,v)\|, \text{ for all } (u,v) \in \kappa\cap\partial\Omega_2. \tag{11}$$

Therefore, it follows from (10), (11) and Lemma 4, A has a fixed point in $\kappa\cap(\overline{\Omega}_2\backslash\Omega_1)$, which is a positive solution of (1). □

Similarly, we can also obtain the following theorem that is in some way a duality of Theorem 1.

Theorem 2. *Suppose* (H1), (H2) *hold, and* $0 < \alpha < 1$, *then we have*

(1) If $0 < f_0, q_{1\infty}, g_0, q_{2\infty} < \infty$, $A_1q_{1\infty} < \alpha\sigma B_1 f_0$, *then for each* $\lambda \in (\frac{1}{\sigma B_1 f_0}, \frac{\alpha}{A_1 q_{1\infty}})$ *and* $\mu \in (0, \frac{1-\alpha}{A_2 q_{2\infty}})$, *the system* (1) *has at least one positive solution.*

(2) If $0 < f_0, q_{1\infty}, g_0, q_{2\infty} < \infty$, $A_2q_{2\infty} < (1-\alpha)\sigma B_2 g_0$, *then for each* $\lambda \in (0, \frac{\alpha}{A_1 q_{1\infty}})$ *and* $\mu \in (\frac{1}{\sigma B_2 g_0}, \frac{1-\alpha}{A_2 q_{2\infty}})$, *the system* (1) *has at least one positive solution.*

Proof. The proof is very similar to the proof of Theorem 1, we omit it here. □

4. Multiplicity results

In this section, we prove the existence of at least two positive solutions for the system (1).

Theorem 3. *Suppose* (H1), (H2) *hold. In addition, assume that there exist four constants* r_1, M, K, α, *where* K *is sufficient small,* $0 < \alpha < 1$, *with* $\alpha B_1 M > A_1 K$, $(1-\alpha)B_2 M > A_2 K$, *such that:*

(1) $q_{10} = q_{1\infty} = 0, q_{20} = q_{2\infty} = 0$;

(2) $f(t,u,v) \geq Mr_1$, *or* $g(t,u,v) \geq Mr_1$, *for* $\sigma r_1 \leq \|(u,v)\| \leq r_1$.

Then for any $\lambda \in \left[\frac{1}{B_1 M}, \frac{\alpha}{A_1 K}\right]$, $\mu \in \left(0, \frac{1-\alpha}{A_2 K}\right]$ *or* $\lambda \in [0, \frac{\alpha}{A_1 K}]$, $\mu \in \left[\frac{1}{B_2 M}, \frac{1-\alpha}{A_2 K}\right]$, *the system* (1) *has at least two positive solutions.*

Proof. We only prove the case of $\lambda \in \left[\frac{1}{B_1 M}, \frac{\alpha}{A_1 K}\right]$, $\mu \in \left(0, \frac{1-\alpha}{A_2 K}\right]$. The other case is similar.

Step 1. By the definition of $q_{10} = q_{20} = 0$, there exists $H_1 \in (0, r_1)$ such that

$$q_1(t,u,v) \leq K(u+v),\ q_2(t,u,v) \leq K(u+v),\ \text{for } u+v \in (0, H_1).$$

Then we have

$$\begin{aligned}
A_\lambda(u,v)(t) &= \lambda \int_a^b G_m(t,s) f(s,u(s),v(s))ds \\
&\leq \lambda \Big(\frac{b-a}{6}\Big)^{m-1} \int_a^b \frac{(s-a)(b-s)}{b-a} p_1(s) q_1(s,u(s),v(s))ds \\
&\leq \lambda \Big(\frac{b-a}{6}\Big)^{m-1} \int_a^b \frac{(s-a)(b-s)}{b-a} p_1(s) ds K(u+v) \\
&\leq \lambda \Big(\frac{b-a}{6}\Big)^{m-1} K\|(u,v)\| \int_a^b \frac{(s-a)(b-s)}{b-a} p_1(s) ds \\
&= \lambda A_1 K \|(u,v)\| \leq \alpha \|(u,v)\|, \\
A_\mu(u,v)(t) &= \mu \int_a^b G_n(t,s) g(s,u(s),v(s))ds \\
&\leq \mu \Big(\frac{b-a}{6}\Big)^{n-1} \int_a^b \frac{(s-a)(b-s)}{b-a} p_2(s) q_2(s,u(s),v(s))ds \\
&\leq \mu \Big(\frac{b-a}{6}\Big)^{n-1} \int_a^b \frac{(s-a)(b-s)}{b-a} p_2(s) ds K(u+v) \\
&\leq \mu \Big(\frac{b-a}{6}\Big)^{n-1} K\|(u,v)\| \int_a^b \frac{(s-a)(b-s)}{b-a} p_2(s) ds \\
&= \mu A_2 K \|(u,v)\| \leq (1-\alpha) \|(u,v)\|.
\end{aligned}$$

Hence,

$$\|A(u,v)\| = \|A_\lambda(u,v)\| + \|A_\mu(u,v)\| \leq \|(u,v)\|.$$

Set $\Omega_1 = \{(u,v) \in \kappa : \|(u,v)\| < H_1\}$, then

$$\|A(u,v)\| \leq \|(u,v)\|,\ \text{for all } (u,v) \in \kappa \cap \partial\Omega_1. \tag{12}$$

Step 2. By the definition of $q_{1\infty} = q_{2\infty} = 0$, there exist $H_2 > r_1$ such that

$$q_1(t,u,v) \leq K(u+v),\quad q_2(t,u,v) \leq K(u+v),\ \text{for } u+v \in [H_2, \infty).$$

Similarly, set $\Omega_2 = \{(u,v) \in \kappa : \|(u,v)\| < H_2\}$, then

$$\|A(u,v)\| \le \|(u,v)\|, \text{ for all } (u,v) \in \kappa \cap \partial\Omega_2. \tag{13}$$

Step 3. Set $\Omega_3 = \{(u,v) \in \kappa : \|(u,v)\| < r_1\}$, then $\forall (u,v) \in \kappa$ with $\|(u,v)\| = r_1$, we have

$$\begin{aligned} A_\lambda(u,v)(t) &= \lambda \int_a^b G_m(t,s) f(s,u(s),v(s))ds \\ &\ge \lambda\theta_m\Big(\frac{3a+b}{4}\Big) \int_{\frac{3a+b}{4}}^{\frac{a+3b}{4}} \frac{(s-a)(b-s)}{b-a} M r_1 ds \\ &= \lambda B_1 M r_1 \ge r_1, \quad \forall t \in \Big[\frac{3a+b}{4}, \frac{a+3b}{4}\Big]. \end{aligned}$$

Then

$$\|A(u,v)\| \ge \|(u,v)\|, \text{ for all } (u,v) \in \kappa \cap \partial\Omega_3. \tag{14}$$

Consequently, from (12)-(14) and Lemma 4, the system (1) has at least two positive solutions $(u_1,v_1) \in \kappa$, $(u_2,v_2) \in \kappa$ with $0 \le \|(u_1,v_1)\| \le r_1 \le \|(u_2,v_2)\|$. □

The following result is an antithesis of Theorem 3.

Theorem 4. *Suppose* (H1), (H2) *hold. In addition, assume that there exist four constants r_1, M, K, α, where K is sufficient large, $0 < \alpha < 1$, with $\alpha B_1 K\sigma > A_1 M$, $(1-\alpha)B_2 K\sigma > A_2 M$, such that:*

(3) $q_1(t,u,v) \le Mr_1$, or $q_2(t,u,v) \le Mr_1$, for $0 \le \|(u,v)\| \le r_1$;

(4) $f_0 = f_\infty = \infty$ or $g_0 = g_\infty = \infty$.

Then for any $\lambda \in \big[\frac{1}{B_1 K\sigma}, \frac{\alpha}{A_1 M}\big]$ and $\mu \in \big(0, \frac{1-\alpha}{A_2 M}\big]$ or $\lambda \in [0, \frac{\alpha}{A_1 M}\big]$ and $\mu \in \big[\frac{1}{B_2 K\sigma}, \frac{1-\alpha}{A_2 M}\big]$, the system (1) *has at least two positive solutions.*

For the convenience of the discussion of the existence of more than two positive solutions for the system (1), we study the problem under a more general case than the assumption of Theorem 3 and Theorem 4.

Let

$$\begin{aligned} \varphi_i(r) &= \sup\{q_i(t,u,v) : t \in [a,b],\ \sigma r \le u+v \le r\},\ i = 1,2. \\ \psi_1(r) &= \inf\{f(t,u,v) : t \in \Big[\frac{3a+b}{4}, \frac{a+3b}{4}\Big],\ \sigma r \le u+v \le r\}. \\ \psi_2(r) &= \inf\{g(t,u,v) : t \in \Big[\frac{3a+b}{4}, \frac{a+3b}{4}\Big],\ \sigma r \le u+v \le r\}. \\ \varphi(r) &= \max\{\varphi_1(r), \varphi_2(r)\},\ \psi(r) = \min\{\psi_1(r), \psi_2(r)\}. \end{aligned}$$

Then, we can obtain the following result.

Theorem 5. *Suppose* (H1) *hold. In addition, assume that there exist three constants $M, K, \alpha, 0 < \alpha < 1$ with $\alpha B_1 M > A_1 K$, $(1-\alpha)B_2 M > A_2 K$ and three constants d_1, d_2, d_3 with $0 < d_1 < d_2 < d_3$, such that one of the following two conditions is satisfied:*

(I) $\varphi(d_1) \le d_1 K,\ \psi(d_2) > d_2 M,$ *and* $\varphi(d_3) \le K d_3$.

(II) $\psi(d_1) \ge d_1 M,\ \varphi(d_2) < d_2 K,$ *and* $\psi(d_3) \ge M d_3$.

Then for any $\lambda \in \left[\frac{1}{B_1 M}, \frac{\alpha}{A_1 K}\right]$, $\mu \in \left(0, \frac{1-\alpha}{A_2 K}\right]$ *or* $\lambda \in (0, \frac{\alpha}{A_1 K}]$, $\mu \in \left[\frac{1}{B_2 M}, \frac{1-\alpha}{A_2 K}\right]$, *the system* (1) *has at least two positive solutions* $(u_1^*, v_1^*), (u_2^*, v_2^*)$ *and* $d_1 \le \|(u_1^*, v_1^*)\| < d_2 < \|(u_2^*, v_2^*)\| \le d_3$.

Proof. We only prove the case of (I) and $\lambda \in \left[\frac{1}{B_1 M}, \frac{\alpha}{A_1 K}\right]$, $\mu \in \left(0, \frac{1-\alpha}{A_2 K}\right]$. The other cases are similar. Let $\Omega_{d_1} = \{(u,v) \in \kappa : \|(u,v)\| < d_1\}$. If $(u,v) \in \partial\Omega_{d_1}$, then $\|(u,v)\| = d_1$. Since $\sigma d_1 \le u + v \le d_1, a \le t \le b$, then we have

$$\begin{aligned}
A_\lambda(u,v)(t) &= \lambda \int_a^b G_m(t,s) f(s, u(s), v(s)) ds \\
&\le \lambda \Big(\frac{b-a}{6}\Big)^{m-1} \int_a^b \frac{(s-a)(b-s)}{b-a} p_1(s) q_1(s, u(s), v(s)) ds \\
&\le \lambda \Big(\frac{b-a}{6}\Big)^{m-1} \int_a^b \frac{(s-a)(b-s)}{b-a} p_1(s) \varphi(d_1) ds \\
&\le \lambda \Big(\frac{b-a}{6}\Big)^{m-1} d_1 K \int_a^b \frac{(s-a)(b-s)}{b-a} p_1(s) ds \\
&\le \lambda A_1 d_1 K \le d_1 \alpha = \alpha \|(u,v)\|, \\
A_\mu(u,v)(t) &= \mu \int_a^b G_n(t,s) g(s, u(s), v(s)) ds \\
&\le \mu \Big(\frac{b-a}{6}\Big)^{n-1} \int_a^b \frac{(s-a)(b-s)}{b-a} p_2(s) q_2(s, u(s), v(s)) ds \\
&\le \mu \Big(\frac{b-a}{6}\Big)^{n-1} \int_a^b \frac{(s-a)(b-s)}{b-a} p_2(s) \varphi(d_1) ds \\
&\le \mu \Big(\frac{b-a}{6}\Big)^{n-1} d_1 K \int_a^b \frac{(s-a)(b-s)}{b-a} p_2(s) ds \\
&\le \mu A_2 d_1 K \le (1-\alpha) d_1 = (1-\alpha)\|(u,v)\|.
\end{aligned}$$

Then

$$\|A(u,v)\| = \|A_\lambda(u,v)\| + \|A_\mu(u,v)\| \le \|(u,v)\|, \text{ for all } (u,v) \in \kappa \cap \partial\Omega_{d_1}. \tag{15}$$

If $(u,v) \in \partial\Omega_{d_2}$, then $\|(u,v)\| = d_2$. Since $\sigma d_2 \le u + v \le d_2,\ t \in \left[\frac{3a+b}{4}, \frac{a+3b}{4}\right]$, we have

$$\begin{aligned}
A_\lambda(u,v)(t) &= \lambda \int_a^b G_m(t,s) f(s, u(s), v(s)) ds \\
&\ge \lambda \theta_m \Big(\frac{3a+b}{4}\Big) \int_{\frac{3a+b}{4}}^{\frac{a+3b}{4}} \frac{(s-a)(b-s)}{b-a} \psi(d_2) ds \\
&= \lambda B_1 d_2 M \ge d_2 = \|(u,v)\|.
\end{aligned}$$

That is

$$\|A(u,v)\| \ge \|(u,v)\|, \text{ for all } (u,v) \in \kappa \cap \partial\Omega_{d_2}. \tag{16}$$

If $(u,v) \in \partial\Omega_{d_3}$, then $\|(u,v)\| = d_3$. Since $\sigma d_3 \le u+v \le d_3,\ a \le t \le b$, we have

$$\begin{aligned}
A_\lambda(u,v)(t) &= \lambda \int_a^b G_m(t,s) f(s,u(s),v(s))ds \\
&\le \lambda\Big(\frac{b-a}{6}\Big)^{m-1} \int_a^b \frac{(s-a)(b-s)}{b-a} p_1(s) q_1(s,u(s),v(s))ds \\
&\le \lambda\Big(\frac{b-a}{6}\Big)^{m-1} \int_a^b \frac{(s-a)(b-s)}{b-a} p_1(s)\varphi(d_3)ds \\
&\le \lambda\Big(\frac{b-a}{6}\Big)^{m-1} d_3 K \int_a^b \frac{(s-a)(b-s)}{b-a} p_1(s)ds \\
&\le \lambda A_1 d_3 K \le d_3\alpha = \alpha\|(u,v)\|,
\end{aligned}$$

$$\begin{aligned}
A_\mu(u,v)(t) &= \mu \int_a^b G_n(t,s) g(s,u(s),v(s))ds \\
&\le \mu\Big(\frac{b-a}{6}\Big)^{n-1} \int_a^b \frac{(s-a)(b-s)}{b-a} p_2(s) q_2(s,u(s),v(s))ds \\
&\le \mu\Big(\frac{b-a}{6}\Big)^{n-1} \int_a^b \frac{(s-a)(b-s)}{b-a} p_2(s)\varphi(d_3)ds \\
&\le \mu\Big(\frac{b-a}{6}\Big)^{n-1} d_3 K \int_a^b \frac{(s-a)(b-s)}{b-a} p_2(s)ds \\
&\le \mu A_2 d_3 K \le (1-\alpha)d_3 = (1-\alpha)\|(u,v)\|.
\end{aligned}$$

Then

$$\|A(u,v)\| = \|A_\lambda(u,v)\| + \|A_\mu(u,v)\| \le \|(u,v)\|, \text{ for all } (u,v) \in \kappa \cap \partial\Omega_{d_3}. \tag{17}$$

From (15), (16), (17) and Lemma 4, the system has at least two positive solutions $(u_1^*, v_1^*) \in \kappa, (u_2^*, v_2^*) \in \kappa$ and $d_1 \le \|(u_1^*, v_1^*)\| < d_2 < \|(u_2^*, v_2^*)\| \le d_3$. □

5. Example

As an example, we consider the existence of positive solutions for the following systems:

$$\begin{cases}
u^{(4)} = \lambda[(u+v)^3 + (u+v)^{\frac{1}{3}}], & t \in [0,1] \\
-v^{(6)} = \mu[(u+v)^2 + (u+v)^{\frac{1}{2}}], & t \in [0,1] \\
u^{(2i)}(0) = u^{(2i)}(1) = 0, & i = 0,1 \\
v^{(2j)}(0) = v^{(2j)}(1) = 0, & j = 0,1,2.
\end{cases} \tag{18}$$

We choose $r_1 = 1, M = 3, K = \frac{16^4 \times 2^{14} \times 2}{11^5}, \alpha = \frac{1}{2}$, then all the conditions in Theorem 4 are satisfied. Therefore, for any $\lambda \in [\frac{99}{128}, 6]$ and $\mu \in (0,36]$ or $\lambda \in (0,6]$ and $\mu \in [27,36]$, (18) has at least two positive solutions $(u_1(t), v_1(t)), (u_2(t), v_2(t))$ with $0 < \|(u_1(t), v_1(t))\| < 1 < \|(u_2(t), v_2(t))\|$.

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