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Majorization theorem for convexifiable functions

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Abstract. In this paper we extend the majorization theorem from convex to covexifiable functions, in particular to smooth functions with Lipschitz derivative, twice continuously differentiable functions and analytic functions.

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1. Introduction

There is a certain intuitive appeal to the vague notion that the components of n-tuple **x** are less spread out, or more nearly equal, than are the components of n-tuple **y**. The notion arises in a variety of contexts, and it can be made precise in a number of ways. But in remarkably many cases, the appropriate statement is that **x** majorizes **y** means that the sum of m largest entries of **y** does not exceed the sum of m largest entries of **x** for all $m = 1, 2, .., n$ with equality for $m = n$. A mathematical origin of majorization is illustrated by the work of Schur [12] on Hadamard's determinant inequality. Many mathematical characterization problems are known to have solutions that involve majorization. A complete and superb reference on the subject are the books [2], [9]. The comprehensive survey by Ando [1] provides alternative derivations, generalizations, and a different viewpoint.

The following theorem is well-known in the literature as the majorization theorem and a convenient references for its proof are ([4, p. 75], [11, p. 320]). This result is due to Karamata [7] and can also be found in [6].

Theorem 1. *Let $\phi : I \to \mathbb{R}$ be a continuous convex function on the interval I and $\boldsymbol{x} = (x_1, .., x_n), \boldsymbol{y} = (y_1, .., y_n)$ two n-tuples such that $x_i, y_i \in I$ $(i = 1, 2, .., n)$. If $\boldsymbol{x}$ majorizes $\boldsymbol{y}$, then the inequality*

$$\sum_{i=1}^{n} \phi(y_i) \leq \sum_{i=1}^{n} \phi(x_i) \tag{1}$$

holds.

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Inequality (1) is known in the litrature as Karamata's inequality. It is a theorem in elementary algebra for convex real-valued functions defined on an interval of the real line and it generalizes the finite form of Jensen's inequality. This majorization ordering is equivalently described in Kemperman's review [8]. An extension of this fact for arbitrary real weights and decreasing n-tuples $\mathbf{x}$ and $\mathbf{y}$ can be found in [5]. General results of this type are due to Dragomir [3] and Niezgoda [10].

We recall some results we will use in further work.

Definition 1 ([16]). *Given a continuous $\phi : I \to \mathbb{R}$ defined on the compact interval $I \subset \mathbb{R}$, consider a function $\varphi : I \times \mathbb{R} \to \mathbb{R}$ defined by $\varphi(x, \alpha) = \phi(x) - \frac{1}{2}\alpha x^2$. If $\varphi(x, \alpha)$ is a convex function on I for some $\alpha = \alpha^*$, then $\varphi(x, \alpha)$ is called a convexification of ϕ and α^* is its convexifier on I. Function ϕ is convexifiable if it has a convexification.*

Remark 1. *If α^* is a convexifier of ϕ, then so is every $\alpha \leq \alpha^*$.*

In order to characterize a convexifiable function, the mid-point acceleration function

$$\Psi(x, y) = \frac{4}{|x - y|} \left(\phi(x) + \phi(y) - 2\phi \left(\frac{x + y}{2} \right) \right), \quad x, y \in I, \ x \neq y$$

was introduced in [16]. There it was shown that a continuous $\phi : I \to \mathbb{R}$ defined on the compact interval $I \subset \mathbb{R}$ is convexifiable on I if and only if its mid-point acceleration function Ψ is bounded from below on $I \times I$.

For two important classes of functions a convexifier α can be given explicitly.

Lemma 1 ([16]). *Given a twice continuously differentiable function $\phi : I \to \mathbb{R}$ on a compact interval I in $\mathbb{R}$. Then $\lambda^* = \min_{x \in I} \phi''(x)$ is a convexifier.*

We say that a continuously differentiable function ϕ has Lipschitz derivative if $|\phi'(x) - \phi'(y)| \leq L|x - y|$ for every $x, y \in I$ and some constant L.

Lemma 2 ([16]). *Given a continuously differentiable function $\phi : I \to \mathbb{R}$ with Lipschitz derivative and a Lipschitz constant L on a compact interval I in $\mathbb{R}$. Then $\alpha = -L$ is a convexifier.*

One can show that every convexifiable function $\phi : I \to \mathbb{R}$ is Lipschitz. This means that a scalar non-Lipschitz function is not convexifiable. However, almost all smooth functions of practical interest are convexifiable; e.g., [16]. In [16], Zlobec gave discrete and integral Jensen's inequality for convexifiable functions and many other interesting results for these functions, e.g., see [13 - 17].

In this note, inequality (1) and its weighted version are extended from convex to convexifiable functions. These include all twice continuously differentiable functions, all once continuously differentiable functions with Lipschitz derivative and all analytic functions.

2. Main results

In this section we extend Karamata's inequality and its weighted version to convexifiable functions.

Theorem 2. *Let $\phi : I \to \mathbb{R}$ be a continuous convexifiable function on the compact interval I and α its convexifier. Let $\boldsymbol{x} = (x_1, ..., x_n)$, $\boldsymbol{y} = (y_1, ..., y_n)$ be two n-tuples such that $x_i,\ y_i \in I$ $(i = 1, ..., n)$ and $\boldsymbol{x}$ majorizes $\boldsymbol{y}$. Then*

$$\sum_{i=1}^{n} \phi(y_i) \le \sum_{i=1}^{n} \phi(x_i) - \frac{\alpha}{2} \sum_{i=1}^{n} \left(x_i^2 - y_i^2\right) \tag{2}$$

Proof. Since ϕ is convexifiable with convexifier α, so $\varphi(x, \alpha) = \phi(x) - \frac{1}{2}\alpha x^2$ is a convex function and $\mathbf{x}$ majorizes $\mathbf{y}$. Therefore by using $\varphi(x, \alpha)$ instead of $\phi(x)$ in (1) we obtain (2). □

Using the fact that for a convex function ϕ one can choose the convexifier $\alpha = 0$, one recovers (1). For a twice continuously differentiable function one can specify $\alpha = \lambda^*$ (by Lemma 1) and for a continuously differentiable function with Lipschitz derivative and its Lipschitz constant L, one can specify $\alpha = -L$ (by Lemma 2).

Remark 2. *If in (2) for $\alpha = \lambda^*$ the function ϕ is convex, then $\lambda^* \ge 0$ and by utilizing $\phi(x) = x^2$ in (1) we obtain $\sum_{i=1}^{n} \left(x_i^2 - y_i^2\right) \ge 0$. So (2) may provide a better bound than (1). Since every analytic function $\phi : I \to \mathbb{R}$ is twice continuously differentiable, (2) holds, in particular, for analytic functions with $\lambda^* = \min\limits_{x \in I} \phi''(x)$.*

In the following theorem we extend Fuch's [5] result for convexifiable function.

Theorem 3. *Let $\phi : I \to \mathbb{R}$ be a continuous convexifiable function on the compact interval I and α be its convexifier. Let $\boldsymbol{x} = (x_1, ..., x_n)$, $\boldsymbol{y} = (y_1, ..., y_n)$ be two decreasing n-tuples such that $x_i,\ y_i \in I$ $(i = 1, ..., n)$, $\boldsymbol{p} = (p_1, p_2, ..., p_n)$ be a real n-tuple such that*

$$\sum_{i=1}^{k} p_i\, y_i \le \sum_{i=1}^{k} p_i\, x_i \ \ \text{for}\ \ k = 1, ..., n-1, \tag{3}$$

and

$$\sum_{i=1}^{n} p_i\, y_i = \sum_{i=1}^{n} p_i\, x_i. \tag{4}$$

Then

$$\sum_{i=1}^{n} p_i \phi(y_i) \le \sum_{i=1}^{n} p_i \phi(x_i) - \frac{\alpha}{2} \sum_{i=1}^{n} p_i \left(x_i^2 - y_i^2\right). \tag{5}$$

Remark 3. *By putting different conditions on the n-tuples $\boldsymbol{x}, \boldsymbol{y}$ and $\boldsymbol{p}$, weighted versions of inequality (1) and their integral versions have been proved e.g [3, 5, 10] and some of the reference therein. Therefore similarly to Theorem 3 we can extend all such results for convexifiable functions.*

Remark 4. *By setting $y_i = \frac{1}{n}\sum_{i=1}^{n} x_i$ $(i = 1, 2, ..., n)$ in (1) we can obtain Jensen's inequality for convex function. So from (2) and its integral version we can obtain the inequalities obtained in [14, 15].*

In the following example we illustrate the basic difference between Karamata's inequality for a convex and a convexifiable function.

Example 1. *Consider* $\phi(t) = \sin t$, $t \in [-\pi, \pi]$ *and* $\boldsymbol{x} = (\pi, \frac{\pi}{2}, -\frac{\pi}{2})$, $\boldsymbol{y} = (\frac{3\pi}{4}, \frac{\pi}{4}, 0)$. *Then* $\boldsymbol{x}$ *majorizes* $\boldsymbol{y}$. *By using these 3-tuples* $\boldsymbol{x}, \boldsymbol{y}$ *and the function* ϕ *in (1) we have* $\sqrt{2} \leq 0$, *i.e inequality (1) is not satisfied as the function* ϕ *is not convex on* $[-\pi, \pi]$. *On the other hand, the function* ϕ *is convexifiable and its convexification is* $\varphi(t, \alpha) = \sin t - \frac{1}{2}\alpha t^2$. *Now using these in (2) we obtain the inequality* $\sqrt{2} \leq -\frac{7\alpha\pi^2}{16}$ *which is valid for any convexifier* $\alpha \leq -0.33$.

A situation where the new bound is sharper than the one provided by Karamata's inequality for a convex function is illustrated in the following example.

Example 2. *Consider* $\phi(t) = t^4$, $t \in [0, 3]$ *and* $\boldsymbol{x} = (2+\lambda, 2-\lambda, \lambda)$, $\boldsymbol{y} = (2, 1+\lambda, 1)$, $\lambda \in [0, 1]$. *Then* $\boldsymbol{x}$ *majorizes* $\boldsymbol{y}$. *Using these* $\boldsymbol{x}, \boldsymbol{y}$ *and the function* $\phi(t) = t^4$ *in (1) and in its extension (2) yield* $(1+\lambda)^4 + 17 \leq (2+\lambda)^4 + (2-\lambda)^4 + \lambda^4$ *and* $(1+\lambda)^4 + 17 \leq (2+\lambda)^4 + (2-\lambda)^4 + \lambda^4 - 6((2+\lambda)^2 + (2-\lambda)^2 - 2\lambda - 6)$, *respectively. The upper bounds are compared to the original function and the new bound is better than the old bound for a convex function.*

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