Math. Commun. **18**(2013), 87–96

Approximation to minimum-norm common fixed point of a semigroup of nonexpansive operators

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Received January 21, 2012; accepted November 28, 2012

Abstract. The purpose of this paper is to introduce a new iterative algorithm for a semigroup of nonexpansive operators in Hilbert space. We prove that the proposed iterative algorithm converges strongly to the minimum-norm common fixed point of the semigroup of nonexpansive operators. The results of this paper extend and improve some known results in the literature.

AMS subject classifications: 41A65, 47H20

Key words: common fixed point, minimum-norm, semigroup, nonexpansive operators

1. Introduction

Many problems in various branches of mathematical and physical sciences can be reduced to finding a common fixed point in a given family of mappings. It is usually called the common fixed point problem (hereinafter referred to as: CFPP), that is

$$\text{Find } \; x \in F := \bigcap_{i \in I} Fix(T_i) \neq \emptyset, \tag{1}$$

where $Fix(T_i)$ denotes the fixed point set of T_i and I denotes the index of mappings T_i. For example, if we take $T_i = P_{C_i}$, for each $i \in I$, then the common fixed point problem becomes a well-known convex feasibility problem (CFP) of finding $x \in \bigcap_{i \in I} C_i \neq \emptyset$, where each C_i is a nonempty closed convex subset of Hilbert space H and $P_\Omega(x)$ is an orthogonal projection of a point $x \in H$ onto a closed convex set $\Omega \subseteq H$ which is defined by

$$P_\Omega(x) := \arg\min\{\|x - z\| \mid z \in \Omega\}, \tag{2}$$

where $\|\cdot\|$ denotes the norm in H. A complete and exhaustive study on algorithms and applications for solving the convex feasibility problem can be found in [3].

Throughout the paper, we always assume that $F \neq \emptyset$. Many iterative algorithms have appeared to solve the CFPP (1). For a finite family of firmly nonexpansive

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mappings $\{T_i\}_{i\in I}$, where $I = \{1, 2, \cdots, N\}$, $N \geq 1$ is an integer. Combettes [7] introduced a simultaneous iterative algorithm as follows:

$$x_{n+1} = \alpha_n x_0 + (1-\alpha_n)\left(\lambda \sum_{i\in I} \omega_i T_i(x_n) + (1-\lambda)x_n\right), \quad n \geq 0, x_0 \in C, \tag{3}$$

where $\{\alpha_n\} \subset (0,1)$ satisfies

$$(i)\ \lim_{n\to\infty} \alpha_n = 0, \quad (ii)\ \sum_{n=0}^{\infty} \alpha_n = +\infty, \quad (iii)\ \sum_{n=0}^{\infty} |\alpha_{n+1} - \alpha_n| < +\infty. \tag{4}$$

$\omega_i \in (0,1]$ for all $i \in I$, $\sum_{i\in I} \omega_i = 1$ and $0 < \lambda \leq 2$. Meanwhile, he defined a sequential algorithm by

$$x_{n+1} = \alpha_n x_0 + (1-\alpha_n)(T_1 \cdots T_N)(x_n), \quad n \geq 0, x_0 \in C, \tag{5}$$

where $\{\alpha_n\}$ is as in (4). He showed that any sequence $\{x_n\}_{n\geq 0}$ generated by both algorithms (3) and (5) converges strongly to $P_F x_0$. Since every firmly nonexpansive mapping is nonexpansive, Bauschke [2] proposed a sequential method to find the common fixed point of a finite family of nonexpansive mappings. This iterative algorithm has the following form.

$$x_{n+1} = \alpha_n u + (1-\alpha_n) T_{[n]} x_n, \quad n \geq 0, u, x_0 \in C, \tag{6}$$

where $[n] = n(mod\ N) + 1$, the mod N function takes values in $\{1, 2, \cdots, N\}$. He proved the sequence generated by (6) converges in norm to $P_F u$ under assumptions on the mappings that

$$F = Fix(T_N \cdots T_1) = Fix(T_1 T_N \cdots T_3 T_2) = \cdots = Fix(T_{N-1} T_{N-2} \cdots T_1 T_N), \tag{7}$$

and $\{\alpha_n\}$ is a sequence of parameters in $(0,1)$ which satisfies the following:

$$(i)\ \lim_{n\to\infty} \alpha_n = 0, \quad (ii)\ \sum_{n=0}^{\infty} \alpha_n = +\infty, \quad (iii)\ \sum_{n=0}^{\infty} |\alpha_n - \alpha_{n+N}| < +\infty. \tag{8}$$

Remark 1. *If $\{T_i\}_{i\in I}$ is a family of firmly nonexpansive mappings, then condition (7) is naturally met (see Proposition 2.2 of [6]). Even to nonexpansive mappings, by the results of [13] and [9], assumption (7) can be simplified by*

$$F = Fix(T_N \cdots T_1). \tag{9}$$

On the other hand, if I is a countable infinite set, Shimoji and Takahashi [11] investigated the following iterative algorithm.

$$x_{n+1} = \alpha_n x_0 + (1-\alpha_n) W_n(\lambda_n x_0 + (1-\lambda_n)x_n), \quad n \geq 0, x_0 \in C, \tag{10}$$

where W_n is a W-mapping defined by (16) below, $\{\alpha_n\} \subset (0,1)$ and $\{\lambda_n\} \subset (0,1)$ satisfy $\lim_{n\to\infty} \alpha_n = \lim_{n\to\infty} \lambda_n = 0$, $\prod_{n=0}^{\infty}(1-\alpha_n)(1-\lambda_n) = 0$ and $\sum_{n=0}^{\infty}(|\alpha_n -$

$\alpha_{n+1}| + |\lambda_n - \lambda_{n+1}|) < +\infty$. They proved the sequence $\{x_n\}_{n\geq 0}$ converges strongly to $P_F x_0$. When I is an unbounded subset of $\mathbb{R}_+$, where $\mathbb{R}_+$ denotes the set of non-negative real numbers. Aleyner and Censor [1] introduced the following algorithm for a family of nonexpansive semigroups $\{T_t \mid t \in I\}$.

$$x_{n+1} = \alpha_n u + (1 - \alpha_n) T_{r_n} x_n, \quad n \geq 0, u, x_0 \in C, \tag{11}$$

where $\{\alpha_n\} \subset (0,1)$ satisfies the condition as in (4) and $\{r_n\}_{n\geq 0} \subset I$ is some given sequence. If $\{T_t \mid t \in I\}$ is a uniformly asymptotically regular semigroup of a nonexpansive operator, they proved the sequence $\{x_n\}_{n\geq 0}$ converges strongly to $P_F u$. Suzuki [14] proved the sequence $\{x_n\}_{n\geq 0}$ generated by (11) converges strongly to $P_F u$ with an assumption that $\{T_t \mid t \in I\}$ is a one-parameter nonexpansive semigroup and the sequences $\{\alpha_n\}$ and $\{r_n\}$ satisfying

(i) $0 < \alpha_n < 1, 0 \leq r_n$ and $s_n := \liminf_{m\to\infty} |t_m - t_n| > 0$, for any $n \geq 0$;

(ii) $\{r_n\}$ is bounded;

(iii) $\lim_{n\to\infty} \alpha_n / s_n = 0$,

since these iterative algorithms not only have strong convergence, but also converge to the projection of the starting point x_0 or any point u onto F. In contrast to the common fixed point problem, it is in addition called the best approximation problem with respect to F. Consider the projection operator $P_F x$

$$P_F x = \arg\min\{\|x - z\| \mid z \in F\},$$

where F is as in (1). Define $x^* := P_F 0 = \arg\min\{\|z\| \mid z \in F\}$, i.e., x^* is the minimum-norm common fixed point of F. If $0 \in C$, then the iterative algorithms (3), (5), (6), (10) and (11) do the job to find the minimum-norm common fixed point of $\bigcap_{i\in I} Fix(T_i)$. In fact, one can let $x_0 = 0$ or $u = 0$. However, if $0 \notin C$, then none of these algorithms work to find the minimum-norm element of F. In order to overcome this difficulty caused by possible exclusion of the origin from C, some authors have applied the metrical projection P_C on the right-hand side of the iterative algorithm (see for example [6, 10 – 12]). The role of the metrical projection P_C is to pull the substituted sequence back to C, then the iterative sequences are well-defined. In these works, Liu and Cui [9] proposed two iterative algorithms, one was sequential; the other is simultaneous.

(i) The sequential method.

$$x_{n+1} = P_C\left((1 - t_n) T_{[n+1]} x_n\right), \quad n \geq 0, x_0 \in C, \tag{12}$$

where $\{t_n\} \subset (0,1)$ satisfies the following properties: (i) $\lim_{n\to\infty} t_n = 0$; (ii) $\sum_{n=0}^{\infty} t_n = +\infty$; (iii) either $\sum_{n=0}^{\infty} |t_n - t_{n+N}| < +\infty$ or $\lim_{n\to\infty} t_n / t_{n+N} = 1$. $T_{[n]} := T_{n \bmod N}$ with the mod N function taking values in the set $\{1, 2, \cdots, N\}$.

(ii) The simultaneous method.

$$x_{n+1} = P_C\left((1 - t_n) \sum_{i=1}^{N} \lambda_i^{(n)} T_i x_n\right), \quad n \geq 0, x_0 \in C, \tag{13}$$

where $\lambda_i^{(n)} > 0$ for all $n \geq 0$, $i = 1, 2, \cdots, N$, and $\sum_{i=1}^{N} \lambda_i^{(n)} = 1$ for all n and satisfy (i) $\sum_{n=0}^{\infty} \sum_{i=1}^{N} |\lambda_i^{(n+1)} - \lambda_i^{(n)}| < +\infty$, $\inf_{n\geq 0} \lambda_i^{(n)} > 0$ for all i; (ii) $\lim_{n\to\infty} t_n = 0$ and $\sum_{n=0}^{\infty} t_n = +\infty$; (iii) either $\sum_{n=0}^{\infty} |t_{n+1} - t_n| < +\infty$ or $\lim_{n\to\infty}(t_n/t_{n+1}) = 1$. Assume that $\{T_i\}_{i=1}^{N}$ satisfy condition (9), they proved that the sequence $\{x_n\}_{n\geq 0}$ generated by the sequential method and the simultaneous method converge strongly to the minimum-norm common fixed point of the mappings $\{T_i\}_{i=1}^{N}$.

Motivated and inspired by the above works, we introduce a new iterative algorithm for finding the minimum-norm common fixed point of a nonexpansive semigroup $\{T_t \mid t \in I\}$. The proposed algorithm combines the iterative algorithm given by Aleyner and Censor [1] and Liu and Cui [9]. The sequence $\{x_n\}$ is generated by the following recursive.

$$x_{n+1} = (1 - \alpha_n)x_n + \alpha_n P_C((1 - t_n)T_{r_n} x_n), n \geq 0, x_0 \in C, \tag{14}$$

where the parameters $\{\alpha_n\}$ and $\{t_n\}$ are sequences in $(0, 1)$, $\{r_n\}_{n\geq 0} \subset I$ is some given sequence. Furthermore, we present a new way to prove the strong convergence of the iterative algorithm (14) under a mild assumption on the parameters and its limit is also the minimum-norm common fixed point of a nonexpansive semigroup $\{T_t \mid t \in I\}$.

2. Preliminaries

In this section we present definitions and some tools that will be used later on in the proof of our main theorem. Throughout this paper, by $\mathbb{R}$ we denote the set of real numbers and by $\mathbb{R}_+$ the set of nonnegative real numbers. Let H be a Hilbert space with inner product $\langle \cdot, \cdot \rangle$ and norm $\| \cdot \|$, respectively. In a Hilbert space, it is known that for all $x, y \in H$ and $\alpha \in \mathbb{R}$,

$$\|\alpha x + (1 - \alpha)y\|^2 = \alpha\|x\|^2 + (1 - \alpha)\|y\|^2 - \alpha(1 - \alpha)\|x - y\|^2. \tag{15}$$

Recall that the orthogonal projection $P_C x$ of x onto C is defined by the following

$$P_C x = \arg \min_{y\in C} \|x - y\|.$$

The orthogonal projection has the following well-known properties. For a given $x \in H$,

(i) $\langle x - P_C x, z - P_C x\rangle \leq 0$, for all $z \in C$;

(ii) $\|P_C x - P_C y\|^2 \leq \langle P_C x - P_C y, x - y\rangle$, for all $x, y \in H$.

In what follows, we give some definitions and lemmas.

Definition 1. *Let C be a nonempty closed convex subset of H. $T : C \to C$ is called*

(i) nonexpansive if $\|Tx - Ty\| \leq \|x - y\|$, for all $x, y \in C$,

(ii) firmly nonexpansive if $\|Tx - Ty\|^2 \le \langle x - y, Tx - Ty\rangle$*, for all* $x, y \in C$*.*

Remark 2. *It is easy to see that the projection operator is firmly nonexpansive, and the firmly nonexpansive mapping is a nonexpansive mapping. The relations between them can be expressed as the visual picture.*

$$\text{Projection operator} \Longrightarrow \text{Firmly nonexpansive} \Longrightarrow \text{Nonexpansive}$$

Definition 2 (See [11])**.** *Let* C *be a nonempty closed convex subset of Banach space* E*. Let* $\{T_i\}_{i=1}^{\infty}$ *be infinite mappings of* C *into themselves and let* $\alpha_1, \alpha_2, \cdots$ *be real numbers such that* $0 \le \alpha_i \le 1$ *for every* i*. For any* $n \ge 1$*, define a mapping* W_n *of* C *into itself as follows:*

$$\begin{aligned} U_{n,n+1} &= I, \\ U_{n,n} &= \alpha_n T_n U_{n,n+1} + (1-\alpha_n)I, \\ U_{n,n-1} &= \alpha_{n-1} T_{n-1} U_{n,n} + (1-\alpha_{n-1})I, \\ &\vdots \\ U_{n,k} &= \alpha_k T_k U_{n,k+1} + (1-\alpha_k)I, \\ U_{n,k-1} &= \alpha_{k-1} T_{k-1} U_{n,k} + (1-\alpha_{k-1})I, \\ &\vdots \\ U_{n,2} &= \alpha_2 T_2 U_{n,3} + (1-\alpha_2)I, \\ W_n = U_{n,1} &= \alpha_1 T_1 U_{n,2} + (1-\alpha_1)I, \end{aligned} \tag{16}$$

where I *is the identity mapping. Such a mapping* W_n *is called a* W*-mapping generated by* $T_n, T_{n-1}, \cdots, T_1$ *and* $\alpha_n, \alpha_{n-1}, \cdots, \alpha_1$*.*

A semigroup of nonexpansive operators could be recognized as special families of nonexpansive operators, see [10] and others.

Definition 3. *Let* I *be an unbounded subset of* $\mathbb{R}_+$ *such that*
(i) $t + s \in I$*, for all* $t, s \in I$*,* *(ii)* $t - s \in I$*, for all* $t, s \in I$
with $t \ge s$*, and let* $\Gamma = \{T_t \mid t \in I\}$ *be a family of self-operators on a nonempty closed convex subset* C *of* H*. The family* Γ *is called a semigroup of nonexpansive operators on* C *if the following conditions hold:*

(i) T_t *is a nonexpansive self-operator on* C*, for all* $t \in I$*,*

(ii) $T_{t+s}x = T_t T_s x$*, for all* $t, s \in I$ *and all* $x \in C$*.*

In addition,

(iii) for each $x \in C$*, the mapping* $t \mapsto T_t x$ *from* $[0, +\infty)$ *into* C *is strongly continuous.*

Then the family of mappings $\{T_t \mid t \in I\}$ *is called a one-parameter strongly continuous semigroup of nonexpansive mappings (a one-parameter nonexpansive semigroup, for short).*

The concept of a uniformly asymptotically regular semigroup of nonexpansive operators can be found in [4, 5].

Definition 4. *Let $\Gamma = \{T_t \mid t \in I\}$ be a semigroup of nonexpansive operators on a nonempty closed convex subset C of H. The family Γ is called a uniformly asymptotically regular semigroup of nonexpansive operators on C if*

$$\lim_{r\to\infty}\left(\sup_{x\in C}\|T_sT_rx - T_rx\|\right) = 0, \tag{17}$$

uniformly for all $s \in I$.

As a matter of fact, condition (17) implies that there exists a monotone sequence $\{r_n\}_{n\geq 0} \subseteq I$ such that

$$0 \leq r_0 \leq r_1 \leq \cdots \leq r_n \leq \cdots, \text{ and } \lim_{n\to\infty} r_n = \infty, \tag{18}$$

and

$$\sum_{n=0}^{\infty}\sup_{x\in C}\|T_sT_{r_n}x - T_{r_n}x\| < +\infty, \tag{19}$$

uniformly for all $s \in I$.

The following demiclosedness principle of a nonexpansive mapping played an important role in our work. We denote strong or weak convergence by " $\to$ " or " $\rightharpoonup$ ", respectively.

Lemma 1. *Let $T : C \to C$ a nonexpansive mapping with $Fix(T) \neq \emptyset$. If $x_n \rightharpoonup x$ and $(I - T)x_n \to 0$, then $x = Tx$.*

In order to prove the main results in this paper, we shall make use of the following lemmas.

Lemma 2 (See [12]). *Let $\{x_n\}$ and $\{y_n\}$ be bounded sequences in a Banach space E and let $\{\beta_n\}$ be a sequence in $[0,1]$ with $0 < \liminf_{n\to\infty}\beta_n \leq \limsup_{n\to\infty}\beta_n < 1$. Suppose $x_{n+1} = \beta_n y_n + (1-\beta_n)x_n$ for all $n \geq 0$ and*

$$\limsup_{n\to\infty}\left(\|y_{n+1} - y_n\| - \|x_{n+1} - x_n\|\right) \leq 0.$$

Then $\lim_{n\to\infty}\|y_n - x_n\| = 0$.

Lemma 3 (See [15]). *Let $\{a_n\}$ be a sequence of nonnegative real sequences satisfying the following inequality:*

$$a_{n+1} \leq (1-\gamma_n)a_n + \gamma_n\delta_n, n \geq 0,$$

where $\{\gamma_n\}$ is a sequence in $(0,1)$ and $\{\delta_n\}$ is a sequence such that

(1) $\sum_{n=0}^{\infty}\gamma_n = +\infty$;

(2) $\limsup_{n\to\infty}\delta_n \leq 0$ *or* $\sum_{n=0}^{\infty}|\gamma_n\delta_n| < +\infty$.

Then $\lim_{n\to\infty} a_n = 0$.

3. Main results

The main result of our work is the next convergence theorem for the iterative algorithm (14). Now, we are in the position to prove the following theorem.

Theorem 1. *Let C be a nonempty closed convex subset of a real Hilbert space H. Let $\Gamma = \{T_t \mid t \in I\}$ be a uniformly asymptotically regular semigroup of nonexpansive operators on C such that $F := \bigcap_{t\in I} Fix(T_t) \neq \emptyset$. Let the sequence $\{x_n\}_{n\geq 0}$ be generated by the iterative algorithm (14), where $\{\alpha_n\}$ and $\{t_n\} \subset (0,1)$ satisfy the conditions:*

(i) $\lim_{n\to\infty} t_n = 0$, $\sum_{n=0}^{\infty} t_n = +\infty$;

(ii) $0 < \liminf_{n\to\infty} \alpha_n \leq \limsup_{n\to\infty} \alpha_n < 1$.

Then any sequence $\{x_n\}_{n\geq 0}$ generated by (14) converges strongly to the minimum-norm common fixed point of F.

Proof. We divide the proof into five steps.

Step 1. We prove that the sequence $\{x_n\}_{n\geq 0}$ is bounded. In fact, take $p \in F$, by (14), we have

$$\begin{aligned}
\|x_{n+1} - p\| &= \|(1-\alpha_n)(x_n - p) + \alpha_n(P_C((1-t_n)T_{r_n}x_n) - p)\| \\
&\leq (1-\alpha_n)\|x_n - p\| + \alpha_n\|(1-t_n)T_{r_n}x_n - p\| \\
&\leq (1-\alpha_n)\|x_n - p\| + \alpha_n(1-t_n)\|x_n - p\| + \alpha_n t_n\|p\| \\
&= (1-\alpha_n t_n)\|x_n - p\| + \alpha_n t_n\|p\| \\
&\leq \max\{\|x_n - p\|, \|p\|\}.
\end{aligned}$$

By induction, we get

$$\|x_n - p\| \leq \max\{\|x_0 - p\|, \|p\|\}, \quad \text{for all } n \geq 0.$$

Hence, $\{x_n\}$ is bounded. So is the sequence $\{T_{r_n}x_n\}$. Let $M > 0$, such that $M \geq \sup_{n\geq 0}\{\|x_n\|, \|T_{r_n}x_n\|\}$.

Set $z_n := P_C((1-t_n)T_{r_n}x_n)$, we obtain

$$\begin{aligned}
\|z_n - p\| &= \|P_C((1-t_n)T_{r_n}x_n) - p\| \\
&\leq \|(1-t_n)T_{r_n}x_n - p\| \\
&\leq (1-t_n)\|x_n - p\| + t_n\|p\| \\
&\leq \max\{\|x_n - p\|, \|p\|\}.
\end{aligned}$$

Since $\{x_n\}$ is bounded, we get that $\{z_n\}$ is also bounded.

Step 2. We show that $\|x_{n+1} - x_n\| \to 0$ as $n \to \infty$. Let $\widetilde{C}$ be any bounded subset of C which contains the sequence $\{x_n\}_{n\geq 0}$. Since $z_n = P_C((1-t_n)T_{r_n}x_n)$, we get

$$\begin{aligned}\|z_{n+1}-z_n\| =&\|P_C((1-t_{n+1})T_{r_{n+1}}x_{n+1})-P_C((1-t_n)T_{r_n}x_n)\|\\ \le&\|(1-t_{n+1})T_{r_{n+1}}x_{n+1}-(1-t_n)T_{r_n}x_n\|\\ \le&\|(1-t_{n+1})T_{r_{n+1}}x_{n+1}-(1-t_{n+1})T_{r_{n+1}}x_n\|\\ &+\|(1-t_{n+1})T_{r_{n+1}}x_n-(1-t_n)T_{r_n}x_n\|\\ \le&(1-t_{n+1})\|x_{n+1}-x_n\|+\|(1-t_{n+1})T_{r_{n+1}}x_n-(1-t_n)T_{r_{n+1}}x_n\|\\ &+\|(1-t_n)T_{r_{n+1}}x_n-(1-t_n)T_{r_n}x_n\|\\ \le&(1-t_{n+1})\|x_{n+1}-x_n\|+|t_n-t_{n+1}|M+(1-t_n)\|T_{r_{n+1}}x_n-T_{r_n}x_n\|.\end{aligned} \tag{20}$$

Since Γ is a semigroup, and by using (18), we are able to rewrite the last term as follows

$$\|T_{r_{n+1}}x_n-T_{r_n}x_n\|=\|T_{r_{n+1}-r_n}T_{r_n}x_n-T_{r_n}x_n\|.$$

It follows that

$$\|z_{n+1}-z_n\|-\|x_{n+1}-x_n\|\le|t_n-t_{n+1}|M+(1-t_n)\sup_{x\in\widetilde{C}}\|T_{r_{n+1}-r_n}T_{r_n}x_n-T_{r_n}x_n\|.$$

By using (19) and condition (i), we deduce that

$$\limsup_{n\to\infty}(\|z_{n+1}-z_n\|-\|x_{n+1}-x_n\|)\le 0.$$

With the help of Lemma 2, we get

$$\lim_{n\to\infty}\|x_n-z_n\|=0.$$

Hence, from (14), we have

$$\lim_{n\to\infty}\|x_{n+1}-x_n\|=\lim_{n\to\infty}\alpha_n\|x_n-z_n\|=0.$$

Step 3. We show that for each fixed $s\in I$, $\|T_sx_n-x_n\|\to 0$ as $n\to\infty$. In fact,

$$\begin{aligned}\|x_n-T_{r_n}x_n\|\le&\|x_n-x_{n+1}\|+\|x_{n+1}-T_{r_n}x_n\|\\ \le&\|x_n-x_{n+1}\|+(1-\alpha_n)\|x_n-T_{r_n}x_n\|\\ &+\alpha_n\|P_C((1-t_n)T_{r_n}x_n)-T_{r_n}x_n\|\\ \le&\|x_n-x_{n+1}\|+(1-\alpha_n)\|x_n-T_{r_n}x_n\|+\alpha_nt_nM,\end{aligned}$$

which implies that

$$\|x_n-T_{r_n}x_n\|\le\frac{\|x_n-x_{n+1}\|}{\alpha_n}+t_nM\to 0 \quad \text{as} \quad n\to\infty. \tag{21}$$

On the other hand, by using (19) and (21), we have

$$\begin{aligned}\|T_sx_n-x_n\|&\le\|T_sx_n-T_sT_{r_n}x_n\|+\|T_sT_{r_n}x_n-T_{r_n}x_n\|+\|T_{r_n}x_n-x_n\|\\ &\le 2\|x_n-T_{r_n}x_n\|+\sup_{x\in\widetilde{C}}\|T_sT_{r_n}x_n-T_{r_n}x_n\|\to 0 \quad \text{as } n\to\infty.\end{aligned} \tag{22}$$

Step 4. We prove that $\limsup_{n\to\infty}\langle x^*-x_n, x^*\rangle \le 0$, where $x^* = P_F 0$. Indeed, we can choose a subsequence $\{x_{n_j}\}$ of $\{x_n\}$ such that

$$\limsup_{n\to\infty}\langle x^*-x_n, x^*\rangle = \lim_{j\to\infty}\langle x^*-x_{n_j}, x^*\rangle.$$

Since $\{x_{n_j}\}$ is bounded, there exists a subsequence of $\{x_{n_j}\}$ which converges weakly to a point $\tilde{x}$. Without loss of generality, we may assume that $\{x_{n_j}\}$ converges weakly to $\tilde{x}$. Therefore, from (22) and Lemma 1, we have $x_{n_j} \rightharpoonup \tilde{x} \in F$. Since $x^* = P_F 0$, it follows from the properties of the projection operator that

$$\limsup_{n\to\infty}\langle x^*-x_n, x^*\rangle = \langle x^*-\tilde{x}, x^*\rangle \le 0. \tag{23}$$

Step 5. Finally, we prove that $x_n \to x^*$. We observe that

$$\begin{aligned}\langle x^*-T_{r_n}x_n, x^*\rangle &= \langle x^*-x_n, x^*\rangle + \langle x_n - T_{r_n}x_n, x^*\rangle\\ &\le \langle x^*-x_n, x^*\rangle + \|x_n - T_{r_n}x_n\|\|x^*\|.\end{aligned}$$

Taking the limsup on both sides of the above inequality, and together with (21), (23), we get

$$\limsup_{n\to\infty}\langle x^*-T_{r_n}x_n, x^*\rangle \le 0.$$

From (15) and (14), we have

$$\begin{aligned}\|x_{n+1}-x^*\|^2 &= \|(1-\alpha_n)(x_n-x^*) + \alpha_n(P_C((1-t_n)T_{r_n}x_n) - x^*)\|^2\\ &\le (1-\alpha_n)\|x_n-x^*\|^2 + \alpha_n\|P_C((1-t_n)T_{r_n}x_n) - x^*\|\\ &\le (1-\alpha_n)\|x_n-x^*\|^2 + \alpha_n\|(1-t_n)(T_{r_n}x_n - x^*) - t_n x^*\|^2\\ &= (1-\alpha_n)\|x_n-x^*\|^2 + \alpha_n(1-t_n)^2\|T_{r_n}x_n - x^*\|^2\\ &\quad + 2\alpha_n(1-t_n)t_n\langle x^*-T_{r_n}x_n, x^*\rangle + \alpha_n t_n^2\|x^*\|^2\\ &\le (1-\alpha_n t_n)\|x_n-x^*\|^2 + 2\alpha_n(1-t_n)t_n\langle x^*-T_{r_n}x_n, x^*\rangle + \alpha_n t_n^2\|x^*\|^2.\end{aligned}$$

It is clear that all conditions of Lemma 3 are satisfied. Therefore, we immediately deduce that $x_n \to x^*$ as $n \to \infty$. This completes the proof. □

Remark 3. *Theorem 1 improves the results of Aleyner and Censor [1] by discarding the assumption that "$\sum_{n=0}^{\infty}|\alpha_{n+1}-\alpha_n| < +\infty$". The proposed iterative algorithm (14) is a sequential algorithm which combines the Krasnoselskii-Mann algorithm with iterative algorithm (12) for solving the minimum-norm common fixed point problem with respect to the common fixed point set of infinitely countable or non-countable families of nonexpansive mappings in a real Hilbert space. Therefore, Theorem 1 also generalizes the corresponding results of Liu and Cui [9] and removes the conditions on $\{t_n\}$ that "$\sum_{n=0}^{\infty}|t_n - t_{n+N}| < +\infty$ or $\lim_{n\to\infty} t_n/t_{n+N} = 1$".*

Acknowledgment

The author would like to thank the reviewers and the editor for their helpful suggestions. This work was supported partly by the National Natural Science Foundations of China (11201216), the Natural Science Foundations of Jiangxi Province

(20114BAB201004) and the Youth Science Funds of The Education Department of Jiangxi Province (GJJ12141).

References

[1] A. Aleyner, Y. Censor, *Best approximation to common fixed points of a semigroup of nonexpansive operators*, J. Nonlinear Convex Anal. **61**(2005), 137–151.

[2] H. H. Bauschke, *The approximation of fixed points of compositions of nonexpansive mappings in Hilbert space*, J. Math. Anal. Appl. **202**(1996), 150–159.

[3] H. H. Bauschke, J. M. Borwein, *On projection algorithms for solving convex feasibility problems*, SIAM Review **38**(1996), 367–426.

[4] M. Budzyńska, T. Kuczumov, S. Reich, *Uniform asymptotic normal structure, the uniform semi-opial property, and fixed points of asymptotically regular uniformly lipschitzian semigroups, Part I*, Abstr. Appl. Anal. **3**(1998), 133–151.

[5] M. Budzyńska, T. Kuczumov, S. Reich, *Uniform asymptotic normal structure, the uniform semi-opial property, and fixed points of asymptotically regular uniformly lipschitzian semigroups, Part II*, Abstr. Appl. Anal. **3**(1998), 247–263.

[6] C. Byrne, *A unified treatment of some iterative algorithms in signal processing and image reconstruction*, Inverse Probl. **20**(2004), 103–120.

[7] P. L. Combettes, *The convex feasibility problem in image recovery*, Adv. Imaging Electron Phys. **95**(1996), 155–270.

[8] Y. L. Cui, X. Liu, *Notes on Browder's and Halpern's methods for nonexpansive mappings*, Fixed Point Theory **10**(2009), 89–98.

[9] Y. L. Cui, X. Liu, *The common minimal-norm fixed point of a finite family of nonexpansive mappings*, Nonlinear Anal. **73**(2010), 76–83.

[10] S. Reich, *The asymptotic behavior of a class of nonlinear semigroups in the Hilbert ball*, J. Math. Anal. Appl. **157**(1991), 237–242.

[11] K. Shimoji, W. Takahashi, *Strong convergence to common fixed points of infinite nonexpansive mappings and applications*, Taiwanese J. Math. **5**(2001), 387–404.

[12] T. Suzuki, *Strong convergence of Krasnoselskii and Manns type sequences for one-parameter nonexpansive semigroups without Bochner integrals*, J. Math. Anal. Appl. **305**(2005), 227–239.

[13] T. Suzuki, *Some notes on Bauschke's condition*, Nonlinear Anal. **67**(2007), 2224–2231.

[14] T. Suzuki, *Some comments about recent results on one-parameter nonexpansive semigroups*, Bull. Kyushu Inst. Tech. **54**(2007), 13–26.

[15] H. K. Xu, *Iterative algorithms for nonlinear operators*, J. London Math. Soc. **66**(2002), 240-256.

[16] X. Yang, Y. C. Liou, Y. H. Yao, *Finding minimum-norm fixed point of nonexpansive mappings and applications*, Math. Probl. Eng. **2011**(2011), Article ID 106450.

[17] Y. H. Yao, H. K. Xu, *Iterative methods for finding minimum-norm fixed points of nonexpansive mappings with applications*, Optimization **60**(6)(2011), 645–658.