

## Area formulas for a triangle in the alpha plane

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**Abstract.** In this paper, we give three area formulas for a triangle in the alpha plane in terms of the alpha distance. The two of them are alpha versions of the standard area formula for a triangle in the Euclidean plane, and the third one is an alpha version of the well-known Heron's formula.

**AMS subject classifications:** 51K05, 51K99

**Key words:** alpha distance, alpha plane, area, Heron's formula, plane geometry, protractor geometry

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### 1. Introduction

The *taxicab metric* was given in a family of metrics of the real plane by Minkowski [12]. Later, Chen [1] developed the *Chinese Checker metric*, and Tian [14] gave a family of metrics,  $\alpha$ -metric (*alpha metric*) for  $\alpha \in [0, \pi/4]$ , which includes the taxicab and Chinese checker metrics as special cases. Then, Gelişgen and Kaya extended the  $\alpha$ -distance to three and  $n$  dimensional spaces in [7] and [8], respectively. Afterwards, Çolakoğlu [2] extended the  $\alpha$ -metric for  $\alpha \in [0, \pi/2)$ . According to the latter, if  $P = (x_1, y_1)$  and  $Q = (x_2, y_2)$  are two points in  $\mathbb{R}^2$ , then for each  $\alpha \in [0, \pi/2)$  and  $\lambda(\alpha) = (\sec \alpha - \tan \alpha)$ , the  $\alpha$ -distance between  $P$  and  $Q$  is

$$d_\alpha(P, Q) = \max\{|x_1 - x_2|, |y_1 - y_2|\} + \lambda(\alpha) \min\{|x_1 - x_2|, |y_1 - y_2|\}, \quad (1)$$

while the well-known Euclidean distance between  $P$  and  $Q$  is

$$d_E(P, Q) = [(x_1 - x_2)^2 + (y_1 - y_2)^2]^{1/2}. \quad (2)$$

Since  $\alpha$ -geometry has a distance function different from that of Euclidean geometry, it is interesting to study the  $\alpha$ -analogues of topics that include the distance concept in Euclidean geometry. In this paper, we give area formulas for a triangle in the alpha plane in terms of the alpha distance.

**Remark 1.** *In this study, we use the usual Euclidean area notion. One can easily see that in the  $\alpha$ -plane, there are triangles whose  $\alpha$ -lengths of corresponding sides are the same, while areas of these triangles are different (see Figure 1).*

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This fact gives rise to a natural question: How can one compute the area of a triangle in the  $\alpha$ -plane? It is obvious that every formula to compute the area of a triangle depends on some parameters, and using different parameters gives different formulas. Here we give three formulas to compute the area of a triangle in the  $\alpha$ -plane, using different parameters.

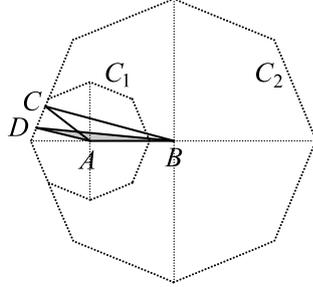
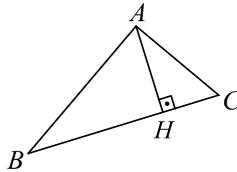


Figure 1:

Let the line  $AB$  be parallel to the  $x$ -axis, let  $C_1$  be an  $\alpha$ -circle with center  $A$  and radius  $b$ ,  $C_2$  an  $\alpha$ -circle with center  $B$  and radius  $b + c$ , and  $C$  and  $D$  two points in  $C_1 \cap C_2$ . For different  $C$  and  $D$  such that  $C$  and  $D$  are not symmetric to the line  $AB$ ,  $Area(ABC) \neq Area(ABD)$ , while  $d_\alpha(A, C) = d_\alpha(A, D)$  and  $d_\alpha(B, C) = d_\alpha(B, D)$ .

## 2. Area of a triangle in $\alpha$ -plane

It is well-known that if  $ABC$  is a triangle with the area  $\mathcal{A}$  in the Euclidean plane, and  $H$  is the point of orthogonal projection of the point  $A$  on the line  $BC$ , then standard area formula for the triangle  $ABC$  is  $\mathcal{A} = \mathbf{ah}/2$ , where  $\mathbf{a} = d_E(B, C)$  and  $\mathbf{h} = d_E(A, H)$  or  $\mathbf{h} = d_E(A, BC)$  (see Figure 2). In this section, we give two  $\alpha$ -versions of standard area formula in terms of  $\alpha$ -distance. Clearly, an  $\alpha$ -version of standard area formula for triangle  $ABC$  would be an equation that relates the two  $\alpha$ -distances  $a$  and  $h$ , where  $a = d_\alpha(B, C)$ ,  $h = d_\alpha(A, H)$  or  $h = d_\alpha(A, BC)$  and area  $\mathcal{A}$  of triangle  $ABC$ . Here, we give two  $\alpha$ -versions of the area formula that depend on one parameter, namely, the slope of the base segment, in addition to the other parameters. Note that the real numbers  $\alpha$  and  $\lambda(\alpha)$  are fixed.

Figure 2:  $d_E(A, H) = d_E(A, BC)$ 

The following equation, which relates the Euclidean distance to the  $\alpha$ -distance between two points in the Cartesian coordinate plane, plays an important role in the first  $\alpha$ -version of the area formula.

**Proposition 1.** *For any two points  $P$  and  $Q$  in the Cartesian plane that do not lie on a vertical line, if  $m$  is the slope of the line through  $P$  and  $Q$ , then*

$$d_E(P, Q) = \rho(m)d_\alpha(P, Q) \tag{3}$$

where  $\rho(m) = (1 + m^2)^{1/2}/(\max\{1, |m|\} + \lambda(\alpha) \min\{1, |m|\})$ . If  $P$  and  $Q$  lie on a vertical line, then by definition,  $d_E(P, Q) = d_\alpha(P, Q)$ .

**Proof.** Let  $P = (x_1, y_1)$  and  $Q = (x_2, y_2)$  with  $x_1 \neq x_2$ ; then  $m = (y_2 - y_1)/(x_2 - x_1)$ . Equation (3) is derived by a straightforward calculation with  $m$  and the coordinate definitions of  $d_E(P, Q)$  and  $d_\alpha(P, Q)$  given in Section 1.  $\square$

Another useful fact that can be verified by direct calculation is:

**Proposition 2.** *For any real number  $m \neq 0$*

$$\rho(m) = \rho(-m) = \rho(1/m) = \rho(-1/m). \tag{4}$$

We first note by Proposition 1 and Proposition 2 that the  $\alpha$ -distance between two points is invariant under all translations, rotations of  $\pi/2$ ,  $\pi$  and  $3\pi/2$  radians around a point, and the reflections about the lines parallel to  $x = 0$ ,  $y = 0$ ,  $y = x$  or  $y = -x$  (see [5]).

The following theorem gives an  $\alpha$ -version of the well-known Euclidean area formula of a triangle:

**Theorem 1.** *Let  $ABC$  be a triangle with the area  $\mathcal{A}$  in the  $\alpha$ -plane,  $H$  orthogonal projection (in the Euclidean sense) of the point  $A$  on the line  $BC$ ,  $m$  the slope of the line  $BC$ , and let  $a = d_\alpha(B, C)$  and  $h = d_\alpha(A, H)$ .*

(i) *If  $BC$  is parallel to a coordinate axis, then*

$$\mathcal{A} = ah/2. \tag{5}$$

(ii) *If  $BC$  is not parallel to any one of the coordinate axes, then*

$$\mathcal{A} = [\rho(m)]^2 ah/2 \tag{6}$$

where  $\rho(m) = (1 + m^2)^{1/2}/(\max\{1, |m|\} + \lambda(\alpha) \min\{1, |m|\})$ .

**Proof.** Let  $\mathbf{a} = d_E(B, C)$  and  $\mathbf{h} = d_E(A, H)$ . Then,  $\mathcal{A} = \mathbf{a}\mathbf{h}/2$ .

(i): If  $BC$  is parallel to a coordinate axis, then clearly  $a = \mathbf{a}$  and  $h = \mathbf{h}$ . Hence,  $\mathcal{A} = ah/2$ .

(ii): Let  $BC$  not be parallel to any one of the coordinate axes, and let the slope of the line  $BC$  be  $m$ . Then, the slope of the line  $AH$  is  $(-1/m)$ . By Proposition 1 and Proposition 2,  $\mathbf{a} = \rho(m)a$ ,  $\mathbf{h} = \rho(m)h$ , hence  $\mathcal{A} = \rho^2(m)ah/2$ .  $\square$

In the  $\alpha$ -plane, the  $\alpha$ -distance from a point  $P$  to a line  $l$  is defined by

$$d_\alpha(P, l) = \min_{Q \in l} \{d_\alpha(P, Q)\} \tag{7}$$

as in the Euclidean plane. It is well-known that in the Euclidean plane, Euclidean distance from a point  $P = (x_0, y_0)$  to a line  $l : ax + by + c = 0$  can be calculated by the following formula:

$$d_E(P, l) = \frac{|ax_0 + by_0 + c|}{(a^2 + b^2)^{1/2}}. \quad (8)$$

In Proposition 4 we give a similar formula for  $d_\alpha(P, l)$ , using  $\alpha$ -circles (see [2]). It is easy to see that if  $\alpha \in (0, \pi/2)$ , then unit  $\alpha$ -circle is an octagon with vertices  $A_1 = (1, 0)$ ,  $A_2 = (\frac{1}{k}, \frac{1}{k})$ ,  $A_3 = (0, 1)$ ,  $A_4 = (\frac{-1}{k}, \frac{1}{k})$ ,  $A_5 = (-1, 0)$ ,  $A_6 = (\frac{-1}{k}, \frac{-1}{k})$ ,  $A_7 = (0, -1)$  and  $A_8 = (\frac{1}{k}, \frac{-1}{k})$ , where  $k = 1 + \lambda(\alpha)$ . If  $\alpha = 0$ , then  $\alpha$ -circle is a square with vertices  $A_1 = (1, 0)$ ,  $A_3 = (0, 1)$ ,  $A_5 = (-1, 0)$  and  $A_7 = (0, -1)$  (see Figure 3). Note that  $A_2$  and  $A_6$  are on the line  $y = x$ ;  $A_4$  and  $A_8$  are on the line  $y = -x$ .

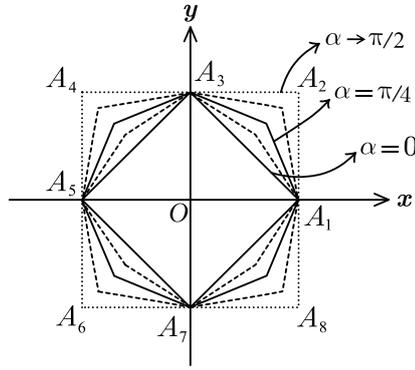


Figure 3: Graph of the unit  $\alpha$ -circle

**Proposition 3.** Given a point  $P = (x_0, y_0)$ , and a line  $l : ax + by + c = 0$  in the  $\alpha$ -plane. Then the  $\alpha$ -distance from the point  $P$  to the line  $l$  can be calculated by the following formula:

$$d_\alpha(P, l) = |ax_0 + by_0 + c| / \max \left\{ |a|, |b|, \frac{|a+b|}{1+\lambda(\alpha)}, \frac{|a-b|}{1+\lambda(\alpha)} \right\}. \quad (9)$$

**Proof.** Clearly, if  $P$  is on the line  $l$ , then equation (9) is true. If  $P$  is not on the line  $l$ , then we expand an  $\alpha$ -circle with center  $P$  and radius 0 until the line  $l$  becomes a *tangent* to the  $\alpha$ -circle, to find the minimum  $\alpha$ -distance from the point  $P$  to the line  $l$  (Here, by *tangent* to an  $\alpha$ -circle with center  $P$  and radius  $r$ , we mean “a line whose  $\alpha$ -distance from  $P$  is equal to  $r$ ”, as in Euclidean plane.). It is easy to see that a line can only be a tangent to an  $\alpha$ -circle at one vertex or two vertices (at one edge). Since corresponding vertices of expanding  $\alpha$ -circle are on lines through  $P$  and parallel to lines  $y = 0$ ,  $x = 0$ ,  $y = x$  and  $y = -x$ , if  $l$  is a tangent to an  $\alpha$ -circle with center  $P$ , then a tangent point is one of points

$$P_1 = \left( \frac{-by_0 - c}{a}, y_0 \right), P_2 = \left( x_0, \frac{-ax_0 - c}{b} \right), P_3 = \left( \frac{bx_0 - by_0 - c}{a + b}, \frac{-ax_0 + ay_0 - c}{a + b} \right)$$

and

$$P_4 = \left( \frac{-bx_0 - by_0 - c}{a - b}, \frac{ax_0 + ay_0 + c}{a - b} \right),$$

where  $P_1, P_2, P_3$  and  $P_4$  are intersection points of the line  $l$  and  $y = y_0, x = x_0, -x + y + x_0 - y_0 = 0$  and  $x + y - x_0 - y_0 = 0$ , respectively (see Figure 4). Therefore,  $d_\alpha(P, l) = \min\{d_\alpha(P, P_i) : i = 1, 2, 3, 4\}$ .

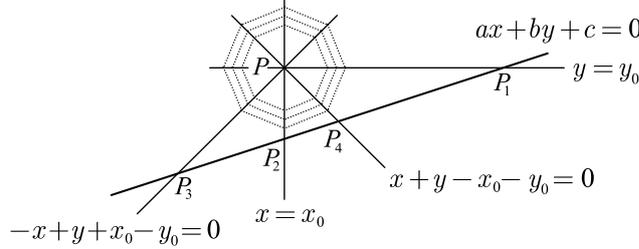


Figure 4:

Since

$$d_\alpha(P, P_1) = \frac{|ax_0 + by_0 + c|}{|a|}, d_\alpha(P, P_2) = \frac{|ax_0 + by_0 + c|}{|b|},$$

$$d_\alpha(P, P_3) = \frac{|ax_0 + by_0 + c|}{|a + b|/(1 + \lambda(\alpha))} \text{ and } d_\alpha(P, P_4) = \frac{|ax_0 + by_0 + c|}{|a - b|/(1 + \lambda(\alpha))},$$

we get

$$d_\alpha(P, l) = \frac{|ax_0 + by_0 + c|}{\max \left\{ |a|, |b|, \frac{|a+b|}{1+\lambda(\alpha)}, \frac{|a-b|}{1+\lambda(\alpha)} \right\}}.$$

□

The following equation, which relates the Euclidean distance to the  $\alpha$ -distance from a point to a line in the Cartesian coordinate plane, plays an important role in the second  $\alpha$ -version of the area formula.

**Proposition 4.** *Given a point  $P$ , and a line  $l$  in the Cartesian plane that is not a vertical line, if  $m$  is the slope of the line  $l$ , then*

$$d_E(P, l) = \tau(m)d_\alpha(P, l) \tag{10}$$

where

$$\tau(m) = \max \left\{ |1|, |m|, \frac{|1 + m|}{1 + \lambda(\alpha)}, \frac{|1 - m|}{1 + \lambda(\alpha)} \right\} / (1 + m^2)^{1/2}.$$

If  $l$  is a vertical line, then  $d_E(P, l) = d_\alpha(P, l)$ .

**Proof.** Let  $P = (x_0, y_0)$  be a point, and  $l : ax + by + c = 0$  be a line in the Cartesian plane. If  $l$  is not a vertical line, then  $b \neq 0$  and  $m = -\frac{a}{b}$ . Using  $m$  in equation (8) and equation (9), one gets  $d_E(P, l) = |ax_0 + by_0 + c| / |b| (1 + m^2)^{1/2}$  and

$$d_\alpha(P, l) = |ax_0 + by_0 + c| / |b| \max \left\{ |1|, |m|, \frac{|1 + m|}{1 + \lambda(\alpha)}, \frac{|1 - m|}{1 + \lambda(\alpha)} \right\}.$$

Hence,  $d_E(P, l) = \tau(m)d_\alpha(P, l)$  where

$$\tau(m) = \max \left\{ |1|, |m|, \frac{|1+m|}{1+\lambda(\alpha)}, \frac{|1-m|}{1+\lambda(\alpha)} \right\} / (1+m^2)^{1/2}.$$

If  $l$  is a vertical line, then  $b = 0$  and  $a \neq 0$ . Therefore,  $d_E(P, l) = |ax_0 + c| / |a|$  and  $d_\alpha(P, l) = |ax_0 + c| / |a|$ , hence  $d_E(P, l) = d_\alpha(P, l)$ .  $\square$

The following theorem gives another  $\alpha$ -version of the well-known Euclidean area formula of a triangle:

**Theorem 2.** *Let  $ABC$  be a triangle with area  $\mathcal{A}$  in the  $\alpha$ -plane,  $m$  the slope of the line  $BC$ , and let  $a = d_\alpha(B, C)$  and  $h = d_\alpha(A, BC)$ .*

(i) *If  $BC$  is parallel to a coordinate axis, then*

$$\mathcal{A} = ah/2. \quad (11)$$

(ii) *If  $BC$  is not parallel to any one of the coordinate axes, then*

$$\mathcal{A} = \sigma(m)ah/2 \quad (12)$$

where

$$\sigma(m) = \max \left\{ |1|, |m|, \frac{|1+m|}{1+\lambda(\alpha)}, \frac{|1-m|}{1+\lambda(\alpha)} \right\} / (\max\{1, |m|\} + \lambda(\alpha) \min\{1, |m|\}).$$

**Proof.** Let  $\mathbf{a} = d_E(B, C)$  and  $\mathbf{h} = d_E(A, BC)$ . Then,  $\mathcal{A} = \mathbf{a}\mathbf{h}/2$ .

(i): If  $BC$  is parallel to a coordinate axis, then clearly  $a = \mathbf{a}$  and  $h = \mathbf{h}$ . Hence,  $\mathcal{A} = ah/2$ .

(ii): Let  $BC$  not be parallel to any one of the coordinate axes, and let the slope of the line  $BC$  be  $m$ . Then, by Proposition 1 and Proposition 5,  $\mathbf{a} = \rho(m)a$ ,  $\mathbf{h} = \tau(m)h$ , hence  $\mathcal{A} = \rho(m)\tau(m)ah/2$ . Since  $\rho(m)\tau(m) = \sigma(m)$ , we get  $\mathcal{A} = \sigma(m)ah/2$ .  $\square$

### 3. Alpha version of Heron's formula

It is well-known that if  $ABC$  is a triangle with the area  $\mathcal{A}$  in the Euclidean plane, and  $\mathbf{a} = d_E(B, C)$ ,  $\mathbf{b} = d_E(A, C)$ ,  $\mathbf{c} = d_E(A, B)$ , and  $\mathbf{p} = (\mathbf{a} + \mathbf{b} + \mathbf{c})/2$ , then  $\mathcal{A} = [\mathbf{p}(\mathbf{p} - \mathbf{a})(\mathbf{p} - \mathbf{b})(\mathbf{p} - \mathbf{c})]^{1/2}$ , which is known as the *Heron's formula*. In this section, we give an  $\alpha$ -version of this formula in terms of the  $\alpha$ -distance. Clearly, an  $\alpha$ -version of the Heron's formula for triangle  $ABC$  would be an equation that relates the three  $\alpha$ -distances  $a$ ,  $b$  and  $c$ , where  $a = d_\alpha(B, C)$ ,  $b = d_\alpha(A, C)$ ,  $c = d_\alpha(A, B)$ , and the area  $\mathcal{A}$  of triangle  $ABC$ . Here, we give an  $\alpha$ -version of the Heron's formula that depend on three new parameters in addition to  $a$ ,  $b$ ,  $c$  and  $\mathcal{A}$ .

We need the following two definitions given in [13] and [11] respectively, to give an  $\alpha$ -version of the Heron's formula:

**Definition 1.** *Let  $ABC$  be any triangle in the  $\alpha$ -plane. Clearly, there exists a pair of lines passing through every vertex of the triangle, each of which is parallel to a coordinate axis. A line  $l$  is called a base line of  $ABC$  if and only if*

- (1)  $l$  passes through a vertex;
- (2)  $l$  is parallel to a coordinate axis;
- (3)  $l$  intersects the opposite side (as a line segment) of the vertex in (1).

Clearly, at least one of vertices of the triangle always has one or two base lines. Such a vertex of the triangle is called a basic vertex. A base segment is a line segment on a base line, which is bounded by a basic vertex and its opposite side.

**Definition 2.** A line with slope  $m$  is called a steep line, a gradual line and a separator if  $|m| > 1$  or  $m \rightarrow \infty$ ,  $|m| < 1$  and  $|m| = 1$ , respectively.

The following theorem gives an  $\alpha$ -version of the Heron's formula:

**Theorem 3.** Let  $ABC$  be a triangle with area  $\mathcal{A}$  in the  $\alpha$ -plane, such that  $C$  is a basic vertex,  $a = d_\alpha(B, C)$ ,  $b = d_\alpha(A, C)$  and  $c = d_\alpha(A, B)$ . Let  $D$  be the intersection point of a base line and  $AB$ , the opposite side of the basic vertex  $C$ . Let  $H_1$  and  $H_2$  be orthogonal projections (in the Euclidean sense) of  $A$  and  $B$  on the base line  $CD$ , respectively. Then,

$$\mathcal{A} = \begin{cases} \frac{l}{2} [2p - c - \lambda(\alpha)(l_1 + l_2)] & ; \text{if } C_1 \text{ is valid} \\ \frac{l}{2\lambda(\alpha)} [2p - c - (l_1 + l_2)] & ; \text{if } C_2 \text{ is valid} \\ \frac{l}{2\lambda(\alpha)} [2p - c + (\lambda(\alpha) - 1)b - (\lambda^2(\alpha)l_1 + l_2)] & ; \text{if } C_3 \text{ is valid} \\ \frac{l}{2\lambda(\alpha)} [2p - c + (\lambda(\alpha) - 1)a - (l_1 + \lambda^2(\alpha)l_2)] & ; \text{if } C_4 \text{ is valid} \end{cases} \quad (13)$$

where

$$p = (a + b + c)/2, l = d_\alpha(C, D), l_1 = d_\alpha(C, H_1), l_2 = d_\alpha(C, H_2),$$

$C_1$  : lines  $AC$  and  $BC$  are not gradual and base line  $CD$  is horizontal, or lines  $AC$  and  $BC$  are not steep and base line  $CD$  is vertical;

$C_2$  : lines  $AC$  and  $BC$  are not steep and base line  $CD$  is horizontal, or lines  $AC$  and  $BC$  are not gradual and base line  $CD$  is vertical;

$C_3$  : line  $AC$  is not gradual, line  $BC$  is not steep and base line  $CD$  is horizontal, or line  $AC$  is not steep, line  $BC$  is not gradual and base line  $CD$  is vertical;

$C_4$  : line  $AC$  is not steep, line  $BC$  is not gradual and base line  $CD$  is horizontal, or line  $AC$  is not gradual, line  $BC$  is not steep and base line  $CD$  is vertical.

**Proof.** Let  $ABC$  be a triangle with area  $\mathcal{A}$  in the  $\alpha$ -plane, such that  $C$  is a basic vertex,  $a = d_\alpha(B, C)$ ,  $b = d_\alpha(A, C)$  and  $c = d_\alpha(A, B)$ . Let  $D$  be the intersection point of a base line and  $AB$ , the opposite side of the basic vertex  $C$ . Let  $H_1$  and  $H_2$  be orthogonal projections of  $A$  and  $B$  on the base line  $CD$ , respectively. And let  $p = (a + b + c)/2$ ,  $l = d_\alpha(C, D)$ ,  $l_1 = d_\alpha(C, H_1)$ ,  $l_2 = d_\alpha(C, H_2)$ ,  $h_1 = d_\alpha(A, H_1)$ ,  $h_2 = d_\alpha(B, H_2)$ . Since the  $\alpha$ -distance between two points is invariant under all translations, rotations of  $\pi/2$ ,  $\pi$  and  $3\pi/2$  radians around a point, and reflections

about the lines parallel to  $x = 0$ ,  $y = 0$ ,  $y = x$  or  $y = -x$ , Figure 5 and Figure 6 represent all triangles for which  $C_1$  holds, Figure 7 and Figure 8 represent all triangles for which  $C_2$  holds, Figure 9 and Figure 10 represent all triangles for which  $C_3$  holds, and finally Figure 11 and Figure 12 represent all triangles for which  $C_4$  holds. In Figure 5 and Figure 6,  $a = h_2 + \lambda(\alpha)l_2$  and  $b = h_1 + \lambda(\alpha)l_1$  by the  $\alpha$ -distance defi-

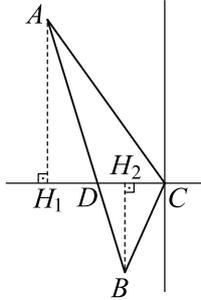


Figure 5:

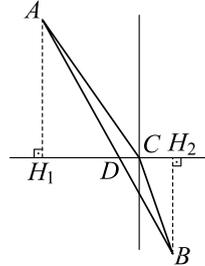


Figure 6:

nition. Since  $A(ABC) = A(ADC) + A(BDC) = \frac{1}{2}(h_1 + h_2)$ , using  $h_1$  and  $h_2$  values, one gets  $\mathcal{A} = \frac{l}{2} [2p - c - \lambda(\alpha)(l_1 + l_2)]$ . In Figure 7 and Figure 8,  $a = l_2 + \lambda(\alpha)h_2$

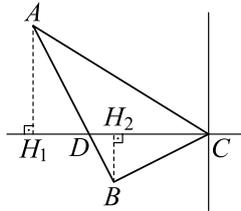


Figure 7:

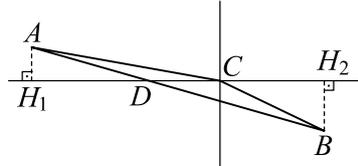


Figure 8:

and  $b = l_1 + \lambda(\alpha)h_1$  by the  $\alpha$ -distance definition. Since  $A(ABC) = A(ADC) + A(BDC) = \frac{l}{2}(h_1 + h_2)$ , using  $h_1$  and  $h_2$  values, one gets  $\mathcal{A} = \frac{l}{2\lambda(\alpha)} [2p - c - (l_1 + l_2)]$ . In Figure 9 and Figure 10,  $a = l_2 + \lambda(\alpha)h_2$  and  $b = h_1 + \lambda(\alpha)l_1$  by the  $\alpha$ -distance

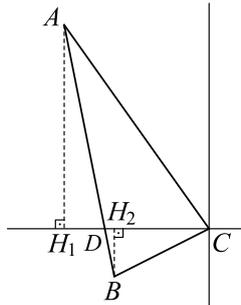


Figure 9:

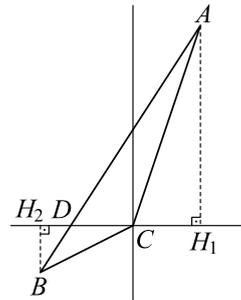


Figure 10:

definition. Since  $A(ABC) = A(ADC) + A(BDC) = \frac{l}{2}(h_1 + h_2)$ , using  $h_1$  and  $h_2$  values, one gets  $\mathcal{A} = \frac{l}{2\lambda(\alpha)} [2p - c + (\lambda(\alpha) - 1)b - (\lambda^2(\alpha)l_1 + l_2)]$ . In Figure 11 and

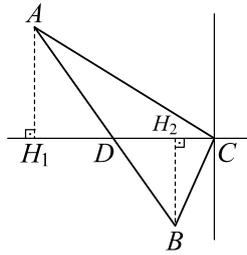


Figure 11:

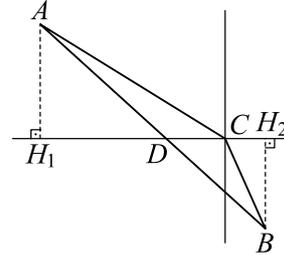


Figure 12:

Figure 12,  $a = h_2 + \lambda(\alpha)l_2$  and  $b = l_1 + \lambda(\alpha)h_1$  by the  $\alpha$ -distance definition. Since  $A(ABC) = A(ADC) + A(BDC) = \frac{1}{2}(h_1 + h_2)$ , using  $h_1$  and  $h_2$  values, one gets  $A = \frac{l}{2\lambda(\alpha)} [2p - c + (\lambda(\alpha) - 1)a - (l_1 + \lambda^2(\alpha)l_2)]$ .  $\square$

**Remark 2.** Since well-known taxicab and Chinese Checker distances are special cases of the  $\alpha$ -distance for  $\alpha = 0$  and  $\alpha = \pi/4$ , respectively, Theorem 3, Theorem 6 and Theorem 7 also give taxicab and Chinese Checker versions of area formulas for a triangle, when  $\alpha = 0$  and  $\alpha = \pi/4$ , respectively (see [10], [13] and [9]). Note that if  $\alpha = 0$ , then  $\lambda(\alpha) = 1$ , and we get simple equations for taxicab plane:  $\rho(m) = (1 + m^2)^{1/2}/(1 + |m|)$ ,  $d_0(P, l) = d_T(P, l) = |ax_0 + by_0 + c| / \max\{|a|, |b|\}$ ,  $\tau(m) = \max\{|1|, |m|\}/(1+m^2)^{1/2}$ ,  $\sigma(m) = \max\{|1|, |m|\}/(1+|m|)$ ,  $A = \frac{l}{2} [2p - c - (l_1 + l_2)]$ .

### References

- [1] G. CHEN, *Lines and circles in taxicab geometry*, Master Thesis, Department of Mathematics and Computer Science, University of Central Missouri, 1992.
- [2] H. B. ÇOLAKOĞLU, *Concerning the alpha distance*, *Algebras Groups Geom.* **8**(2011), 1–14.
- [3] H. B. ÇOLAKOĞLU, R. KAYA, *Chinese checker versions of the Pythagorean theorem*, *Int. J. Contemp. Math. Sci.* **4**(2009), 61–69.
- [4] H. B. ÇOLAKOĞLU, Ö. GELİŞGEN, R. KAYA, *Pythagorean theorems in the alpha plane*, *Math. Commun.* **14**(2009), 211–221.
- [5] H. B. ÇOLAKOĞLU, R. KAYA, *A generalization of some well-known distances and related isometries*, *Math. Commun.* **16**(2011), 21–35.
- [6] B. DIVJAK, *Notes on taxicab geometry*, *KoG. Croat. Soc. Geom. Graph.* **5**(2000), 5–9.
- [7] Ö. GELİŞGEN, R. KAYA, *On  $\alpha$ -distance in three dimensional space*, *Appl. Sci.* **8**(2006), 65–69.
- [8] Ö. GELİŞGEN, R. KAYA, *Generalization of  $\alpha$ -distance to n-dimensional space*, *KoG. Croat. Soc. Geom. Graph.* **10**(2006), 33–35.
- [9] Ö. GELİŞGEN, R. KAYA, *CC-version of the Heron's formula*, *Missouri J. Math. Sci.* **21**(2009), 94–110.
- [10] R. KAYA, *Area formula for taxicab triangles*, *Pi Mu Epsilon Journal* **12**(2006), 213–220.
- [11] E. F. KRAUSE, *Taxicab geometry*, Addison–Wesley, Menlo Park, California, 1975; Dover Publications, New York, 1987.
- [12] H. MINKOWSKI, *Gesammelte Abhandlungen*, Chelsea Publishing Co., New York, 1967.
- [13] M. ÖZCAN, R. KAYA, *Area of a triangle in terms of the taxicab distance*, *Missouri J. Math. Sci.* **15**(2003), 178–185.

- [14] S. TIAN, *Alpha distance—A generalization of Chinese checker distance and taxicab distance*, Missouri J. Math. Sci. **17**(2005), 35–40.