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Cubic surfaces and q-numerical ranges

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Abstract. Let A be an $n \times n$ complex matrix and $0 \leq q \leq 1$. The boundary of the q-numerical range of A is the orthogonal projection of a hypersurface defined by the dual surface of the homogeneous polynomial

$$F(t, x, y, z) = \det(t\, I_n + x(A + A^*)/2 + y(A - A^*)/(2i) + z\, A^* A).$$

We construct different types of cubic surfaces S_F corresponding to the homogeneous polynomial $F(t, x, y, z)$ induced by some 3×3 matrices. The degree of the boundary of the Davis-Wielandt shell of a 3×3 upper triangular matrix is determined by the cubic surface S_F.

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1. Introduction

Let A be an $n \times n$ complex matrix and $0 \leq q \leq 1$. The q-numerical range of A is defined and denoted as

$$W_q(A) = \{\zeta^* A \xi : \xi, \zeta \in \mathbf{C}^n, \xi^* \xi = \zeta^* \zeta = 1, \zeta^* \xi = q\},$$

where ξ^* denotes the transpose of the coordinate-wise complex conjugate of the vector $\xi \in \mathbf{C}^n$. It is well known (see [18]) that $W_q(A)$ is a convex subset of C. Its star-shaped generalization is studied in [15]. When $q = 1$, $W_q(A)$ reduces to the classical numerical range $W(A) = \{\xi^* A \xi : \xi \in \mathbf{C}^n, \xi^* \xi = 1\}$. For $n = 3$, there has been a number of interesting papers on their numerical ranges ([3, 5, 6, 16]). Furthermore, a comprehensive study of the numerical ranges of 3×3 matrices can be found in [7, 8] which classify the shapes of the numerical range via the homogeneous polynomial

$$F(t, x, y) = \det(t\, I_n + x(A + A^*)/2 + y(A - A^*)/(2i)),$$

where A^* stands for the Hermitian adjoint of A.

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The study of the q-numerical range is closely related to the so-called *Davis-Wielandt shell* of $A \in M_n$ which is defined as

$$DW(A) = \{(\xi^* A\xi, \xi^* A^* A\xi) : \xi \in \mathrm{C}^n, \xi^*\xi = 1\}.$$

(see [4, 10]). Consider the homogeneous polynomial

$$F(t, x, y, z) = \det(t\, I_n + x(A + A^*)/2 + y(A - A^*)/(2i) + z\, A^* A), \qquad (1)$$

which defines the algebraic variety $S_F = \{[(t, x, y, z] \in \mathbf{CP}^3 : F(t, x, y, z) = 0\}$. Let $G(t, x, y, z) = 0$ be the dual surface of S_F. We consider a hypersurface in the 4-dimensional Euclidean space

$$S = \{(x, y, u, v) \in \mathbf{R}^4 : u^2 + v^2 = h(x + iy)^2\},$$

where $h(z) = \sup\{w \in \mathbf{R} : (z, w) \in DW(A)\}$. Define an orthogonal projection π_q of $\mathbf{R}^4$ onto $\mathbf{C} \cong \mathbf{R}^2$ by

$$\pi_q((x, y, u, v)) = (qx + \sqrt{1 - q^2}u) + i(qy + \sqrt{1 - q^2}v).$$

Then the range $W_q(A)$ is given by $W_q(A) = \pi_q(S)$ (cf. [4]). Every boundary point (z, w) of $DW(A)$ satisfies $G(1, \Re(z), \Im(z), w) = 0$ or the point lies on a multi-tangent of the surface $G(1, \Re(z), \Im(z), w) = 0$. If the surface $F(t, x, y, z) = 0$ has no singular point, then the range $W_q(A)$ is given by

$$\pi_q\{(x, y, u, v) \in \mathbf{R}^4 : G(1, x, y, x^2 + y^2 + u^2 + v^2) = 0\}.$$

The range $W_q(A)$ is essentially determined by the form $G(t, x, y, z)$, and hence by the form $F_A(t, x, y, z) = F(t, x, y, z)$. If we replace A by UAU^* for some unitary matrix U, the associated form $F_{UAU^*}(t, x, y, z)$ coincides with $F_A(t, x, y, z)$. Thus the range $W_q(A)$ is invariant under a unitary similarity. The relation $W_q(A) = \pi_q(A)$ is rewritten as

$$W_q(A) = \{qz + \sqrt{1 - q^2}wh(z) : z \in W(A), w \in \mathbf{C}, |w| \le 1\}.$$

Furthermore, if the boundary of the range $DW(A)$ has a flat portion on the plane $a_1x_1 + a_2x_2 + a_3x_3 + a_0 = 0$, then the real point (a_0, a_1, a_2, a_3) is a singular point of the surface S_F. Thus the number of the flat portions of the boundary of the range $DW(A)$ is less than or equal to the number of the singular points of the surface S_F (cf. [5]). The analysis of the degree of the boundary equation of $W_q(A)$ is closely related to the study of the singularities of the surface S_F.

Cubic surfaces is a classical subject in algebraic geometry. Schläfli [17] gave a foundation of its classification theory (see also [2, 9]). It is of great interest in computer aided geometric design (cf. [1, 14]). In this paper, we study the Davis-Wielandt shells of certain 3×3 upper triangular matrices from a viewpoint of the types of singularities occuring on the cubic surfaces S_F corresponding to the matrices.

2. Singular points of cubic surfaces

Let $F(t,x,y,z)$ be an irreducible complex cubic form in the polynomial ring $\mathbf{C}[t,x,y,z]$. Suppose that $(t,x,y,z)=(1,x_0,y_0,z_0)$ is a singular point of the algebraic surface S_F, that is, $F(1,x_0,y_0,z_0)=F_t(1,x_0,y_0,z_0)=F_x(1,x_0,y_0,z_0)=F_y(1,x_0,y_0,z_0)=F_z(1,x_0,y_0,z_0)=0$.

In this case, we assume that

$$F(1,x_0+x,y_0+y,z_0+z) = \alpha_{11}x^2+\alpha_{22}y^2+\alpha_{33}z^2+2\alpha_{12}x\,y +2\alpha_{13}x\,z+2\alpha_{23}y\,z+F_3(x,y,z), \tag{2}$$

where $F_3(x,y,z)$ is homogeneous of degree 3. If the cubic surface S_F has non isolated singularities, then the singularity set is a line (cf. [2] page 252, [13]). A fundamental classification of a singularity is provided by the types of the quadratic form $\alpha_{11}x^2+\alpha_{22}y^2+\alpha_{33}z^2+2\alpha_{12}x\,y+2\alpha_{13}x\,z+2\alpha_{23}y\,z$. Consider the symmetric matrix

$$\alpha = \begin{bmatrix} \alpha_{11} & \alpha_{12} & \alpha_{13} \\ \alpha_{12} & \alpha_{22} & \alpha_{23} \\ \alpha_{13} & \alpha_{23} & \alpha_{33} \end{bmatrix}$$

corresponding to the coefficients of the quadratic terms in (2). Firstly we consider an exceptional case $\alpha_{11}=\alpha_{22}=\alpha_{33}=0$, $\alpha_{12}=\alpha_{13}=\alpha_{23}=0$, or equivalently $F(t,x,y)=F_3(x,y,z)$. If the irreducible cubic curve $F_3(x,y,z)=0$ has no singular point, the surface S_F has no singular point. If $F_3(x,y,z)=0$ has a node or a cusp, then the surface S_F has a line of singularities.

Secondly we consider a generic case $\alpha\neq 0$. In this case, if the surface S_F has a singular point $(1,x_0,y_0,z_0)$, then the surface $F(t,x_0t+x,y_0t+y,z_0t+z)=0$ is expressed as $t\,(a_{11}x^2+a_{22}y^2+a_{33}z^2+2a_{12}xy+2a_{13}xz+2a_{23}yz)+F_3(x,y,z)=0$. For instance, we assume that $a_{33}\neq 0$. The surface S_F has a rational parametrization

$$t = -\frac{F_3(x,y,1)}{a_{33}+2a_{13}x+2a_{23}y+a_{11}x^2+2a_{12}xy+a_{22}y^2}.$$

If the matrix α is non-singular, the point $(1,x_0,y_0,z_0)$ is called an *ordinary double point* (also called A_1 point). If α is singular with rank r, for $r=2$, the singular point $(1,x_0,y_0,z_0)$ is called a *biplanar double point* (or a binode). If $r=1$, the singular point $(1,x_0,y_0,z_0)$ is called a *uniplanar double point* (or a unode). If $r=0$, or $\alpha=0$, the singular point $(1,x_0,y_0,z_0)$ is a called a *triple point.* Biplanar double points are classified into four types. Suppose that $(1,x_0,y_0,z_0)$ is a biplanar double point of S_F. By changing the variables, we may assume that $\alpha_{13}=\alpha_{23}=\alpha_{31}=\alpha_{32}=\alpha_{33}=0, \alpha_{11}\alpha_{22}-\alpha_{12}^2\neq 0$. Under theses assumptions, if $F_3(0,0,1)\neq 0$, the point $(1,x_0,y_0,z_0)$ is a biplanar double point A_2. We are interested in the real cubic form $F(t,x,y,z)$ given by (1), which is *hyperbolic* with respect to $(1,0,0,0)$, that is, the cubic equation $F(t,x_0,y_0,z_0)=0$ in t has 3 real roots counting multiplicities for every $(x_0,y_0,z_0)\in\mathbf{R}^3$. If we replace A^*A in equation (1) by an arbitrary 3×3 hermitian matrix K, we can construct a real irreducible hyperbolic form F for which the surface $F(t, x, y, z) = 0$ has non-isolated singularities. An example is given by

$$A = \begin{bmatrix} 1+i & 0 & 0 \\ 0 & 1+i & 0 \\ 0 & 0 & 3i \end{bmatrix}, \quad K = \begin{bmatrix} 0 & 2 & 0 \\ 2 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}.$$

In the sequel, we shall treat the case the cubic surface $F(t, x, y, z)$ has isolated singularities. For more singularity classification of cubic surfaces, we refer the reader to Bruce and Wall [2] and references therein. There are 21 types of cubic surfaces in referring to isolated singularities listed on the webpage of Labs [11]. Nice models of cubic surfaces can be found on the webpage [12]. In Section 3, we show that the following 6 typical types of cubic surfaces actually occur as surfaces S_F corresponding to some matrices:

[I]: no singularity;

[II]: one ordinary double point A_1;

[IV]: two ordinary double points $2A_1$;

[VIII]: three ordinary double points $3A_1$;

[IX]: two biplanar double points $2A_2$;

[XVII]: two biplanar double points $2A_2$ and one ordinary double point A_1.

Let A be a 3×3 matrix, and $F(t, x, y, z)$ the corresponding homogeneous polynomial. In [2], the *class* of an irreducible cubic surface S_F with isolated singularities is defined. It is the number of tangent hyperplanes of S_F passing through a generic point. The class number is given by

$$12 - \sum_j \nu(P_j), \tag{3}$$

where P_j is a singular point of S_F and $\nu(P)$ is a positive number depending on the type of singularity at the point P. In particular, $\nu(P) = 2$ if P is an A_1 point, and $\nu(P) = 3$ if P is an A_2 point.

Notice that every boundary point P of the Davis-Wielandt shell of a matrix lies in the dual surface of the cubic surface if P does not lie on a flat portion. An algorithm for computing the boundary of the Davis-Wielandt shell of a 3×3 matrix can be found in [4, 5]. The class number (3) of the cubic surface S_F is exactly the degree of the boundary generating surface $G(1, x, y, z) = 0$ of the Davis-Wielandt shell of A.

3. Upper triangular matrices

We deal with the Davis-Wielandt shell of a 3×3 upper triangular matrix using the cubic surface S_F.

Theorem 1. *Let A be the matrix given by*

$$A = \begin{bmatrix} 2 & 3+\epsilon & 0 \\ 0 & 0 & 2 \\ 0 & 0 & -2 \end{bmatrix},$$

for $\epsilon = \pm 1$.

(*i*) *If $\epsilon = +1$, then the surface S_F has no singular points. The cubic surface is of type [I], and the boundary generating surface of the Davis-Wielandt shell $DW(A)$ lies in a polynomial surface of degree 12.*

(*ii*) *If $\epsilon = -1$, the surface S_F has an ordinary double point at $(t,x,y,z) = (1,0,0,-\frac{1}{8}$. The cubic surface is of type [II], and the boundary generating surface of the Davis-Wielandt shell $DW(A)$ lies in a polynomial surface of degree 10.*

Proof. Firstly we treat case (i), that is $\epsilon = +1$. The derivative of the form $F(t,x,y,z)$ with respect y is given by

$$F_y(t,x,y,z) = -2y\,(5t - 6x + 36z). \tag{4}$$

Hence, if S_F has a singular point, it lies on a hyperplane $y = 0$ or a hyperplane $z = (-5t + 6x)/36$. We compute the resultant $R_1(x,z)$ of F_t and F_x with respect to t, and the resultant $R_2(x,z)$ of F_t and F_z with respect to t under the assumption that $y = 0$. We obtain that

$$R_1(x,z) = -72(3x^2 - 16z^2)(9x^2 + 24xz + 80z^2), \tag{5}$$
$$R_2(x,z) = 192(3x + 20z)^2(3x^2 + 8xz - 144z^2), \tag{6}$$

which are products of linear factors. The equation $R_1(x,z) = R_2(x,z) = 0$ implies $x = z = 0$. Since $F(1,0,0,0) = 1 \neq 0$, the surface S_F has no singular points on the hyperplane $y = 0$. Next we compute the resultant $R_3(x,z)$ of F_t and F_x with respect to t, and the resultant $R_4(x,z)$ of F_t and F_z with respect to t under the assumption that $z = (-5t + 6x)/36$. Then we have

$$R_3(x,z) = \frac{256}{729}(1521x^4 + 906x^2y^2 + 121y^4), \tag{7}$$

$$R_4(x,z) = \frac{4096}{729}(729x^4 + 886x^2y^2 + 81y^4). \tag{8}$$

These are also products of linear factors. The equation $R_3(x,y) = R_4(x,y) = 0$ implies $x = y = 0$. Because $F(1,0,0,0) = 1 \neq 0$, the surface S_F has no singular points.

Secondly we treat case (ii), that is $\epsilon = -1$. The form $F(t,x,y,z)$ associated with A satisfies the equation

$$F(1,x,y,-\frac{1}{8} + z) = -\frac{9}{2}x^2 - \frac{1}{2}y^2 + 64z^2 - 12(x^2 + y^2)z, \tag{9}$$

and hence $(1,0,0,-\frac{1}{8})$ is an ordinary double point of the cubic surface. On the hyperplane $t=0$ at infinity, we have

$$F_x(0,x,y,z) = -24xz, F_y(0,x,y,z) = -24yz, F(0,x,y,z) = -12(x^2+y^2)z.$$

These relations imply that the cubic surface S_F has no singular point on $t=0$. On the affine 3-space $t=1$, the equation

$$F(1,x,y,z) = 1+16z-6x^2-2y^2+64z^2-12x^2z-12y^2z \tag{10}$$

implies that the resultant of F_x and F_y with respect z,x,y are respectively given by

$$-192xy, \quad -4y(1+6z), \quad -12x(1+2z).$$

Since $F(1,0,y,-1/6) = 1/9$, $F(1,x,0,-1/2) = 9$, the singular point $(1,x,y,z)$ of S_F necessarily satisfies $x=y=0$. Then $F(1,0,0,z) = 1+16z+64z^2 = (1+8z)^2$, and thus $z = -\frac{1}{8}$.

The class numbers (3) of (i) and (ii) are respectively $12 = 12-0$ and $10 = 12-2$, which are the degrees of the boundary generating surface of the respective $DW(A)$. □

Theorem 2. *Let A be the upper triangular matrix given by*

$$A = \begin{bmatrix} 1 & a & 0 \\ 0 & 1 & b \\ 0 & 0 & 0 \end{bmatrix},$$

$a>0,\ b>0$. *Then*

- (i) *The corresponding cubic surface S_F has no singular points on the plane at infinity $t=0$.*
- (ii) *If $a=\sqrt{1+b^2}$, the surface S_F has an ordinary double point at $(t,x,y,z) = (1,2/b^2,0,-1/b^2)$ and a pair of imaginary ordinary double points $(t,x,y,z) = (1,(1-b^2)/b^2,\pm i(b^2+1)/b^2,-1/b^2)$. The cubic surface is of type [VIII], and the boundary generating surface of the Davis-Wielandt shell $DW(A)$ lies in a polynomial surface of degree 6.*
- (iii) *If $a\neq\sqrt{1+b^2}$, the surface S_F has a pair of imaginary ordinary double points $(t,x,y,z) = (1,(1-b^2)/b^2,\pm i(b^2+1)/b^2,-1/b^2)$. The cubic surface is of type [IV], and the boundary generating surface of the Davis-Wielandt shell $DW(A)$ lies in a polynomial surface of degree 8.*

Proof. Let $g(t,x,y,z) = 4F(t,x,y,z)$. Firstly, we show that the surface $g(t,x,y,z) = 0$ has no singular points on the plane $t=0$. We have

$$g(0,x,y,z) = -b^2(x^2+y^2)(x+(a^2+1)z).$$

Consider $g_z(0,x,y,z) = 0$, we may assume that $(x,y,z) = (0,0,1)$ or $(x,y,z) = (1,\pm i,z)$. If $(x,y,z) = (0,0,1)$ then $g_t(0,0,0,1) = 4(a^2b^2+b^2+1) \neq 0$, and thus $(x, y) = (1, \pm i)$. The condition $g_y(0, x, y, z) = 0$ implies that $z = -1/(a^2 + 1)$ and hence $g_x(0, x, y, z) = -b^2(3 - 1 - 2/(a^2 + 1) = -2b^2(a^2 + 1 - 1)/(a^2 + 1) = -2a^2b^2/(a^2 + 1) \neq 0$. This shows that the surface $g(t, x, y, z) = 0$ has no singular points on the plane $t = 0$.

Next, we deal with singular points of the surface $g(t, x, y, z) = 0$ on the affine space $t = 1, (x, y, z) \in \mathbf{C}^3$. Assume that $(1, x, y, z)$ is a singular point of $g(t, x, y.z) = 0$. Then

$$g_y(1, x, y, z) = -2y(b^2x + (a^2b^2 + b^2)z + a^2 + b^2) = 0. \tag{11}$$

Suppose

$$b^2x + (a^2b^2 + b^2)z + a^2 + b^2 = 0 \tag{12}$$

in (11). Solve (12) for $x = x(z)$. Then the equation $g(1, x(z), y, z) = 0$ becomes $4a^4(b^2z + 1)^2/b^4 = 0$. Thus $z = -1/b^2$, and $x = (1 - b^2)/b^2$. Further, we solve

$$g_z(1, (1 - b^2)/b^2, y, -1/b^2) = -\frac{a^2 + 1}{b^2}(b^4y^2 + (1 + b^2)^2) = 0$$

in y. Then $y = \pm i((b^2 + 1)/b^2)$. Conversely the point $(t, x, y, z) = (1, (1 - b^2)/b^2, \pm i(b^2 + 1)/b^2, -1/b^2)$ satisfies the condition for singularity. We conclude that the singular points $(1, x, y, z)$ with $y \neq 0$ are $(x, y, z) = ((1 - b^2)/b^2, \pm i(b^2 + 1)/b^2, -1/b^2)$.

Lastly, we deal with singular points of the surface $g(t, x, y, z) = 0$ on the plane $y = 0$. In this plane, $g_y(1, x, 0, z) = 0$ holds. Assume that $(1, x, 0, y)$ is a singular point of $g(t, x, y, z) = 0$. We solve the equation

$$g_z(1, x, 0, z) = 8(a^2b^2 + b^2 + 1)z - (a^2b^2 + b^2)x^2 + (4b^2 + 8)x + 4a^2 + 8 = 0$$

in $z = z(x)$. Then the resultant of $g(1, x, 0, z(x))$ and $g_x(1, x, 0, z(x))$ with respect to x is given by

$$-\frac{a^{12}b^8(a^2 + 1)^2(b^2 + 1)^4(a^2 - 1 - b^2)^2}{16(a^2b^2 + b^2 + 1)^5}$$

which does vanish if and only if $a = \sqrt{b^2 + 1}$. Thus in case (iii), $a \neq \sqrt{b^2 + 1}$, the cubic surface is of type [IV].

We assume $a = \sqrt{b^2 + 1}$ in (ii). Applying an Euclidean algorithm for $g(1, x, 0, z(x))$ and $g_x(1, x, 0, z(x))$ with respect to x, we obtain that their common divisor $b^2x - 2 = 0$. Then the singular point $(1, x, 0, z(x))$ on the plane $y = 0$ satisfies $x = 2/b^2$ and $z(x)$ is given by $z = -1/b^2$. Thus the surface S_F has three ordinary double points, the cubic surface is of type [VIII].

The class numbers (3) of (i) and (ii) are respectively $12 = 12 - 0$ and $10 = 12 - 2$, which are the degrees of the boundary generating surfaces of the respective $DW(A)$. □

We consider the cubic form corresponding to a nilpotent matrix

$$A = \begin{bmatrix} 0 & a & c \\ 0 & 0 & b \\ 0 & 0 & 0 \end{bmatrix}.$$

We assume $a \neq 0$, and we may assume $a = 1$. We also assume that $b > 0$ and $c \in \mathbf{R}$. We deal with the matrix

$$A = \begin{bmatrix} 0 & 1 & c \\ 0 & 0 & b \\ 0 & 0 & 0 \end{bmatrix}, \ b > 0, \ c \in \mathbf{R}, \tag{13}$$

Theorem 3. *Let A be the matrix as in (13).*

(*i*) *If $b = 1$, the surface S_F has two biplanar double points $(0, 1, i, c/b)$ and $(0, 1, -i, c/b)$, and one ordinary double point $(1, 2c, 0, c^2 - 1)$. The cubic surface is of type [XVII], and the boundary generating surface of the Davis-Wielandt shell $DW(A)$ lies in a polynomial surface of degree 4.*

(*ii*) *If $b \neq 1$, the surface S_F has two biplanar double points $(0, 1, i, c/b)$ and $(0, 1, -i, c/b)$. The cubic surface is of type [IX], and the boundary generating surface of the Davis-Wielandt shell $DW(A)$ lies in a polynomial surface of degree 6.*

Proof. By direct computations, a pair of points $(t, x, y, z) = (0, 1, i, c/b), (0, 1, -i, c/b)$ are biplanar double points of type A_2, and the surface S_F has no other singular points on the plane $t = 0$ at infinity. We examine singular points on the affine 3-space : $t = 1$, $(x, y, z) \in \mathbf{C}^3$. For $b = 1$, the surface S_F has an ordinary double point at $(t, x, y, z) = (1, 2c, 0, c^2 - 1)$. The cubic surface is of type [XVII].

For $0 < b, b \neq 1$, we will show that there is no singular point of the surface S_F on the affine 3-space $t = 1$, $(x, y, z) \in \mathbf{C}^3$, and thus the cubic surface is of type [IX]. We define the polynomial $g(x, y, z)$ and compute that

$$\begin{aligned} g(x, y, z) &= 4F(1, x, y, z) \\ &= bcx^3 + bcxy^2 - b^2(x^2 + y^2)z - (b^2 + c^2 + 1)(x^2 + y^2) \\ &\quad +4b^2z^2 + -4bcxz + 4(b^2 + c^2 + 1)z + 4. \end{aligned}$$

Then $g_y(x, y, z) = -2y(-bcx + b^2z + b^2 + c^2 + 1) = 0$. Suppose (x_0, y_0, z_0) is a singular point of $g = 0$ with $y_0 \neq 0$. We set $h = -bcx + b^2z + b^2 + c^2 + 1$. We find the resultant of g and h with respect to z is $4b^4 \neq 0$. Thus there is no such a singular point. So we assume that $(x_0, 0, z_0)$ is a singular point of $g = 0$. Then $g_z(x_0, 0, z_0) = 8b^2z - b^2x^2 - 4bcx + 4b^2 + 4c^2 + 4$. Solve $g_z(x, 0, z) = 0$ with respect to z, and substitute the solution $z = k(x)$ into $g_x(x, 0, z) = 0$, we obtain $(bx - 2c)(b^2x^2 - 4bcx + 4b^2 + 4c^2 + 4) = 0$. We set $m(x) = b^2x^2 - 4bcx + 4b^2 + 4c^2 + 4$. Then the resultant $m(x)$ and $g(x, 0, k(x))$ with respect to x is $16b^8 \neq 0$. This implies that $bx - 2c = 0$, we have $x = 2c/b$, and then $z = (2c^2 - b^2 - 1)/(2b^2)$. Hence $F = -(b - 1)^2(b + 1)^2/b^2 \neq 0$, and thus the surface $F = 0$ has no singular point in the affine 3-space.

The class numbers (3) of (*i*) and (*ii*) are respectively $4 = 12 - 2 \times 3 - 2$ and $6 = 12 - 2 \times 3$, which are the degrees of the boundary generating surfaces of the respective $DW(A)$. □

Remark 1. *We have found in Theorems 1-3 six types of cubic surfaces related with the Davis-Wielandt shell $DW(A)$ of 3×3 matrices. It is open whether there exist cubic surfaces other than types (I), (II), (IV), (VIII), (IX), (XVII) related with some 3×3 matrices.*

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