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On multiple conclusion deductions in classical logic

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Abstract. Kneale observed that Gentzen's calculus of natural deductions NK for classical logic is not symmetric and has unnecessarily complicated hypothetical inference rules. Kneale proposed inference rules with multiple conclusions as a basis for a symmetric natural deduction calculus for classical logic. However, Kneale's informally presented calculus is not complete. In this paper, we define a calculus of *multiple conclusion* natural deductions (MCD) for classical propositional logic based on Kneale's multiple conclusion inference rules. For MCD we present elementary proof search that produces proofs in normal form. MCD proof search is motivated and explained as being a notational variant of Smullyan's analytic tableaux method in its initial part and a notational variant of refutation proofs based on Robinson's resolution in its final part. We consider MCD to have semantic motivation of both its inference rules and its proof search. This is unusual for the natural deduction calculi as they are syntactically motivated. Syntactic motivation is adequate for intuitionistic logic but not a natural fit for truth-functional classical propositional logic.

AMS subject classifications: 03B05, 03F03

Key words: multiple conclusion natural deductions, Kneale's developments, analytic deductions, classical propositional logic

1. Introduction

Among the well-established formalisms of classical logic (natural deductions, sequent deductions and analytic tableaux) tableaux and sequent calculi are strongly related: a closed tableau is regarded as an upside-down notational variation of a cut-free sequent proof of G3cp calculus (see e.g. [6, 12]). However, conversion of NK natural deductions to sequent proofs (or tableaux) and back is not as direct and trivial. Additionally, we must take into account that a desired property of such conversions is to preserve the normality of the proof. It is the consensus that natural deduction calculi are inadequate for proof search because they lack deep symmetries (see e.g. [1, 4]). In this paper, we present a multiple conclusion natural deduction calculus for classical logic with symmetric inference rules. Symmetric inference rules allow for elementary proof search that is strongly related to the analytic tableaux method and Robinson's resolution.

The calculus of multiple conclusion deductions for classical logic has been informally proposed by Kneale in [10]. However, Kneale's version of multiple conclusion

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calculus is not complete. Variants of Kneale's calculus have been proposed and discussed by Shoesmith and Smiley in [15], Šikić in [19] and later by Ungar in [20]. Here we define the calculus of multiple conclusion deductions (MCD) as a complete variant of Kneale's calculus for classical propositional logic (CPL) with the objective to demonstrate that MCD allows for elementary proof search that produces proofs in normal form and that is strongly related to proof search of the analytic tableaux method and the resolution method.

Proof search for MCD has two phases: analysis and synthesis. Analysis, the initial phase of the proof search, is presented syntactically but also explained as a notational variant of the semantic analysis of the analytic tableaux method (cf. [18]). Synthesis, the second phase of proof search, is concerned with proof assembly and it is presented as an application of Robinson's resolution. Completeness of MCD proof search follows from the completeness of the resolution as a proof procedure. Finally, completeness and the normal form theorem for MCD calculus follow from the completeness of MCD proof search.

The approach to proof search is similar to the simultaneous bottom-up (from conclusions to premisses) and top-down approach (from premisses to conclusions) of Sieg and Byrnes [4] and Ferrari and Fiorentini [16]. Because of the symmetry and local inference rules present in MCD, the formulation of proof search for MCD is simpler than in [16] and [4], where authors work around the technical difficulties imposed by non-symmetric single-conclusion inferences.

1.1. Notation

The language consists of standard logical connectives $\neg$ (negation), $\wedge$ (conjunction), $\vee$ (disjunction) and $\rightarrow$ (material conditional). Letters $X, Y, Z, \dots$ are reserved for propositional variables, whereas letters A, B, C, $\dots$ are reserved for metaformulas. Sets of formulas are denoted by capital Greek letters Γ and Δ. Sets are also denoted in the cedent style (of sequents): we write Γ, A instead of $\Gamma \cup \{A\}$; we also write Γ, Δ instead of $\Gamma \cup \Delta$. By $\neg\Delta$ we denote the set of negations of formulas in Δ, and by $A \equiv B$ we denote the logical equivalence of formulas A and B.

1.2. Kneale's inference rules

In [10], W. Kneale observed that hypothetical inference rules, namely introductions $(\mathsf{I}{\rightarrow})_G$, $(\mathsf{I}\neg)_G$ and elimination $(\mathsf{E}\vee)_G$, of Gentzen's NK calculus are unnecessarily complicated (see Fig. 1). Hypothetical inference rules are not true inference rules (see [14]). Hypothetical inference rules are *proof rules* of NK and NJ – as their premisses are not formulas but proofs (or sequents). This single-conclusion form of inference rules is a good fit for intuitionistic logic, but not a natural fit for truth-functional classical propositional logic (see [1] for full discussion).

Kneale proposed inference rules with multiple conclusions (Fig. 2) as a foundation for a simpler calculus of deductions of classical logic. Introduction inference rules are listed in the left column in Fig. 2 and elimination inference rules are listed in the right column in Fig. 2. Each inference rule has an obvious major formula – the formula whose principal connective is eliminated or introduced. The other formulas

$$(\mathsf{I}{\to})_G\ \frac{\begin{array}{c}\not{A}\\ \vdots\\ B\end{array}}{A\to B}\qquad (\mathsf{I}\neg)_G\ \frac{\begin{array}{c}\not{A}\\ \vdots\\ \bot\end{array}}{\neg A}\qquad (\mathsf{E}{\vee})_G\ \frac{A\vee B\qquad \begin{array}{c}\not{A}\\ \vdots\\ C\end{array}\qquad \begin{array}{c}\not{B}\\ \vdots\\ C\end{array}}{C}$$

Figure 1: *Hypothetical inference rules of NK*

in Kneale's inference are called minor formulas.

Note the subformula property – a minor formula is a subformula of a major formula in an inference. Also note the absence of premises in (I¬), (I→); and the absence of conclusions in (E¬) inference rules. Rules (I¬) and (I→) need no premises. Rule (I¬) reflects the classical principle of bivalence. Likewise, (E¬) inference rule, reflecting the principle of non-contradiction, has contradictory premisses and therefore no conclusion.

$$(\mathsf{I}{\wedge})\ \frac{A\qquad B}{A\wedge B}\qquad\qquad (\mathsf{E}{\wedge})\ \frac{A\wedge B}{A}\qquad \frac{A\wedge B}{B}$$

$$(\mathsf{I}{\vee})\ \frac{A}{A\vee B}\qquad \frac{B}{A\vee B}\qquad\qquad (\mathsf{E}{\vee})\ \frac{A\vee B}{A\qquad B}$$

$$(\mathsf{I}{\to})\ \frac{B}{A\to B}\qquad \frac{}{A\qquad A\to B}\qquad\qquad (\mathsf{E}{\to})\ \frac{A\qquad A\to B}{B}$$

$$(\mathsf{I}\neg)\ \frac{}{A\qquad \neg A}\qquad\qquad (\mathsf{E}\neg)\ \frac{A\qquad \neg A}{}$$

Figure 2: *Kneale's multiple conclusion inference rules*

1.3. Kneale's calculus

Proofs in the NK and NJ calculi are rooted formula trees based on single conclusion inferences. On the other hand, graphs representing multiple conclusion proofs may branch in both directions (up and down). For this reason, the underlying graph structure is somewhat more complicated.

Definition 1 (Formula graph)**.** *A formula graph* Π *is a directed acyclic graph (DAG) with a bipartition* $(\mathcal{F}, \mathcal{S})$*, where* $\mathcal{F}$ *is a nonempty set of formula-labeled nodes.*

Nodes in $\mathcal{F}$ *are called* formula nodes *or* formulas *for short, and nodes in* $\mathcal{S}$ *are called* inference strokes*.* *A* formula occurrence of A *in* Π *is a node labeled with formula* A*.*

We say that node **u** precedes node **v** in a formula graph Π if there exists a directed path from **u** to **v**. We draw formula graphs with edges directed downward – "from top to bottom": formula A is drawn above B if A precedes B in Π. A source (sink) in Π is a formula node of zero indegree (outdegree). A formula graph Π with a sink that is an occurrence of A is denoted by Π/A. Likewise, a formula graph Π with a source occurrence of A is denoted by A/Π. In a graphical context, formula graphs Π/A and A/Π are denoted by $\genfrac{}{}{0pt}{}{\Pi}{A}$ and $\genfrac{}{}{0pt}{}{A}{\Pi}$, respectively. A formula node in Π is external if it is a source or a sink. Other formula nodes are internal formula nodes of Π.

Kneale's inferences are examples of formula graphs. Formula graphs that correspond to Kneale's inferences (instances of inference rules) are displayed in Fig. 3.

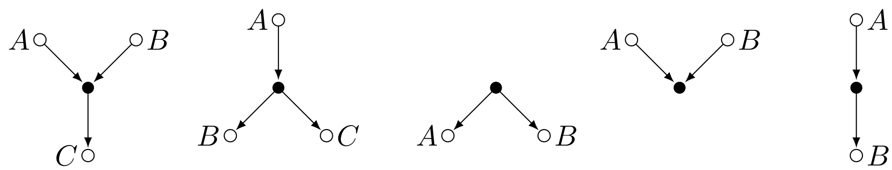

Figure 3: *Formula graphs of Kneale's inferences: white nodes are formula nodes, black nodes represent inference strokes*

Formula graphs in this paper are displayed in the familiar style of natural deductions. Inference strokes are drawn as horizontal lines (as usual). Directed edges are implicit (not drawn) and the assumed direction is top-down. A graph may branch up or down at the inference node. The type of branching is clear from the type of inference rule of a particular inference stroke – an inference with multiple premises branches up and an inference with multiple conclusions branches down.

Example 1. *Kneale described multiple conclusion proofs as an assembly of inferences. So, for example, a proof of modus tollens can be assembled as follows.*

The following inferences

$$\frac{A \to B \qquad A}{B}, \qquad \frac{\neg B \qquad B}{}$$

can be joined over B to obtain the formula graph:

$$\frac{\neg B \qquad \dfrac{A \to B \qquad A}{B}}{} \quad .$$

This formula graph can be further joined over A with an inference

$$\frac{}{A \qquad \neg A}$$

to finally obtain the formula graph Π

$$\frac{\neg B \qquad \dfrac{A \to B \qquad \overline{A \qquad\qquad} \neg A}{B}}{} \quad .$$

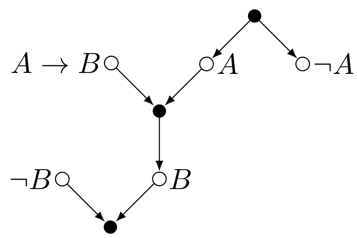

Figure 4: *Formula graph Π explicitly drawn as a DAG*

Formula graph Π is an example of Kneale's proof of $\neg B, A \to B \vdash \neg A$ (modus tollens). Premises of Π are sources $\neg B$ and $A \to B$. Conclusion of Π is the sink $\neg A$.

Formula graph Π is displayed as a DAG in Fig. 4.

Definition 2 (Join). *Let Π_1/A and A/Π_2 be disjoint formula graphs. A join of Π_1/A and A/Π_2 over A is a formula graph obtained by a vertex contraction of a sink formula occurrence of A in Π_1 with a source formula occurrence of A in Π_2.*

Now we give a formal definition of Kneale's proofs which he named developments.

Definition 3. *Kneale's developments are defined as follows.*

(i) *A singleton formula graph A is Kneale's development.*

(ii) *Let Π_1/A, Π_2/B, C/Π_3 and D/Π_4 be Kneale's developments.*

(a) *If* $\dfrac{A \qquad B}{}$ *is Kneale's inference, then the formula graph*

$$\begin{array}{cc} \Pi_1 & \Pi_2 \\ A & B \\ \hline \end{array}$$

is Kneale's development.

(b) *If* $\dfrac{A \qquad B}{C}$ *is Kneale's inference, then the formula graph*

$$\begin{array}{c} \begin{array}{cc} \Pi_1 & \Pi_2 \\ A & B \\ \hline \end{array} \\ C \\ \Pi_3 \end{array}$$

is Kneale's development.

(c) *If* $\dfrac{A}{C}$ *is Kneale's inference, then the formula graph*

$$\begin{array}{c} \Pi_1 \\ \dfrac{A}{C} \\ \Pi_3 \end{array}$$

is Kneale's development.

(*d*) *If* $\dfrac{A}{C \qquad D}$ *is Kneale's inference, then the formula graph*

$$\begin{array}{c} \Pi_1 \\ \dfrac{A}{C \qquad D} \\ \Pi_3 \qquad \Pi_4 \end{array}$$

is Kneale's development.

(*e*) *If* $\dfrac{}{C \qquad D}$ *is Kneale's inference, then the formula graph*

$$\begin{array}{c} \dfrac{}{C \qquad D} \\ \Pi_3 \qquad \Pi_4 \end{array}$$

is Kneale's development.

The source formula node of Kneale's development Π *is called a premise node of* Π. *Formula* A *is a premise of* Π *if a formula occurrence of* A *is a premise node of* Π. *Conclusion nodes and conclusions are defined likewise.*
Kneale's development Π *is said to be Kneale's development of* Δ *from* Γ *if the set of* Π*'s premises is a subset of* Γ *and the set of* Π*'s conclusions is a subset of* Δ.

$$\dfrac{\dfrac{A \vee B}{A \qquad B}}{A \wedge B}$$

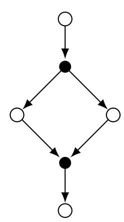

Figure 5: *Absurd pattern (in deductive and explicit style)*

The absurd pattern displayed in Fig. 5 is *not* Kneale's development. It is not possible to build such a graph as a sequence of joins of Kneale's developments. It features what Shoesmith and Smiley [15] named a *circuit*, as they wanted to emphasize that in such a graph there are multiple paths between some formula nodes. Therefore, Shoesmith and Smiley [15] referred to Kneale's developments as circuit-free, because paths between formula nodes in Kneale's development are unique.

Kneale's calculus of developments is sound but incomplete. For example, classical tautology – distributive law $A \vee (B \wedge C) \to (A \vee B) \wedge (A \vee C)$ is not provable with Kneale's developments. Multiple sink or source occurrences of a formula present a problem for the calculus of Kneale's developments.

2. MCD calculus

Let Π_1/A be Kneale's development with multiple conclusion occurrences of A. Let A/Π_2 be Kneale's developments with multiple premise occurrences of A. A join of Π_1/A and A/Π_2 can not "cut" A as a premise and a conclusion of the join of Π_1/A and A/Π_2. Additional occurrences of the same formula should be discarded as premises (conclusions) after the join.

Definition 4 (Contraction)**.** *Let Π be a formula graph with sink (source) occurrences of A. Let Π' be a formula graph obtained by the vertex contraction of all sink (source) occurrences of A. We say that Π' is a contraction of A in Π. Formula A is said to be contracted in Π'.*

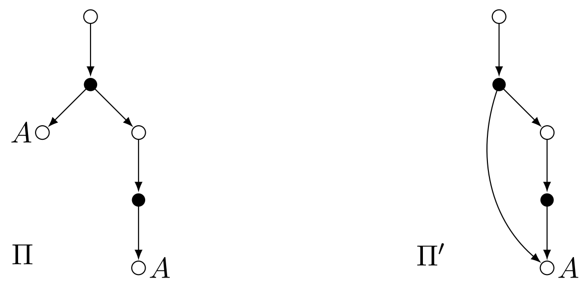

Figure 6: *Π' is a contraction of A in Π*

An example of contraction is displayed explicitly in Fig. 6. To denote contraction and contracted nodes for formula graphs displayed in the deductive style we borrow the notation used for discharging hypotheses in Fig. 7. Note that occurrences of a formula are indexed and all occurrences except a single one are discharged.
Now we define MCD calculus as an extension of Kneale's calculus.

Definition 5 (MCD deductions)**.**

(*i*) *Kneale's development is an MCD deduction.*

(*ii*) *Contraction of an MCD deduction is an MCD deduction.*

(*iii*) *Join of MCD deductions is an MCD deduction.*

Premise nodes, premises, conclusion nodes and conclusions of an MCD deduction are defined as in Definition 3.

$$\frac{A \vee (A \wedge B)}{\cancel{A}_{(1)} \qquad \dfrac{A \wedge B}{A_{(1)}}}$$

Figure 7: *Contraction of A denoted in the deductive style*

Definition 6 (MCD proof)**.**
Let Γ, Δ be sets of formulas. An MCD proof of $\Gamma \vdash \Delta$ is an MCD deduction with a set of premises $\Gamma' \subseteq \Gamma$ and a set of conclusions $\Delta' \subseteq \Delta$. The resulting logical calculus is called the calculus of multiple conclusion deductions and we denote it by MCD.

We say that MCD deductions Π_1 and Π_2 are equivalent if they prove the same $\Gamma \vdash \Delta$ sequents.

Remark 1. *Contraction of MCD deductions corresponds to Gentzen's structural rule of contraction in LK (see [7]), where cedents are not sets but finite sequences of formulas.*

Remark 2. *From now on, we assume that every join operation is preceded by a contraction of the same node. Let Π_1/A be a proof of $\Gamma_1 \vdash A, \Delta_1$ and A/Π_2 a proof of $\Gamma_2, A \vdash \Delta_2$, where $A \notin \Delta_1, \Gamma_2$. Then the join of Π_1/A and A/Π_2 is a proof of a cut-sequent*

$$\Gamma_1, \Gamma_2 \vdash \Delta_1, \Delta_2 \ .$$

2.1. Normal form theorem and completeness of MCD

We may consider MCD proofs to be assembled from inferences (by joins, as hinted by Definition 5 and Example 1). Every formula node in an MCD deduction is contained in some inference. An internal formula node is contained in two *consecutive inferences*, i.e. as a conclusion in one and as a premise in another inference.

We say that an MCD deduction Π is normal or in normal form if no formula occurrence is a major formula in two consecutive inferences, or if introduction inference is never immediately followed by elimination inference (cf. [14]).

Theorem 1 (MCD normal form)**.**
If $\Gamma \vdash \Delta$ is provable in MCD, then there is an MCD deduction in normal form that proves $\Gamma \vdash \Delta$.

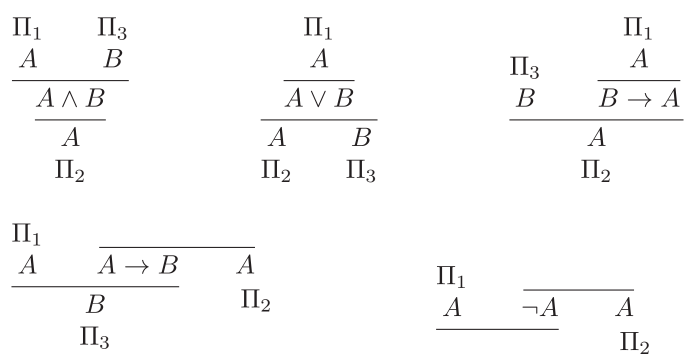

Figure 8: *Kneale's developments that are not in normal form*

Proof of Kneale's developments (see [15]). The normal form theorem has a simple proof for the calculus of Kneale's developments. Kneale's development that is not in normal form corresponds to a formula graph listed in Fig. 8. Due to the commutativity of $\wedge$ and the commutativity of $\vee$, there are only five possibilities.
Each Kneale's development displayed in Fig. 8 can be reduced to stronger and smaller Kneale's development – by performing a join of Π_1/A and A/Π_2. Eventually, a Kneale's development in normal form is reached that is a proof of Δ from Γ. $\square$

Contraction complicates the decomposition of an MCD deduction. Consequently, the previous proof does not translate to the case of MCD deductions.

Theorem 2 (MCD completeness)**.** *MCD calculus is complete:*

Δ *is provable from* Γ *in MCD if and only if* $\Gamma \models \Delta$.

Proof of soundness. Kneale's inferences are valid. Proof rules of MCD (contraction and join) are valid because they correspond to the contraction and cut of Gentzen's LK. □

Both the normal form theorem and the completeness theorem of MCD follow from the completeness of MCD proof search (Theorem 3) described in the following section.

Remark 3 (On duality and vertical symmetry)**.** *The* $\{\neg, \wedge, \vee\}$*-fragment of MCD is vertically symmetric – dual connectives have pairs of vertically symmetric inference rules:* (I$\wedge$) *with* (E$\vee$), (E$\wedge$) *with* (I$\vee$)*; and* (I$\neg$) *with* (E$\neg$) *of the self-dual negation* $\neg$ *(see Fig. 2). Contraction and join, the proof rules of MCD, are also vertically symmetric.*

Let Π *be an MCD proof of* $\Gamma \vdash \Delta$. *Let* Π^d *be a formula graph obtained by reversing the orientation of arcs in* Π *(i.e. by turning* Π *upside-down) and by substituting the* A^d *(dual of* A*) for every formula* A *in* Π. *Then* Π^d *is an MCD proof of* $\Delta^d \vdash \Gamma^d$, *where* Γ^d *and* Δ^d *denote the sets of dual formulas of* Γ *and* Δ*, respectively.*

3. Proof search in MCD

3.1. Overview

In this section, we present elementary proof search for MCD calculus. MCD proof search is divided in two parts: *analysis* and *synthesis*. Proof search starts with analysis, where we search for the fragments needed for the final proof. Analysis is followed by synthesis, where the final MCD proof of $\Gamma \vdash \Delta$ is assembled (if it exists).

In analysis, we pursue a naïve approach: we analytically search for a set of deductions in normal form that have either a premise in Γ or a conclusion in Δ. Afterwards, we try to assemble an MCD deduction Π that proves $\Gamma \vdash \Delta$. We justify this approach by showing how it is related to Smullyan's analytic tableaux method and Robinson's resolution.

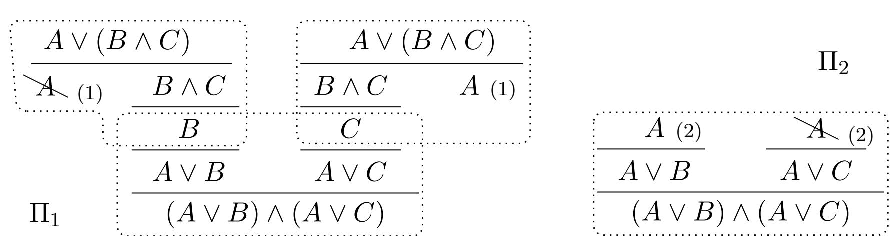

Figure 9: *A join of MCD deductions* Π_1/A *and* A/Π_2 *obtains the MCD proof of* $A \vee (B \wedge C) \vdash (A \vee B) \wedge (A \vee C)$

An example of an assembly of an MCD deduction of $(A \vee B) \wedge (A \vee C)$ from $A \vee (B \wedge C)$ is shown in Fig. 9. These building blocks (placed in dotted frames for emphasis) are obtained from the analysis of the premise $A \vee (B \wedge C)$ and the analysis of the conclusion $(A \vee B) \wedge (A \vee C)$.

3.2. Analysis

Throughout for the remainder of the section, let us assume that Γ and Δ are finite and disjoint sets of formulas.

Definition 7 (Analytic deductions)**.**
Let Π be an MCD deduction. We say Π is analytic if Π is a singleton formula graph or if:

- (i) Π *is in normal form; and*
- (ii) Π *has (exactly) one source or sink formula occurrence of M such that every formula in Π is a subformula of M. (We say that such M is the major formula of deduction Π. All other external formula occurrences are minor or minor formulas.)*

An analytic deduction M/Π with a major formula M is a downward analytic deduction of M. An upward analytic deduction Π of M is defined likewise. An analytic deduction Π is complete if all of its minor formulas are atomic.

Let Π be a complete analytic deduction of M. We say that Π is $\boldsymbol{\Gamma \vdash \Delta}$ *development if it is a downward analytic deduction of $M \in \Gamma$ or if it is an upward analytic deduction of $M \in \Delta$.*

In the following sections, we show that if $\Gamma \models \Delta$ holds, then $\Gamma \vdash \Delta$ developments are sufficient for the assembly of an MCD proof of $\Gamma \vdash \Delta$.

Definition 8. *Let A be a non-atomic minor conclusion of an analytic deduction Π. Let Σ be an elimination inference whose major formula is A. Let Π' be a join of Π/A and A/Σ over A. We say that Π' is analysis down of (a formula occurrence of) A in Π/A. Analysis up is defined likewise.*

Atomic formulas in Γ, Δ need not be analyzed. Let Π be an analytic deduction with a non-atomic minor node A. In case an appropriate inference rule with A as a major formula has two forms, analysis (up or down) of A in Π are two formula graphs – one for each join.

Let us illustrate the search for analytic deductions with an example.

Example 2.
Let Π be an analytic deduction with a minor node $A \to B$.

- (i) *If $A \to B$ is a conclusion of Π, then we join $\Pi/(A \to B)$ with* (E$\to$) *inference to obtain the following MCD deduction*

$$\dfrac{\begin{array}{c}\Pi\\ A \to B\end{array} \qquad A}{B},$$

as analysis down of $\Pi/(A \to B)$.

(*ii*) *If* $A \to B$ *is a premise of* Π, *then the following MCD deductions*

$$\begin{array}{ll} \multicolumn{2}{c}{\overline{\qquad\qquad\qquad\quad}} \\ A & A \to B \\ & \Pi \end{array}, \qquad \begin{array}{c} B \\ \hline A \to B \\ \Pi \end{array}$$

are (both) analysis up of $(A \to B)/\Pi$.

The search for the set of $\Gamma \vdash \Delta$ developments (i.e. the analytic part of MCD proof search) is described as follows:

STEP 1. Let S be a set of analysis down of Γ formulas and analysis up of Δ formulas.

STEP 2. Replace any $\Pi \in S$ that is not complete with analysis (up or down, as appropriate) of Π.

STEP 3. Repeat Step 2 until all $\Pi \in S$ are complete.

Note that $\Gamma \vdash \Delta$ developments are Kneale's developments.

3.2.1. Search for analytic deductions is semantic analysis

The search for analytic deductions described in the previous section is a syntactic procedure. In this section, we explain the search for analytic deductions as a semantic analysis of Smullyan's analytic tableaux method.

Let Π be an analytic deduction of M and let minor nodes of Π be occurrences of $A_1, A_2, \ldots, A_n$. Premises and conclusions in an analytic deduction Π can be rearranged and "read" analytically, i.e. from the major formula toward minor formulas as follows:

$$M' \models A'_1 \vee \cdots \vee A'_n, \tag{1}$$

where

$$M' = \begin{cases} M, & \text{if } M \text{ is a premise} \\ \neg M, & \text{otherwise} \end{cases}, \tag{2}$$

and

$$A'_i = \begin{cases} A_i, & \text{if } A_i \text{ is a conclusion} \\ \neg A_i, & \text{otherwise} \end{cases}. \tag{3}$$

We say that M' and A'_i are *analytically signed* (relative to Π).

Lemma 1 (α and β analysis)**.** *The following holds:*

(*i*) *Let* Π *be an instance of a* (I$\wedge$), (E$\vee$), (E$\to$) *inference rule with a major formula* M. *Then*

$$M' \equiv B'_1 \vee B'_2,$$

where M *is a major formula and* B_1, B_2 *are minor formulas of* Π.

(*ii*) *Let* Π_1 *be an instance of an* ($\mathsf{E}\wedge$)*,* ($\mathsf{I}\vee$) *or* ($\mathsf{I}\rightarrow$) *inference rule with a major formula* M*. Let* Π_2 *be the other inference of the same inference rule. Then*

$$M' \equiv A_1' \vee A_2',$$

where A_1 *and* A_2 *are minor formulas of* Π_1 *and* Π_2*, respectively.*

Proof of Lemma 1. We use Smullyan's unified notation for the classification of propositional formulas. Recall that a propositional formula is either an α or β formula with

$$\alpha \equiv \alpha_1 \wedge \alpha_2 \qquad \text{and} \qquad \beta \equiv \beta_1 \vee \beta_2, \tag{4}$$

where α_i and β_i are (possibly negations of) immediate subformulas of α and β, respectively, for $i = 1, 2$. Note that M' is a β formula in claim (i), and M' is an α formula in claim (ii) of Lemma 1. It is easy to check that B_1', B_2', A_1' and A_2' are exactly β_1, β_2, α_1 and α_2 formulas. □

Therefore, analysis down of Π/A can be regarded as a semantic analysis of A. Likewise, analysis up of A/Π is regarded as semantic analysis of $\neg A$.

Lemma 1 omits the trivial case of the ($\mathsf{I}\neg$) and ($\mathsf{E}\neg$) inference. Analysis of the $\neg A$ formula proceeds by joining with the appropriate ($\mathsf{E}\neg$) or ($\mathsf{I}\neg$) inference and can therefore proceed (correctly) with the analysis of A in the opposite direction.

3.2.2. Clausal form

Definition 9. *Let* Π *be* $\Gamma \vdash \Delta$ *development.*
We define $\mathtt{d}(\Pi)$ *as follows:*

(*i*) *If* Π *is a singleton formula graph with a formula occurrence of* A*, then*

$$\mathtt{d}(\Pi) = A',$$

where $A' = A$ *if* $A \in \Gamma$ *and* $A' = \neg A$ *otherwise (if* $A \in \Delta$*).*

(*ii*) *For a non-singleton deduction* Π*, let* $u_1, \ldots, u_n$ *be minor external formulas of* Π *and let node* u_i *be an occurrence of a formula* A_i *for* $i = 1, \ldots, n$*. Then we define*

$$\mathtt{d}(\Pi) = A_1' \vee \cdots \vee A_n',$$

where A_i' *are signed as in* (3)*.*

If Π is a complete analytic deduction, then $\mathtt{d}(\Pi)$ is a clause – a disjunction of literals.

Proposition 1. *Let* S *be a set of all* $\Gamma \vdash \Delta$ *developments. Formula*

$$\bigwedge_{\Pi \in S} \mathtt{d}(\Pi) \tag{5}$$

is logically equivalent to $\Gamma, \neg\Delta$*.*

Proof. Let Π_0 be an analytic deduction with a minor non-atomic node v. Let v be a formula occurrence of A. Then $d(\Pi_0) = F \vee A'$, where F is a (possibly empty) disjunction of the remaining analytically signed minor formulas in Π_0.

1. If A' is an α formula, then analysis of v in Π are analytic deductions Π_1 and Π_2 such that

$$\mathtt{d}(\Pi_1) \wedge \mathtt{d}(\Pi_2) = (F \vee \alpha_1) \wedge (F \vee \alpha_2)\ . \tag{6}$$

2. If A' is a β formula, then analysis of v in Π is an analytic deduction Π_1 such that

$$\mathtt{d}(\Pi_1) = (F \vee \beta_1 \vee \beta_2)\ . \tag{7}$$

Note that both (6) and (7) are equivalent to $\mathtt{d}(\Pi_0)$.

We define $\mathtt{c}(S) = \bigwedge_{\Pi \in S} \mathtt{d}(\Pi)$ for a set of analytic deductions S. Let S_0 be the initial set of deductions in the search form $\Gamma \vdash \Delta$ developments (Step 1 of analysis). The following holds

$$\mathtt{c}(S_0) \equiv \Gamma, \neg\Delta\ .$$

Let S' be a set obtained in a single-step analysis of set S, i.e. by replacing $\Pi_0 \in S$ with analysis of v in Π_0 (Step 2 of analysis). Then

$$\mathtt{c}(S) \equiv \mathtt{c}(S'),$$

because conjunction appearing on the right-hand side can be obtained by replacing $\mathtt{d}(\Pi_0)$ with the equivalent (6) or (7). Let S_f be a set of all $\Gamma \vdash \Delta$ developments (obtained in a finite number of analysis steps). We conclude that the equivalence $\mathtt{c}(S_f) \equiv \Gamma, \neg\Delta$ holds. □

In other words, a by-product of the search for $\Gamma \vdash \Delta$ developments is the clausal form of $\Gamma, \neg\Delta$.

Remark 4. *A complete analytic tableaux of a formula gives an analogous conversion to disjunctive normal form (DNF) – each conjunction is formed by the literals of a specific open branch (see [18]).*

3.3. Synthesis

The objective of synthesis, the second part of MCD proof search, is to join $\Gamma \vdash \Delta$ developments to obtain a proof of Δ from Γ. In the simplest of cases we can do this "by hand". In the general case, proof assembly is equivalent to finding a resolution refutation proof that shows that $\Gamma, \neg\Delta$ is not satisfiable.

Without loss of generality, we may assume that Γ and Δ do not contain atomic formulas as any atom A can be replaced by the logical equivalent of $A \wedge A$ or $A \vee A$. A join of deductions over an atomic formula is called an atomic join. To obtain a proof of $\Gamma \vdash \Delta$ one must construct a sequence of atomic joins of $\Gamma \vdash \Delta$ developments that produces a deduction without external atomic nodes.

Let Π be a finite atomic join of $\Gamma \vdash \Delta$ developments. For Π we define

$$\partial(\Pi) = X_1' \vee \cdots \vee X_n', \tag{8}$$

where X_i' are analytically signed external occurrences of atoms of Π (as before in (1) and (3)). If Π has no external atomic nodes, then $\partial(\Pi)$ is defined to be the empty clause Λ. Informally, we may think that $\partial(\Pi)$ measures how far Π is from being a proof of $\Gamma \vdash \Delta$. The goal of synthesis is to find Π for which $\partial(\Pi) = \Lambda$.

Let us denote the atomic join of analytic deductions Π_1 and Π_2 by $\Pi_1 * \Pi_2$. The following holds:

$$\partial(\Pi_1 * \Pi_2) = \mathrm{Res}(\partial(\Pi_1), \partial(\Pi_2)) \ , \tag{9}$$

where $\mathrm{Res}(\partial(\Pi_1), \partial(\Pi_2))$ denotes the resolvent of clauses $\partial(\Pi_1)$ and $\partial(\Pi_2)$. Informally, we may say that ∂, regarded as a mapping, is a "homomorphic" extension of d to atomic joins of $\Gamma \vdash \Delta$ developments.

To find Π such that $\partial(\Pi) = \Lambda$ we can translate the resolution-based derivation of Λ from the clausal form of $\Gamma \vdash \Delta$ developments as follows:

(i) translate every clause on the top of the resolution derivation tree to a corresponding $\Gamma \vdash \Delta$ development,

(ii) translate every resolution inference to an atomic join of corresponding deductions of input clauses.

Finally, the empty resolvent is translated to an MCD proof of $\Gamma \vdash \Delta$.

Example 3. *Let us illustrate the search for the MCD proof of* $X \vee (Y \wedge Z) \vdash (X \vee Y) \wedge (X \vee Z)$*.*

1. *Analysis*

   With analysis we obtain analytic deductions $\Pi_1, \ldots, \Pi_6$ *with their respective clauses:*

$$X \vee Y, \quad \neg Y \vee \neg Z, \quad Z \vee X, \quad \neg X \quad \neg X \vee \neg Z \quad \neg X \vee \neg Y. \tag{10}$$

2. *Synthesis*

$$\dfrac{\dfrac{\dfrac{X \vee Y \qquad \neg Y \vee \neg Z}{X \vee \neg Z} \qquad X \vee Z}{X} \qquad \neg X}{\Lambda}$$

Figure 10: *A proof of inconsistency of clausal form* (10)

i. *Refutation tree of clausal form* (10) *is displayed in Fig. 10.*

ii. *A proof* Π *is obtained through the translation of the refutation tree to*

$$\Pi = ((\Pi_1 * \Pi_2) * \Pi_3) * \Pi_4 \ .$$

Proof Π *is displayed in Fig. 11. For emphasis, analytic deductions* $\Pi_1, \ldots,$ Π_4 *needed for the assembly are enclosed in dotted frames.*

Note that turning Π *upside down (see Remark 3) yields a (dual) proof of* $(X \wedge Y) \vee (X \wedge Z) \vdash X \wedge (Y \vee Z)$*.*

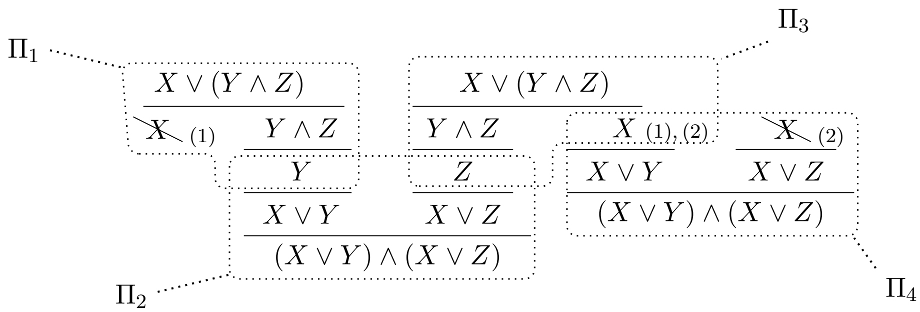

Figure 11: *MCD proof of* $X \vee (Y \wedge Z) \vdash (X \vee Y) \wedge (X \vee Z)$

3.4. Summary of MCD proof search

MCD proof search for $\Gamma \vdash \Delta$ is a proof-search procedure for MCD calculus. It is conducted in two phases. Analysis, the initial phase of MCD proof search, is a search for $\Gamma \vdash \Delta$ developments. A by-product of the analysis is a clausal form of $\Gamma, \neg\Delta$. Analysis is followed by synthesis – the second phase of MCD proof search. In synthesis, the proof of $\Gamma \vdash \Delta$ is assembled if $\Gamma, \neg\Delta$ is not satisfiable, i.e. if $\Gamma \models \Delta$. Synthesis of MCD proof search can be based on the resolution-based proof procedure (e.g. Davis-Putnam algorithm).

Remark 5. *Another approach to synthesis not considered here is to use a closed clausal tableaux to obtain a refutation-tree proof of* $\Gamma \models \Delta$ *(see [11]).*

Now we state the main theorem on MCD proof search.

Theorem 3. *MCD proof search is a complete procedure for MCD calculus. MCD deductions obtained by MCD proof search are in normal form.*

Proof. Completeness of MCD proof search follows from the completeness of Robinson's resolution method.

$\Gamma \vdash \Delta$ developments are by definition in normal form. Atomic join of deductions in normal form is again a deduction in normal form (as an atom is always a minor formula in any inference). Therefore, a proof obtained by MCD proof search is in normal form. □

Proof strategies of tableaux method and resolution-based proof search easily translate to proof search for MCD as steps in analysis and synthesis completely correspond. In synthesis, for example, we can ignore $\Gamma \vdash \Delta$ developments which correspond to clauses that would be ignored in the Davis-Putnam algorithm (e.g. clauses with pure literals, superset clauses ignored by subsumption rule; see [9]).

4. Conclusion

The motivation for this work was the notion that natural deduction proofs are very different from other proof formalisms of classical logic. This is an aftereffect of the fact that presentation of natural deduction calculi is traditionally inclined to intuitionistic logic. Classical natural deduction calculi have a steep learning curve with a significant gap between the ability to read a proof and the ability to construct a proof (to have a proof strategy). Other CPL formalisms (resolution and tableaux, especially truth-tables) have a simple proof strategy. Following Kneale's idea, this paper presents a simple and symmetric multiple conclusion calculus MCD for the CPL. Local inference rules of MCD calculus reflect the truth-functional nature of classical connectives. Proof rules of MCD, join and contraction, are also simple. Simplicity of both the inference and proof rules of MCD allows for elementary proof search with semantic motivation accessible to the first-time students of classical logic.

MCD proof search is divided into two phases: analysis and synthesis. In analysis, the necessary proof fragments are "grown" analytically by joins of formula graphs with increasingly simpler inferences. In synthesis, the obtained proof fragments are matched and assembled into a final proof, if such a proof exists. Both phases in proof search are just notational variations of well established proof formalisms of classical logic. The analysis is a notational variant of the semantic analysis of the analytic tableaux method. The synthesis is an application of Robinson's resolution. From these equivalences there follow standard metalogical results: completeness of MCD proof search and consequently completeness and normal form theorem of MCD.

Multiple conclusion sequents were already introduced by Gentzen in [8], who noted their coherence and formal elegance but discarded them as unnatural. The vague notion of "naturalness" is given to natural deduction calculi to reflect that introducing and discharging hypotheses mimic the mathematicians' proof technique based on making and dropping assumptions. We argue that MCD is natural in this sense: introduction of an assumption in an MCD proof is equivalent to choosing a specific path among several alternatives in the deduction. An assumption in an MCD deduction of $\Gamma \vdash \Delta$ is an internal formula node. An assumption in an MCD deduction is "discharged" if all directed paths end within the set of conclusions Δ, i.e. if the assumption disappears from the set of conclusions.

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